On the geometric Langlands duality

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Outline

This lecture will give an overview on the following topics:

1. Langlands duality for reductive groups
2. The geometric Satake equivalence
3. The oper – center correspondence
4. The local geometric Langlands philosophy (after Frenkel–Gaitsgory)

Reductive Groups

We fix the following:

k an algebraically closed field

G a connected reductive algebraic group over k

$T \subset G$ a maximal torus

Then we define:

$X = \mathrm{Hom}(T, k^\times)$ the characters of T

$X^\vee = \mathrm{Hom}(k^\times, T)$ the cocharacters of T

$R \subset X$ the roots of G with respect to T

$R^\vee \subset X^\vee$ the coroots of G with respect to T

Langlands dual groups

The data $(X, R, X^\vee, R^\vee)$ is called the *root datum* associated to the group G.

Theorem

The group G is determined, up to isomorphism, by its root datum.

Surprise

There exists a connected reductive algebraic group $G^\vee$ over k with root datum $(X^\vee, R^\vee, X, R)$.

$G^\vee$ is called the *Langlands dual group* corresponding to G.

Problems and Questions

Problem

The duality $G \leftrightarrow G^\vee$ uses the artificial datum $(X, R, X^\vee, R^\vee)$.

Question

Is there a direct connection between G and $G^\vee$?

Two quite different answers

- The geometric Satake equivalence.
- The center – oper correspondence of Feigin and Frenkel.

We first discuss the geometric Satake equivalence. It relates the topology of the affine Grassmannian of G to the representation theory of $G^\vee$.

The affine Grassmannian

We fix the following:

G a connected reductive algebraic group over $\mathbb{C}$,

$\mathcal{O}$ $= \mathbb{C}[[t]]$ the ring of formal power series,

$\mathcal{K}$ $= \mathbb{C}((t))$ the field of fractions of $\mathcal{O}$,

$G(\mathcal{O})$ $= \mathrm{Hom}(\mathrm{Spec}(\mathcal{O}), G)$, the *$\mathcal{O}$-points in G*,

$G(\mathcal{K})$ $= \mathrm{Hom}(\mathrm{Spec}(\mathcal{K}), G)$, the *$\mathcal{K}$-points in G* (the *loop group*).

- The quotient $\mathcal{G}\mathrm{r} = G(\mathcal{K})/G(\mathcal{O})$ is called the *affine Grassmannian*.
- $\mathcal{G}\mathrm{r}$ can be considered as a limit of finite dimensional projective varieties via closed embeddings.

The geometric Satake equivalence

Note that $G(\mathcal{O})$ acts on $\mathcal{G}\mathrm{r} = G(\mathcal{K})/G(\mathcal{O})$ by left translations.

Theorem (Mirković–Vilonen, Drinfeld, Lusztig, Ginzburg, ...)

The sheaf cohomology functor yields an equivalence

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}, k) \cong G_k^\vee\text{-mod}$$

of tensor categories.

$\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}, k)$ is the category of *perverse sheaves* on $\mathcal{G}\mathrm{r}$ with coefficients in k that are constructible with respect to the stratification of $\mathcal{G}\mathrm{r}$ by $G(\mathcal{O})$-orbits.

$G_k^\vee$-mod is the category of rational representations of the algebraic group $G_k^\vee$ over k.

Digression: Perverse sheaves

Let X be an orientable compact smooth manifold of dimension n.

Theorem (Poincaré duality)

There is an isomorphism

$$\mathrm{H}^l(X,k) \cong \mathrm{H}_{n-l}(X,k)$$

for all $l \geq 0$.

If X is singular, then Poincaré duality does not necessarily hold.
But this can be repaired.

A new homology theory

In the 1970's, Goresky and MacPherson introduced a new homology theory.
To a possibly singular, complex variety X they associated the *intersection homology* $\mathrm{IH}_\bullet(X, k)$.

Theorem (Poincaré duality for intersection homology)

If X is projective of complex dimension n, then there is an isomorphism

$$\mathrm{IH}_l(X, k) \cong \mathrm{IH}_{2n-l}(X, k)$$

for all $l \geq 0$.

One can interpret (co-)homology theories in terms of sheaves.

Sheaf cohomology

Let $D(X,k)$ be the derived category of sheaves of k-vector spaces on X. Let

$$\mathrm{H}^\bullet(\cdot)\colon D(X,k) \to k\text{-mod}^{gr}$$

be the sheaf cohomology functor (i.e. the derived functor of the global sections functor). Then the cohomology of X is the sheaf cohomology of the constant sheaf $\underline{k}_X$:

$$\mathrm{H}^\bullet(\underline{k}_X) \cong \mathrm{H}^\bullet(X,k).$$

There is a similar result for intersection homology.

