Patterson Sullivan Distributions

J.H. & M. Schröder
(Paderborn)

§1 Motivation

X Riem. mfd. $\xrightarrow{\text{cl.}}$ S^*X sphere bundle, geodesic flow

X Riem. mfd. $\xrightarrow{\text{q.m.}}$ $L^2(X)$ wave group flow $e^{it\sqrt{\Delta}}$

qu. | Limit

Relations:

- trace formulae (Selberg, Gutzwiller)
- Lewis correspondence (Lewis-Zagier) ⇝ Maass forms
- Wigner distr. vs Patterson-Sullivan dist. (Anantharaman-Zelditch) ⇝ quantum ergodicity

Wigner distribution = "microlocal lift" ③

Δ Laplace - Beltrami op.

φ Δ - eigenfunction

$a \in C^\infty(S^*X)$ "symbol function"

$Op(a) : L^2(X) \longrightarrow L^2(X)$ "quantization"

$$W_\varphi(a) = \langle Op(a)\varphi, \varphi \rangle_{L^2(X)}$$

Quantum ergodicity : what are the weak* limits ?
(in $\mathcal{D}'(S^*X)$; flow invariant)

Anantharaman - Zelditch : ④

$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ Poincaré disk

$\Gamma \leq SU(1,1)$ discrete cocompact

$X = \Gamma \backslash \mathbb{D}$

- $\exists$ dist. $\widehat{PS}_\lambda$ on S^*X s.th. $W_\varphi = \widehat{PS}_\lambda + O(|\lambda|^{-1})$
- $\exists$ intertwiner $W_\varphi \rightsquigarrow \widehat{PS}_\lambda$
- $\widehat{PS}_\lambda$ appears as residue of a dynamical zeta fct.

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§2 Setting: Loc. symm. spaces

$D = G/K$ sym. space, non-cpt., rank 1

$$G = KAN \quad \text{Iwasawa}$$
$$M = Z_K(A)$$
$$P = MAN$$
$$B = K/M = G/P$$

geodesics $\leftrightsquigarrow$ pts in $B^{(2)} = B \times B \setminus \mathrm{diag}(B) \cong G/MA$

sphere bundle: $SD \cong G/M \cong D \times B \cong G/K \times K/M$

geodesic flow: (dim $A = 1$)
$$\left.\begin{array}{l} G/M \times A \longrightarrow G/M \\ (gM, a) \longmapsto gaM \end{array}\right\}$$

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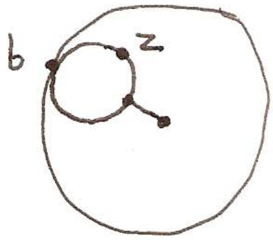
horocycle bracket: $\langle \cdot , \cdot \rangle : D \times B \longrightarrow \mathfrak{a} = \mathrm{Lie}(A)$
$$(z, b) \longmapsto -H(g^{-1}k)$$



$$\left.\begin{array}{l} H : KAN \longrightarrow \mathfrak{a} \\ \quad kan \longmapsto \log a \end{array}\right\} \text{Iwasawa projection}$$

$$(z, b) \leftrightsquigarrow (gK, kM)$$

plane waves: $e_{\lambda, b}(z) = e^{(i\lambda + \rho)\langle z, b \rangle}$

$$\lambda \in \mathfrak{a}^*, \quad b \in B, \quad z \in D$$

later: $X = \Gamma \backslash D$

$\Gamma \leq G$ discrete cocompact

§ 3 Invariant ΨDO calculus

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Idea : • plane waves are eigenfunctions of invariant DO's
• eigenvalues give the symbols
• expand functions as superposition of plane waves ($\rightsquigarrow$ diagonalize the operator)

Facts : • invariant symbols give invariant operators
• suitable symbol class (order 0) $\rightsquigarrow$ L^2-continuity

§ 4 Patterson - Sullivan distributions

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Boundary values: $\mathcal{E}_\lambda(D) = \{ \varphi \in C^\infty(D) \mid \Delta\varphi = -(|\lambda|^2 + |\rho|^2)\varphi \}$

$$P_\lambda : \mathcal{A}'(B) \xrightarrow[\cong]{} \mathcal{E}_\lambda(D)$$

$\mathcal{A}'(B)$: analytic functionals

$$P_\lambda(T)(z) := \int_B e^{(i\lambda+\rho)\langle z,b\rangle}\, T(db)$$

P_λ: Poisson transform

Facts: • $P_\lambda(T_\varphi) = \varphi$ "reasonable"

$\Rightarrow T_\varphi \in \mathcal{D}'(B)$ is a boundary value of φ

• suitable continuity of T_φ if $\varphi \in \mathcal{E}_\lambda(D)^\Gamma$

On $B^{(2)}$: $\quad ps_\lambda(db, db') := d_\lambda(b,b')\, T_\varphi(db)\, T_\varphi(db')$

$$\varphi \in \mathcal{E}_\lambda(D)^\Gamma, \quad d_\lambda(b,b') = d_\lambda(gMA)$$
$$= e^{(i\lambda+\rho)(H(g)+H(gw))}$$

$$\left(\text{for } SU(1,1): \quad d_\lambda(b,b') = |b-b'|^{1+2i\lambda} \right)$$

