Invariant differential operators on reductive symmetric superspaces

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partly joint work with J. HILGERT *and* M. ZIRNBAUER

Classical setup

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As a first step, understand the commutative algebra $\mathbf{D}(X)^G$.

CHEVALLEY's restriction theorem

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$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$	CARTAN decomposition
$\mathfrak{a} \subset \mathfrak{p}$	CARTAN subspace
$W \subset \mathrm{GL}(\mathfrak{a})$	WEYL group

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Theorem (CHEVALLEY)

$S(\mathfrak{p})^{\mathfrak{k}} \to S(\mathfrak{a})^W$ is an isomorphism, and $S(\mathfrak{a})^W$ is affine.

HARISH-CHANDRA's isomorphism

There is a s.e.s.

$$0 \longrightarrow (\mathfrak{U}(\mathfrak{g})\mathfrak{k})^{\mathfrak{k}} \longrightarrow \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} \longrightarrow \mathbf{D}(X)^G \longrightarrow 0$$

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$\Gamma : \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} \to S(\mathfrak{a})^W$ *is surjective with kernel* $(\mathfrak{U}(\mathfrak{g})\mathfrak{k})^{\mathfrak{k}}$.
It induces an algebra isomorphism $\mathbf{D}(X)^G \to S(\mathfrak{a})^W$.

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$$X_0 = G_0/K_0 \hookrightarrow X = G/K$$

where

(G, K)	symmetric pair of complex Lie supergroups
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ZIRNBAUER thus embeds ten of CARTAN’s infinite series.

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$A\vert A$	$\mathrm{Gl}(m\vert n)$	A	A
$A\mathrm{I}\vert A\mathrm{II}$	$\mathrm{Gl}(m\vert 2n)/\mathrm{Osp}(m\vert 2n)$	$A\mathrm{I}$	$A\mathrm{II}$
$A\mathrm{II}\vert A\mathrm{I}$	$\mathrm{Gl}(m\vert 2n)/\mathrm{Osp}(m\vert 2n)$	$A\mathrm{II}$	$A\mathrm{I}$
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$C\vert BD$	$\mathrm{Osp}(m\vert 2n)$	C	BD
$C\mathrm{I}\vert D\mathrm{III}$	$\mathrm{Osp}(2m\vert 2n)/\mathrm{Gl}(m\vert n)$	$C\mathrm{I}$	$D\mathrm{III}$
$D\mathrm{III}\vert C\mathrm{I}$	$\mathrm{Osp}(2m\vert 2n)/\mathrm{Gl}(m\vert n)$	$D\mathrm{III}$	$C\mathrm{I}$
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PROGRAMME (AA, HILGERT, ZIRNBAUER): Develop harmonic analysis!

Invariant differential operators on X—setup

As before

$$\mathbb{C}[T^*X]^G \cong S(\mathfrak{p})^{\mathfrak{k}} \quad \text{and} \quad \mathbf{D}(X)^G \cong \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}}/(\mathfrak{U}(\mathfrak{g})\mathfrak{k})^{\mathfrak{k}}$$

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Assumptions:

- $(\mathfrak{g}, \mathfrak{k})$ is *(strongly) reductive*
  ($\Rightarrow \mathfrak{g} = \bigoplus\{$basic classical simple ideals $\neq A(1|1)\}$)
- $(\mathfrak{g}, \mathfrak{k})$ of *even type*, *i.e.* there exists an *even* CARTAN *subspace*:

$$\mathfrak{a} \subset \mathfrak{p}_{0,\text{semi-simple}} \quad \text{and} \quad \mathfrak{a} = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{a})$$

Let

$\Sigma_i = \Sigma(\mathfrak{g}_i : \mathfrak{a})$	even/odd roots
$\bar{\Sigma}_1$	$\lambda \in \Sigma_1,\ \lambda, 2\lambda \notin \Sigma_0$
W_0	even WEYL group
$W_\lambda = \exp \mathbb{R} T_\lambda \subset \mathrm{GL}(\mathfrak{a})$	$\langle h', T_\lambda(h) \rangle = \lambda(h')\lambda(h)$

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Theorem (AA–Hilgert–Zirnbauer 2010)

Let $W \subset \mathrm{GL}(\mathfrak{a})$ be generated by $W_0 \cup \bigcup_{\lambda \in \bar{\Sigma}_1} W_\lambda$.
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REMARKS.

- For $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{k} \oplus \mathfrak{k}, \mathfrak{k})$ ('group type'), this is due to SERGEEV (1999), KAC (1984), GORELIK (2004).
- In that case, W_λ is a one-parameter group of translations.