Deligne's construction

In 1976, Deligne defined an up to isomorphism unique object $\mathrm{IC}(X,k) \in D(X,k)$ with

$$\mathrm{H}^\bullet(\mathrm{IC}(X,k)) \cong \mathrm{IH}_\bullet(X,k).$$

$\mathrm{IC}(X,k)$ is called the *intersection cohomology sheaf* on X with coefficients in k. The definition of $\mathrm{IC}(X,k)$ gave rise to a remarkable subcategory

$$\mathrm{Perv}(X,k) \subset D(X,k)$$

of *perverse sheaves* on X.

Bibliography on perverse sheaves

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- A. Arabia, *Correspondance de Springer*, Institut de Mathématique de Jussieu, Univ. Paris 7, http://www.institut.math.jussieu.fr/˜arabia/math/Pervers.pdf.
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The oper–center correspondence

Let G be a simple complex algebraic group, $B \subset G$ a Borel subgroup. We define:

$\mathfrak{b} \subset \mathfrak{g}$ the Lie algebras of B and G,

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}K$ the associated affine Kac–Moody algebra with relations

$$[K, \widehat{\mathfrak{g}}] = 0,$$
$$[X \otimes t^n, Y \otimes t^m] = [X, Y] \otimes t^{m+n} + \delta_{m,-n} m k(X, Y) K,$$

where k denotes the Killing-form.

Definition

A $\widehat{\mathfrak{g}}$-module M is called *smooth*, if each element in M is annihilated by $t^N \mathfrak{g}[[t]]$ for N big enough.

We denote by $\widehat{\mathfrak{g}}$-mod the category of smooth $\widehat{\mathfrak{g}}$-modules.

The oper–center correspondence

Let $\kappa \in \mathbb{C}$. A $\widehat{\mathfrak{g}}$-module M is called *of level* $\kappa \in \mathbb{C}$ if K acts on M as multiplication with κ.
Let

$$U_\kappa := U(\widehat{\mathfrak{g}})/\langle K - \kappa.1\rangle.$$

Each smooth $\widehat{\mathfrak{g}}$-module of level κ is acted upon by the completion

$$\widetilde{U}_\kappa := \varprojlim U_\kappa / U_\kappa t^N \mathfrak{g}[[t]].$$

Let $Z_\kappa \subset \widetilde{U}_\kappa$ be the center.

- There is a certain value crit $\in \mathbb{C}$, called the *critical level*.
- If $\kappa \neq$ crit, then $Z_\kappa = \mathbb{C}$.

The center at the critical level

Theorem (Feigin–Frenkel)

There is a natural isomorphism

$$Z_{\mathrm{crit}}(\widehat{\mathfrak{g}}) \cong \operatorname{Fun} \operatorname{Op}_{\mathfrak{g}^\vee}(\mathbb{D}^\times).$$

$\mathbb{D}^\times$ $= \operatorname{Spec} \mathcal{K}$ is the infinitesimal punctured disc.

$\operatorname{Op}_{\mathfrak{g}^\vee}(\mathbb{D}^\times)$ is the ind-scheme of $\mathfrak{g}^\vee$-*opers* (Beilinson–Drinfeld) on $\mathbb{D}^\times$.

- $\operatorname{Fun} \operatorname{Op}_{\mathfrak{g}^\vee}(\mathbb{D}^\times)$ is a polynomial algebra in infinitely many variables.
- Very roughly, it is the symmetric algebra of the $\mathbb{C}$-vector space $\operatorname{Lie} T^\vee \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$.

Next we discuss the notion of a $\mathfrak{g}^\vee$-oper.

Opers and local systems

The group $G^\vee(\mathcal{K})$ acts by *gauge transformations* on the set $\mathfrak{g}^\vee(\mathcal{K})\,dt$ of connections on the trivial $G^\vee$-bundle on $\mathbb{D}^\times$.

Example ($G^\vee = \mathrm{GL}_n(\mathbb{C})$, $\mathfrak{g}^\vee = \mathfrak{gl}_n(\mathbb{C})$)

The gauge action is given by

$$g.(A\,dt) = \left(gAg^{-1} + \frac{\partial}{\partial t}(g)g^{-1}\right)\,dt$$

for $g \in \mathrm{GL}_n(\mathcal{K})$, $A \in \mathfrak{gl}_n(\mathcal{K})$.

We denote by

$$\mathrm{LocSys}_{\mathfrak{g}^\vee}(\mathbb{D}^\times) = \mathfrak{g}^\vee(\mathcal{K})\,dt/G^\vee(\mathcal{K})$$

the set of gauge equivalence classes.

Opers and local systems

Suppose that $G^\vee$ is of adjoint type and fix a Borel subgroup $B^\vee \subset G^\vee$.

