Modular Representation Theory of Endomorphism Rings

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Assumptions: Let $G$ be a finite group and $k$ be an algebraically closed field with characteristic $p > 0$. 
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**Definition and Remark**

Let $X$ be an indecomposable $kG$-module.

1. Let $H \leq G$. Then $X$ is called $H$-projective, if $X \mid (X_H)^G$.
2. Any minimal subgroup $Q$ of $G$ such that $X$ is $Q$-projective is called a vertex of $X$.
3. Vertices are $p$-subgroups of $G$. The set of vertices of $X$ is a conjugacy-class of $G$. 
Alperin’s Weight Conjecture

Theorem (Green Correspondence)

Let $Q \leq G$ be a $p$-subgroup and $N_G(Q) \leq H \leq G$. Then there is a one-to-one-correspondence between

$$\{X : X \text{ indecomposable } kG\text{-module with vertex } Q\} \cong \{Y : Y \text{ indecomposable } kH\text{-module with vertex } Q\}$$

given via

1. $X_H = f(X) \oplus Z$, where each indecomposable direct summand of $Z$ has a vertex in $\{Q^g \cap H : g \in G \setminus H\}$,

2. $Y^G = g(Y) \oplus W$, where each indecomposable direct summand of $W$ has a vertex in $\{Q^g \cap Q : g \in G \setminus H\}$. 
Definition

A weight for $G$ is a tupel $(Q, S)$, where $Q$ is a $p$-subgroup of $G$ and $S$ is a simple $kN_{G}(Q)$-module with vertex $Q$. In this case, we call $S$ a weight module and the Green correspondent $g(S)$ of $S$ in $G$ a weight Green correspondent (WGC).
Alperin’s Weight Conjecture

**Definition**

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- Weights are understood up to the natural equivalence $\sim$ induced by conjugation of $G$.
- $(Q, S)$ is a weight if and only if $S$ is a simple and projective $kN_G(Q)/Q$-module.
- Any WGC is (isomorphic to) a direct summand of the permutation module $k^G_P$, where $P \in \text{Syl}_p(G)$.
Conjecture (Alperin’s Weight Conjecture, 1987)

The number of simple $kG$-modules (up to isomorphism) is equal to the number of weights for $G$ (up to equivalence).
Let $Y := k^G_P = \bigoplus_{i=1}^n Y_i$ be a decomposition of $k^G_P$ into indecomposable direct summands and put $E := \text{End}_{kG}(Y)$.

For $\varphi \in E$, $\varphi(y)$ denotes the image of $y \in Y$ under $\varphi$.

The Hom-Functor $\text{Hom}_{kG}(Y, -) : \text{mod-}kG \to \text{mod-}E$ induces a decomposition of $E_E$ into the PIMs $P_i := \text{Hom}_{kG}(Y, Y_i)$ (Fitting Correspondence).

Of course, $\text{hd}(P_i)$ is a simple $E$-module, but $\text{soc}(P_i)$ is in general not simple.
Computational Experiments

With the computer algebra programs GAP, MeatAxe:
With the computer algebra programes GAP, MeatAxe:

- Construct $Y = k^G_P$, $Y = \bigoplus_{i=1}^n Y_i$, $E_E$, $E_E = \bigoplus_{i=1}^n P_i$.
- Analyze $\text{hd}(Y_i)$, $\text{soc}(Y_i)$, $\text{hd}(P_i)$, $\text{soc}(P_i)$.
- Determine character table, the Cartan- and decomposition matrix of $E$. 
Computational Experiments

With the computer algebra programs GAP, MeatAxe:

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- Determine character table, the Cartan- and decomposition matrix of $E$.