Let $(\mathfrak{g},\mathfrak{k}) = (\mathfrak{k}\oplus\mathfrak{k},\mathfrak{k})$ (group type), $\mathfrak{k} = \mathfrak{osp}(2|2)$. Then

$$S(\mathfrak{a})^W = \mathbb{C}[L] \qquad (L = \text{super-Laplacian})$$

Super CHEVALLEY—examples

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Let $\mathfrak{g} = \mathfrak{osp}(2|2)$ with an 'exceptional' involution. Then

$$S(\mathfrak{a})^W = \mathbb{C}[z,w]/(z^3 - w^2)$$

is the ring of regular functions on a cusp.

Super HARISH-CHANDRA isomorphism

Let

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'Group type' case is due to KAC (1984), GORELIK (2004).

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- Main point: Show that $\Gamma(\mathfrak{U}(\mathfrak{g})^{\mathfrak{k}}) \subset S(\mathfrak{a})^W$.
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  — If one defines ϕ_μ by Harish-Chandra's formula, then $\phi_\mu(1) = 0$. Why is $\phi_\mu \neq 0$?

Sketch of (part of) the proof—part I

- Main point: Show that $\Gamma(\mathfrak{U}(\mathfrak{g})^{\mathfrak{k}}) \subset S(\mathfrak{a})^W$.
  Theorem then follows from CHEVALLEY.
- HARISH-CHANDRA's idea:
  - Choose a non-compact real form (G, K).
  - Let $\phi_\mu(a) = \int_K L_a^* H^*(e^{\mu-\varrho})$, $H : G \to \mathfrak{a}$ IWASAWA projection.
  - Then $D\phi_\mu = \Gamma(D)(\mu) \cdot \phi_\mu$ ($D \in \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}}$) and $\phi_{w\mu} = \phi_\mu$ ($w \in W$).
  - Hence the claim, because $\phi_\mu(1) \neq 0$.
- Problems (in super setting):
  - A simple complex Lie superalgebra does *not* always possess a compact real form (SERGANOVA).
  - How to define IWASAWA decomposition?
  - Theory of invariant Berezin integration?
  - If one defines ϕ_μ by Harish-Chandra's formula, then $\phi_\mu(1) = 0$. Why is $\phi_\mu \neq 0$?
  - For $w = \exp \operatorname{ad} y \in W_\lambda$, $y \notin N_{\mathfrak{k}}(\mathfrak{a})$, but $y \in \mathfrak{U}(\mathfrak{k})_0$. How to prove $\phi_{w\mu} = \phi_\mu$?

Sketch of (part of) the proof—part II

- Non-compact real forms: Complexify structure sheaves! Formally:

  $$\text{real supermanifolds} \quad \rightsquigarrow \quad \textit{cs}\ \text{manifolds (Bernstein)}$$

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  In this category, a global IWASAWA decomposition exists.
- Invariant integration: volume forms and densities have analogues and unimodularity criteria, *etc.*, carry over (AA–HILGERT).
- Replacement for ϕ_μ:
  Let ψ be NA-invariant, $\int_K \psi = 1$ (in general, $\psi \not\equiv$ const.). Set

$$\phi_\mu^\psi(a) = \int_K L_a^*[H^*(e^{\mu-\varrho}) \cdot \psi]$$

  Then

$$\phi_\mu^\psi(1) = 1 \qquad \text{and} \qquad D\phi_\mu^\psi = \Gamma(D)(\mu) \cdot \phi_\mu^\psi$$

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($c : G \times G \to G$ = conjugation)

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  - Finally, use conjugation invariance of F_f.

Thanks!

REFERENCES.

AA, J. Hilgert, and M.R. Zirnbauer. Chevalley's restriction theorem for reductive symmetric superpairs.
J. Algebra, 323(4):1159–1185, 2010.

AA. The Harish-Chandra isomorphism for reductive symmetric superpairs.
arxiv:0017942 [math.RT], 40 pp.

Group type superpairs. $\mathfrak{k}$ a f.d. *contragredient* Lie superalgebra,

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Zirnbauer's examples. $\mathfrak{g} = \mathfrak{gl}(p + q|r + s)$ or $\mathfrak{g} = \mathfrak{osp}(p + q|r + s)$ with r, s even.

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EXCEPTIONAL $\mathfrak{osp}(2|2q)$. $\mathfrak{g} = \mathfrak{osp}(2|2q)$,

$$\theta\colon x = \begin{pmatrix} a & 0 & -w'^t & z'^t \\ 0 & -a & -w^t & z^t \\ z & z' & A & B \\ w & w' & C & -A^t \end{pmatrix} \mapsto \theta(x) = \begin{pmatrix} -a & 0 & -w^t & z^t \\ 0 & a & -w'^t & z'^t \\ z' & z & A & B \\ w' & w & C & -A^t \end{pmatrix}$$

In particular, $\mathfrak{k}_0 = \mathfrak{sp}(2q,\mathbb{C})$. $(\mathfrak{g},\mathfrak{k})$ is reductive and of even type.