On SD : $\quad \langle f, PS_\lambda \rangle_{SD} = \int_{B^{(2)}} \mathcal{R}f(b,b')\, ps_\lambda(db, db')$

Radon

On SX : $\quad \langle a, PS_\lambda \rangle_{S(\Gamma\backslash D)} = \langle a\chi, PS_\lambda \rangle_{SD}$

smooth
fundamental
cut-off

§5 Main results & open problems

$$a \in C^\infty(S(X)), \quad \varphi \in \mathcal{E}_\lambda(D)^\Gamma \subseteq C^\infty(X)$$

- $W_\varphi(a) = \langle L_\lambda(a\chi), PS_\lambda \rangle_{SD}$

  $(L_\lambda f)(g) = \int_N e^{-(i\lambda+\rho)(H(nw))} f(gn)\, dn$

- $W_\varphi(a) = \langle a\chi, c(\lambda) PS_\lambda \rangle_{SD} + O(|\lambda|^{-1})$

- similar results for off-diagonal Wigner distr.

  $\langle Op(a)\varphi, \psi \rangle$

- relation to dynamical zeta function still needs to be checked.

  difficulties:
  - proof via transfer operators (needs symbolic dynamics)
  - proof via Selberg trace formula (needs explicit knowledge of the decomp. of $L^2(\Gamma\backslash G)$)

- higher rank: completely open

<u>Wikipedia</u>: Quantum Chaos

$\boxed{\text{WP1}}$

(M,ω) classical state space, $H \in C^\infty(M)$ Hamiltonian

$\mathbb{P}(\mathcal{H})$ quantization; $\hat{H} \in \mathcal{L}(\mathcal{H})_{s.a.}$

Question: if time evolution for (M,ω,H) is chaotic, what can be said about $e^{it\hat{H}}$?

Slogan: is there chaos in the quantum world?

Wikipedia : Quantum Ergodicity

$\boxed{\text{WP2}}$

Thm (Shnirelman, Zelditch, Colin de Verdière)

X cpt Riem. mfd ; geodesic flow ergodic

(ϕ_n) onb for $L^2(X)$; $\Delta\phi_n = \lambda_n\phi_n$, $\lambda_n \to \infty$

$\Rightarrow \exists\ S \subseteq \mathbb{N}$ of density 1 $\left(\frac{\#(S \cap [0,N])}{N} \xrightarrow[N\to\infty]{} 1 \right)$

s.th

$$\widehat{W}_{\phi_n} \xrightarrow[\substack{n\to\infty \\ \in \\ S}]{} \text{Liouville meas.}$$

Wikipedia : Quantum unique ergodicity conjecture (Rudnik-Sarnak)

$\boxed{\text{WP3}}$

- no exceptional sequences in Shnirelmans Thm (if sectional curvature is negative)
- checked for arithmetic surfaces and Hecke eigenfunctions by Lindenstrauss & Sundararajan
- analog false for almost all stadii (Hassell)

Wikipedia: Pseudodifferential operators (ψDO) $\boxed{\text{WP4}}$

on $\mathbb{R}^n$: $\quad L = \sum_\nu a_\nu \partial^\nu \quad , \quad \partial^\nu = \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial^{\nu_n}}{\partial x_n^{\nu_n}} \; , \; a_\nu \in C^\infty(\mathbb{R}^n)$

$$L(e^{i\langle \xi, \cdot \rangle})(x) = \sum_\nu a_\nu(x)\, \xi^\nu e^{i\langle \xi, x\rangle}$$ symbol $\sigma_L(x,\xi)$

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{i\langle \xi, x\rangle}\, d\xi$$ Fourier inversion

$$(Lf)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, \sigma_L(x,\xi)\, e^{i\langle \xi, x\rangle}\, d\xi$$

Wikipedia: Dynamical zeta functions $\boxed{\text{WP5}}$

Riemann ZF: $\quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \in \mathbb{P}} \frac{1}{1-p^{-s}}$

Ruelle ZF: X cpt. Riem. mfd, only count. many closed geod. γ_k

$$\zeta_{Ru}(s) = \prod_{k \in \mathbb{N}} \frac{1}{1 - e^{-h\ell(\gamma_k)s}}$$, h const. (top. pressure)
$\ell(\gamma_k)$ Length of γ_k

Selberg ZF: $X = \Gamma \backslash \mathcal{H}$
upper half plane

$$Z_\Gamma(s) = \prod_{\{\gamma\} \in \mathbb{P}_\Gamma} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell(\gamma)}\right)$$

$\mathbb{P}_\Gamma$ primitive hyperbolic conjugacy classes in Γ

$\ell(\gamma)$ Length of resulting geodesic

<u>Fact</u>: $\zeta_{Ru}(s) = \dfrac{Z_\Gamma(s+1)}{Z_\Gamma(s)}$