Definition

A $\mathfrak{g}^\vee$-*oper* on $\mathbb{D}^\times$ is an equivalence class of the following data:

- A principal $G^\vee$-bundle $\mathcal{F}$ on $\mathbb{D}^\times$,
- a $B^\vee$-bundle reduction $\mathcal{F}_{B^\vee}$ of $\mathcal{F}$,
- a connection ∇ on $\mathcal{F}$ that satisfies *Griffith's transversality condition* with respect to $\mathcal{F}_{B^\vee}$.

The equivalence classes are taken with respect to the $B^\vee(\mathcal{K})$-gauge action. We denote by

$$\mathrm{Op}_{\mathfrak{g}^\vee}(\mathbb{D}^\times)$$

the space of opers.

Griffith's transversality

Example ($G^\vee = \mathrm{GL}_n(\mathbb{C})$, $B^\vee = \{\text{upper triangular matrices}\}$)

A $G^\vee$-oper on $\mathbb{D}^\times$ is given by a $B^\vee(\mathcal{K})$-equivalence class of connections of the form

$$\begin{pmatrix} * & & & * \\ + & * & & \\ & \ddots & \ddots & \\ 0 & & + & * \end{pmatrix} dt,$$

where the $+$'s denote non-zero elements in $\mathcal{K}$.

There is a forgetful map

$$\gamma \colon \mathrm{Op}_{\mathfrak{g}^\vee}(\mathbb{D}^\times) \to \mathrm{LocSys}_{\mathfrak{g}^\vee}(\mathbb{D}^\times)$$

sending a $B^\vee(\mathcal{K})$-equivalence class to a $G^\vee(\mathcal{K})$-equivalence class. We now discuss the most basic example of the classical local Langlands correspondence.

Representations of loop groups

Let q be a prime power and $F = \mathbb{F}_q((t))$. Consider the group $G(F)$ as a topological group (a basis of open neighbourhoods of 1 are the *congruence subgroups*

$$K_N = \{g \in G(\mathbb{F}_q[[t]]) \mid g \equiv 1 \mod t^N\} \text{ for } N > 0).$$

An irreducible complex representation V of $G(F)$ is called

smooth, if any vector is stabilized by K_N for some $N > 0$,

unramified, if it is smooth and $V^{K_0} \neq 0$.

The space V^{K_0} of K_0-invariants is acted upon by the Hecke algebra $H(G(F), K_0)$ (the algebra of K_0-biinvariant functions on $G(F)$ with the convolution product).

The classical Satake isomorphism

If V is unramified, then $\dim_{\mathbb{C}} V^{K_0} = 1$ and $V \mapsto V^{K_0}$ yields

$$\left\{\begin{matrix}\text{irreducible unramified} \\ \text{representations of } G(F)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{characters} \\ \text{of } H(G(F), K_0)\end{matrix}\right\}.$$

Theorem (classical Satake isomorphism)

There is a ring isomorphism $H(G(F), K_0) \to \operatorname{Rep} G_{\mathbb{C}}^{\vee}$ (the Grothendieck ring of $G_{\mathbb{C}}^{\vee}$-representations).

Hence we have a bijection

$$\left\{\begin{matrix}\text{irreducible unramified} \\ \text{representations of } G(F)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{characters} \\ \text{of } \operatorname{Rep} G^{\vee}(\mathbb{C})\end{matrix}\right\}.$$

We have another bijection

$$\left\{\begin{matrix}\text{semisimple conjugacy} \\ \text{classes in } G^{\vee}(\mathbb{C})\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{characters} \\ \text{of } \operatorname{Rep} G^{\vee}(\mathbb{C})\end{matrix}\right\}$$

$$\gamma \mapsto (V \mapsto \operatorname{Tr}(\gamma, V)).$$

The unramified local Langlands duality

We have hence constructed a bijection

$$\left\{\begin{matrix}\text{semisimple conjugacy}\\ \text{classes in } G^\vee(\mathbb{C})\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{irreducible unramified}\\ \text{representations of } G(F)\end{matrix}\right\}$$

$$\gamma \mapsto V_\gamma.$$

If $v \in V_\gamma^{K_0}$ and $X \in H(G(F), K_0)$ corresponds to $W_X \in \operatorname{Rep} G_{\mathbb{C}}^\vee$ via the Satake isomorphism, then

$$X * v = \operatorname{Tr}(\gamma, W_X) v.$$

(Hecke eigenfunction property.)

