BASICS OF THE STRUCTURE OF SPHERICAL VARIETIES

GUIDO PEZZINI

Abstract. This is an overview of basic properties of spherical varieties, based on a talk given at the Mathematisches Forschungsinstitut Oberwolfach during the MiniWorkshop "Spherical varieties and Automorphic representations", May 2013.

Let G be a reductive connected complex algebraic group, and X a normal irreducible complex algebraic variety equipped with an action of G.

In the paper [4], D. Luna and T. Vust introduced an invariant of such an action, with the goal of giving a sort of measure to which extent the properties of the G-action determine the geometry of the variety. This invariant, called the *complexity* of X and denoted by $c(X)$, is defined as the minimal codimension of a B-orbit on X, where B is a Borel subgroup of G. The idea is that the lower the complexity is, the more influence the symmetries of X induced by G have on the geometry of X itself.

Under this point of view varieties of complexity zero are the simplest cases of G-varieties, yet include many varieties that are classically known and studied in the theory of reductive groups. More precisely, *normal* varieties of complexity zero are called *spherical varieties*, and have been studied extensively in the last 30 years.

The assumption of normality is of technical nature, but so fundamental for the theory that only in the last few years the first results on some non-normal complexity zero varieties appeared in the literature (see e.g. [2]). For the purposes of this report, we may underline that a useful consequence of normality is that X is covered by quasi-affine G-stable open subsets (see [6, Lemma 8]).

Examples of spherical varieties are complete homogeneous spaces $X = G/P$, where P is a parabolic subgroup of G; toric varieties, where in this case $G = (\mathbb{G}_m)^n$ is an algebraic torus; symmetric homogeneous spaces $X = G/G^\theta$ where $\theta\colon G \to G$ is an involution. We report some other examples.

(1) $G = \mathrm{SL}(2) \times \mathrm{SL}(2) \times \mathrm{SL}(2)$ and $X = G/\mathrm{diag}(\mathrm{SL}(2))$ (notice that such a homogeneous space is not spherical if a semisimple group of rank higher than 1 is used instead of $\mathrm{SL}(2)$).
(2) $X = G/U$ where U is a maximal unipotent subgroup of G.
(3) $X = \mathrm{SL}(3)/H$ with $H = TU_{\alpha_1+\alpha_2}$, where $T \subset B$ is a maximal torus, α_1, α_2 are simple roots associated to T and B, and $U_{\alpha_1+\alpha_2}$ is the one dimensional unipotent subgroup of B associated to $\alpha_1 + \alpha_2$.
(4) The projective space of 2-by-2 matrices $\mathbb{P}(M_{2\times 2})$, with $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$ acting by left and right multiplication.

Sphericity is equivalent to various other properties, and has strong consequences on the G- and B-action on X. We summarize in the next two theorems some basic facts, and for other equivalent definition of sphericity we refer to Chapter 5 of [7].

Theorem 1 ([9])**.** *Let X be a normal irreducible G-variety. If X is affine, then X is spherical if and only if the ring of regular functions $\mathbb{C}[X]$ is a multiplicity-free G-module, i.e. any two distinct irreducible submodules are non-isomorphic. In general, X is spherical if and only if the space of global sections $\Gamma(X, \mathcal{L})$ is a multiplicity-free G-module, for all linearized line bundle $\mathcal{L}$ on X.*

Theorem 2 ([8] and [1])**.** *Let X be a spherical G-variety. Then G and B have a finite number of orbits on X, and the closure of any G-orbit is a spherical G-variety.*

We remark that in general the inequality $c(Z) \leq c(X)$ holds for all Z closed, irreducible, B-stable subset of X, without assuming X spherical. This implies the finiteness of the number of B- and G-orbits whenever $c(X) = 0$. On the other hand, if Y is also such a subset and $Z \subset Y$ holds, then the inequality $c(Z) \leq c(Y)$ is *not* true in general. A counterexample is found in the variety $\mathbb{P}(M_{2\times 2})$ under the action of $G = \mathrm{SL}(2)$ by left multiplication.