Observation

For almost all groups analyzed so far:

$$|\{\text{simple constituents of } \text{soc}(E_E)\}| \approx |\{\text{simple } kG\text{-modules}\}| \approx |\{\text{weights for } G\}|$$
Computational Experiments

With the computer algebra programs GAP, MeatAxe:

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Observation

For almost all groups analyzed so far:

\[
\left| \left\{ \text{simple constituents of } \text{soc}(E_E) \right\} / \cong \right| \quad \bigg| \bigg| \quad \left| \left\{ \text{simple } kG\text{-modules} \right\} / \cong \right| = \left| \left\{ \text{weights for } G \right\} / \cong \right|
\]

Exception: $M_{11}$ in characteristic 3.
From this, the following questions and ideas arise:

- Generalize analysis to more general modules than $k^G_P$, e.g. $\bigoplus k^G_{Q_i}$ for certain $p$-subgroups $Q_i$ of $G$.
- Which role does the $\text{Hom}$-functor, and especially its evaluation at a simple $kG$-module play?
- Characterize families of groups for which the observation hold.
$M_{11}$ in characteristic 3

$p = 3$, $G = M_{11}$, $P \in \text{Syl}_3(G)$, $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$

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Generalize $k^G_P$ to $k^G_P \oplus k^G_Q$, where $|Q| = 3$.

$Y := (k^G_P \oplus k^G_Q)/(\text{Defect 0, multiplicities})$
\[ Y := \left( k_P^G \oplus k_Q^G \right) / \text{(Defect 0, multiplicities)} \]

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$M_{11}$ in characteristic 3

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$\mathcal{F} := \text{Hom}_{kG}(Y, -)$ evaluated at simple $kG$-module - what can happen?
\( \mathcal{F} := \text{Hom}_{kG}(Y, -) \) evaluated at simple \( kG \)-module - what can happen?

- In general: \( \mathcal{F}(S) \) for a simple \( kG \)-module \( S \) not simple.
  For example: \( A_6 \) in characteristic 5: \( \mathcal{F}(8) = \begin{pmatrix} 1 \cr 1 \end{pmatrix} \).

- Even \( \text{soc}(\mathcal{F}(S)) \) may not be simple.
  For example: \( M_{11} \) in characteristic 3. Then \( \mathcal{F}(10_3) = 1_5 \oplus 2_3 \).
Assumptions for this section:
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- Consider some $kG$-module $Y$ such that
  \[ \{ S : S \mid \text{hd}(Y) \text{ simple} \} / \cong = \{ T : T \mid \text{soc}(Y) \} / \cong. \]

- Let $Y = \bigoplus_{i=1}^{n} Y_i$ be a decomposition into indecomposable direct summands.

- Assume: Head of $Y_i$ is simple for all $1 \leq i \leq n$.

- $Y_i \cong Y_j \iff i = j$.

- Set $E := \text{End}_{kG}(Y)$. 

Natalie Naehrig
Modular Representation Theory of Endomorphism Rings
Hom-Functor

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Remark

There is a partial order '$\geq_*$' on $\{Y_i : 1 \leq i \leq n\}$, given via:

$Y_i \geq_* Y_j$, if and only if there is a surjection $\varphi : Y_i \to Y_j$. 
Hom-Functor

Notation:

- Fix $S = \text{hd}(Y_i)$ for some $1 \leq i \leq n$.
- Then $i\varphi_* : Y_i \to S$ may be considered as an element of $F(S) = \text{Hom}_{kG}(Y, S) \leq E$. 

Natalie Naehrig
Modular Representation Theory of Endomorphism Rings
Hom-Functor

Notation:

- Fix $S = \text{hd}(Y_i)$ for some $1 \leq i \leq n$.
- Then $i\varphi_* : Y_i \rightarrow S$ may be considered as an element of $\mathcal{F}(S) = \text{Hom}_{kG}(Y, S) \leq E$.

Lemma

(a) Let $\langle \psi \rangle_k \leq \mathcal{F}(S)$ be simple $E$-submodule. Then $\psi$ is of the form $\psi = i\varphi_*$ for some $1 \leq i \leq n$ such that $\text{hd}(Y_i) \cong S$.

(b) The following are equivalent.

- (i) $Y_i$ is maximal w.r.t. $\succeq_*$ for some $1 \leq i \leq n$.
- (ii) $\langle i\varphi_* \rangle_k$ is a simple $E$-submodule of $\mathcal{F}(S)$.
Theorem

Let $S$ be a simple socle constituent of $Y$ and assume (possibly after renumbering) that $\text{hd}(Y_k) \cong S$ for all $1 \leq k \leq l$ for some $l \leq n$ and that $\text{hd}(Y_k) \not\cong S$ for all $l + 1 \leq k \leq n$.