Conjecture (classical local Langlands, non-precise version)

There is a bijection

$$\left\{\begin{matrix}\textit{admissible homomorphisms}\\ \operatorname{Gal}(\overline{F}/F) \to G^\vee(\mathbb{C})\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{irreducible smooth}\\ \textit{representations of } G(F)\\ \textit{over } \mathbb{C}\end{matrix}\right\}$$

A categorification (unramified local geometric Langlands)

For the local *geometric* Langlands conjectures, replace

- $F = \mathbb{F}_q((t))$ by $\mathbb{C}((t))$,
- the space of functions on $G(F)/K_0$ by the D-module category $D(\mathcal{G}\mathrm{r})$-mod on $\mathcal{G}\mathrm{r} = G(\mathcal{K})/G(\mathcal{O})$,
- the Hecke algebra $H(G(F), K_0)$ by the *Hecke category* $D_{G(\mathcal{O})}(\mathcal{G}\mathrm{r})$-mod $\cong \mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}\mathrm{r})$,
- the Hecke algebra action on functions by the convolution action of $D_{G(\mathcal{O})}(\mathcal{G}\mathrm{r})$-mod on $D(\mathcal{G}\mathrm{r})$-mod.
- Hecke eigenfunctions by Hecke eigensheaves: $\mathcal{F} \in D(\mathcal{G}\mathrm{r})$-mod is called a *Hecke eigensheaf*, if for any $\mathcal{G} \in D_{G(\mathcal{O})}(\mathcal{G}\mathrm{r})$-mod with corresponding $G^\vee(\mathbb{C})$-representation $V_\mathcal{G}$ we have

$$\mathcal{G} * \mathcal{F} = \underline{V_\mathcal{G}} \otimes_\mathbb{C} \mathcal{F}.$$

($\underline{V_\mathcal{G}}$ is the vector space underlying $V_\mathcal{G}$).

The local geometric Langlands philosophy for loop groups (after Frenkel–Gaitsgory)

- Replace representations of Galois groups by local systems.
- Replace $G(F)$-representations by *categories* endowed with a $G(\mathcal{K})$-action.

Idea!

The category $\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}$ *is fibred over the scheme* $\text{Op}_{\widehat{\mathfrak{g}}^\vee}(\mathbb{D}^\times)$:

$$\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \to \text{Op}_{\widehat{\mathfrak{g}}^\vee}(\mathbb{D}^\times).$$

(This means that $\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}$ is a $Z_{\text{crit}} = \text{FunOp}_{\widehat{\mathfrak{g}}^\vee}(\mathbb{D}^\times)$-linear category).

For any oper χ denote by $\widehat{\mathfrak{g}}\text{-mod}_\chi$ the corresponding *fiber*. It is the category of all objects on which Z_{crit} acts via the character χ.

The local geometric Langlands philosophy for loop groups (after Frenkel–Gaitsgory)

By the Langlands philosophy, the space of Langlands parameters should be a space of local systems, i.e. there should be a universal category $\mathcal{C}$, fibred over $\mathsf{LocSys}_{\mathfrak{g}^\vee}(\mathbb{D}^\times)$, such that the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathfrak{g}}\text{-mod}_{\mathsf{crit}} & \longrightarrow & \mathsf{Op}_{\widehat{\mathfrak{g}}^\vee}(\mathbb{D}^\times) \\ \downarrow & & \downarrow \gamma \\ \mathcal{C} & \longrightarrow & \mathsf{LocSys}_{\mathfrak{g}^\vee}(\mathbb{D}^\times) \end{array}$$

This philosophy motivates the following

Conjecture (precise and non-philosophical)

If $\chi, \chi' \in \mathsf{Op}_{\widehat{\mathfrak{g}}^\vee}(\mathbb{D}^\times)$ are such that $\gamma(\chi) = \gamma(\chi')$, then we have an equivalence $\widehat{\mathfrak{g}}\text{-mod}_\chi \cong \widehat{\mathfrak{g}}\text{-mod}_{\chi'}$ of categories.

The case of the trivial local system (again the unramified situation)

Definition

An oper is called *regular*, if the underlying local system is trivial.

In the case of the trivial local system, there is another naturally associated category: the category $D_{\text{crit}}(\mathcal{G}\mathrm{r})\text{-mod}^{\mathit{Hecke}}$ of *critically twisted Hecke eigensheaves* on $\mathcal{G}\mathrm{r}$.

Conjecture

For any regular oper χ, there is an equivalence

$$D_{\text{crit}}(\mathcal{G}\mathrm{r})\text{-mod}^{\mathit{Hecke}} \cong \widehat{\mathfrak{g}}\text{-mod}_{\chi}$$

of categories.

A known instance

Let $I \subset G(\mathcal{O})$ be the Iwahori subgroup and $I^0 \subset I$ its unipotent radical.

Theorem (Frenkel–Gaitsgory)

For any regular oper χ, there is an equivalence

$$D_{\mathrm{crit}}(\mathcal{G}\mathrm{r})\text{-mod}^{Hecke, I^0} \cong \widehat{\mathfrak{g}}\text{-mod}_\chi^{I^0}.$$

The categories on both sides are the I^0-equivariant subcategories. On the right hand side, this can be thought of as a critical level version of category $\mathcal{O}$.