Several discrete invariants of a spherical G-variety X can be naturally defined. Due to the central role of Borel subgroups in the representation theory of G, most invariants involve the action of B:

(1) the set of B-eigenvalues of rational functions (on X) that are B-eigenvectors; it is a subgroup, denoted by $\Lambda(X)$, of the group of characters of B, and its rank is by definition the *rank* of X as a spherical variety;
(2) the vector space $\Lambda^*_{\mathbb{Q}}(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$;
(3) the (finite) set $\mathcal{D}(X)$ of all B-stable but not G-stable prime divisors of X, called *colors*.

A spherical variety X contains a dense G-orbit, which we denote by X_0, and X is also called an *embedding* of X_0. Then the above invariants actually depend only on X_0; this can be made precise also for colors, e.g. replacing a color by its generic point.

Notice that a B-eigenvector $f_\chi \in \mathbb{C}(X)$ is determined by its B-eigenvalue $\chi \in \Lambda(X)$ up to multiplication by a constant. Moreover, any $\mathbb{Q}$-valued discrete valuation $\nu\colon \mathbb{C}(X)^* \to \mathbb{Q}$ (over the constant functions) induces an element $\rho(\nu)$ of $\Lambda^*_{\mathbb{Q}}(X)$, by requiring that $\rho(\nu)$ take the value $\nu(f_\chi)$ on $\chi \in \Lambda(X)$.

One may apply this construction to the valuation associated with any prime divisor on X, but the advantage of considering valuations is that they are defined on $\mathbb{C}(X_0) = \mathbb{C}(X)$ regardless of whether they come from some prime divisor.

Under this point of view G-invariant valuations are particularly useful in describing the difference set $X \setminus X_0$: since it is G-stable, the valuation associated to any prime divisor contained in $X \setminus X_0$ is G-invariant.

Invariant valuations are also strictly related to the *little Weyl group*, a crucial invariant of a spherical variety. The first result in this direction is the following.

Theorem 3 ([5])**.** *Let X be a spherical variety. Then*

$$V(X) = \{\rho(\nu) \mid \nu \textit{ is a } G\textit{-invariant valuation } \}$$

is a convex polyhedral cone in $\Lambda^*_{\mathbb{Q}}(X)$.

In analogy with the classification of toric varieties, embeddings of a fixed spherical G-homogeneous space X_0 can be classified by means of families of convex cones in the vector space $\Lambda^*_{\mathbb{Q}}(X_0)$. Some data has to be added however, taking into account the behaviour of colors. We outline this classification, referring to [3] and [4] for details.

The role of affine toric varieties is played here by *simple* spherical varieties, which are by definition spherical varieties with a unique closed G-orbit.

Indeed, in general if $Y \subseteq X$ is any G-orbit, then

$$X_{Y,G} = \{x \in X \mid \overline{Gx} \supseteq Y\}$$

is open in X, quasi-projective and G-stable, spherical and simple with unique closed orbit X. Therefore X is covered by simple spherical varieties.

Now to any simple spherical variety X with open orbit X_0 and closed orbit Y we associate two objects:

(1) the cone $\mathcal{C}_X$ generated in $\Lambda^*_{\mathbb{Q}}(X_0)$ by the image of the valuations associated to all B-stable prime divisors containing Y;

(2) the set $\mathcal{D}_X$ of colors containing Y.

The couple $(\mathcal{C}_X, \mathcal{D}_X)$ is called the *colored cone* of X. "Admissible" colored cones are defined combinatorially in [3, §3].

Theorem 4. *Let X_0 be a spherical G-homogeneous space. The map $X \mapsto (\mathcal{C}_X, \mathcal{D}_X)$ is a bijection between simple embeddings of X_0 (up to G-equivariant isomorphisms that are the identity on X_0) and colored cones in $\Lambda^*_{\mathbb{Q}}(X_0)$.*

If X is not simple, then we consider its G-orbits $Y_1, \ldots, Y_n$, and the set of the colored cones

$$\mathcal{F}_X = \{(\mathcal{C}_{X_{Y_i,G}}, \mathcal{D}_{X_{Y_i,G}}) \mid i \in \{1, \ldots, n\}\}$$

This set is called the *colored fan* of X, and admissible colored fans are also defined combinatorially in [3, §3].

Theorem 5. *Let X_0 be a spherical G-homogeneous space. The map $X \mapsto \mathcal{F}_X$ is a bijection between embeddings of X_0 (up to G-equivariant isomorphisms that are the identity on X_0) and colored fans in $\Lambda^*_{\mathbb{Q}}(X_0)$.*

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