(a) The following are equivalent:

(i) The $E$-submodule $\text{soc}(\mathcal{F}(S))$ has exactly $r$ non-isomorphic constituents.

(ii) Among $\{Y_k : 1 \leq k \leq l\}$ there are exactly $r$ maximal elements with respect to $\leq_*$.

(b) Let $\langle \psi \rangle_k \leq E_E$ be simple $E$-submodule. Then there is $1 \leq i \leq n$ such that $\langle \psi \rangle \cong_E \langle \varphi_\ast \rangle$.
Corollary

Assume that for each simple module occurring in the head of $Y$ there is exactly one maximal element with respect to $\leq^*$. 
Corollary

Assume that for each simple module occurring in the head of $Y$ there is exactly one maximal element with respect to $\leq^*$. Then $\text{soc}(\mathcal{F}(S))$ is a simple $E$-module for each simple constituent $S$ of $\text{hd}(Y)$. Moreover,

$$\{S : S \mid \text{hd}(Y) \text{ simple}\}/\cong \rightarrow \{T : T \mid \text{soc}(E) \text{ simple}\}/\cong$$

$$S \mapsto \text{soc}(\mathcal{F}(S))$$

is a bijection.
For which families of groups does observation hold?
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Joint work with Gerhard Hiss.
Indecomposable Liftable Modules in Cyclic Blocks

For which families of groups does observation hold?
Joint work with Gerhard Hiss.
Assumptions: Let \( \mathcal{B} \) be cyclic \( kG \)-block with defect group \( Q \).
Question 3*: Which indecomposable \( \mathcal{B} \)-modules are liftable?
For which families of groups does observation hold?  
Joint work with Gerhard Hiss.  
Assumptions: Let $B$ be cyclic $kG$-block with defect group $Q$.  

Question 3*: Which indecomposable $B$-modules are liftable?  

Kupisch 1969: Description of indecomposable $B$-modules in terms of paths on the Brauer tree:
Paths of Type I:

Brauer tree, with multiplicity $m = 3$ of exceptional vertex:
Paths of Type I:
Choose path:
Paths of Type I:

Choose marking on path and multiplicity $t = 2 \leq m = 3$: 

\[
egin{array}{ccccccccc}
& E_1 & A_2 & A_1 & E_2 & \chi_1 & B_2 & B_1 & \chi_0 & E_3 & \chi_3
\end{array}
\]
Indecomposable Liftable Modules in Cyclic Blocks

Paths of Type I:

Choose marking on path and multiplicity $t = 2 \leq m = 3$:

From this, we get the following indecomposable $kG$-module:
Paths of Type I:

Choose marking on path and multiplicity $t = 2 \leq m = 3$: 

$$\cdot \quad E_1 \chi_0 \quad E_2 \chi_1 \quad A_2 \quad A_1 \quad E_3 \chi_3$$
Paths of Type I:

Choose marking on path and multiplicity $t = 2 \leq m = 3$:

Note: The module $Y$ is of Type I if and only if, $\text{soc}(Y)$ and $\text{hd}(Y)$ have no common constituents.
Paths of Type II:

Brauer tree, with multiplicity $m = 3$ of exceptional vertex:
Indecomposable Liftable Modules in Cyclic Blocks

Paths of Type II:

Choose path:

\[ \begin{align*}
E_1 & \rightarrow E_2 & \rightarrow E_3 \\
B_1 & \rightarrow B_2 & \rightarrow \Lambda \\
A_1 & \rightarrow A_2 & \rightarrow \ldots \ldots
\end{align*} \]
Indecomposable Liftable Modules in Cyclic Blocks

Paths of Type II:

Choose marking on path and multiplicity $t = 2 \leq m = 3$:

From this, we get the following indecomposable $kG$-module:
Indecomposable Liftable Modules in Cyclic Blocks

Paths of Type II:

Choose marking on path and multiplicity $t = 2 \leq m = 3$:

From this, we get the following indecomposable $kG$-module:

$$E_2 \quad B_2$$

$$A_1 \quad E_2$$
Paths of Type II:

Choose marking on path and multiplicity $t = 2 \leq m = 3$:

From this, we get the following indecomposable $kG$-module:
Paths of Type II:

Choose marking on path and multiplicity $t = 2 \leq m = 3$:

From this, we get the following indecomposable $kG$-module:

Note: The module $Y$ is of Type II if and only if, $\text{soc}(Y)$ and $\text{hd}(Y)$ have common constituents.
Let $\tau$ be a path of Type I on the Brauer tree of $\mathfrak{B}$. Then the indecomposable module $X$ constructed from $\tau$ is liftable if and only if $\tau$ is as in the following figure and one of the following cases occurs:

(a) $\chi_1$ is not the exceptional vertex. Then the character of a lift of $X$ equals $\chi_1$. 

\[ \begin{array}{ccc}
\cdots & \circ & \circ & \circ & \cdots \\
\chi_0 & E_0 & \chi_1 & E_1 & \\
\end{array} \]
Theorem

Let \( \tau \) be a path of Type I on the Brauer tree of \( B \). Then the indecomposable module \( X \) constructed from \( \tau \) is liftable if and only if \( \tau \) is as in the following figure and one of the following cases occurs:

(a) \( \chi_1 \) is not the exceptional vertex. Then the character of a lift of \( X \) equals \( \chi_1 \).

(b) \( \chi_1 \) is the exceptional vertex, which has multiplicity \( m \). Each of \( E_0 \) and \( E_1 \) occurs \( t \) times as composition factor of \( X \) for some \( 1 \leq t \leq m \). The character of any lift of \( X \) is a sum of \( t \) distinct exceptional characters.
Note: If $X$ of Type I, then

- $X$ is uniserial in both cases.
- The head of $X$ is either isomorphic to $E_0$ or $E_1$.
- If the head of $X$ is isomorphic to $E_0$, the successor of $E_1$ around $\chi_1$ equals $E_0$, and if the head of $X$ is isomorphic to $E_1$, the successor of $E_0$ around $\chi_1$ equals $E_1$. 

Theorem

Let $\tau$ be a path of Type II on the Brauer tree. Then the indecomposable module $X$ constructed from $\tau$ is liftable if and only if $\tau$ satisfies one the following three cases:

Case 1: The vertex $\chi_0$ is a leaf of the Brauer tree.
Let $\tau$ be a path of Type II on the Brauer tree. Then the indecomposable module $X$ constructed from $\tau$ is liftable if and only if $\tau$ satisfies one of the following three cases:

**Case 1:** The vertex $\chi_0$ is a leaf of the Brauer tree.

**Case 2:** Either $E_{u0}$ is in the head or in the socle of $X$. If $E_{u0}$ is in the head of $X$, the successor of $E_0$ around $\chi_0$ is $E_{u0}$. If $E_{u0}$ is in the socle of $X$, the successor of $E_{u0}$ around $\chi_0$ is $E_0$.
Let $\tau$ be a path of Type II on the Brauer tree. Then the indecomposable module $X$ constructed from $\tau$ is liftable if and only if $\tau$ satisfies one of the following three cases:

Case 3: The successor of $E_{u0}$ around $\chi_0$ is $E_0$ and $E_{u0}$ is in the socle of $X$. 

![Diagram of Brauer tree with nodes and edges labeled as $\chi_{u0}$, $E_{u0}$, $\chi_{d0}$, $E_{d0}$, $\chi_0$, $E_0$, $\chi_1$, $\chi_s$, $E_s$, $\chi_\Lambda$, and $t$. The edge from $E_{u0}$ to $\chi_0$ is highlighted.]
Theorem

Let $\tau$ be a path of Type II on the Brauer tree. Then the indecomposable module $X$ constructed from $\tau$ is liftable if and only if $\tau$ satisfies one of the following three cases:

Case 3: The successor of $E_{u0}$ around $\chi_0$ is $E_0$ and $E_{u0}$ is in the socle of $X$.

The character of any lift of $X$ equals $\chi_0 + \chi_1 + \cdots + \chi_s + \mu$, where $\mu$ is a sum of $t$ distinct exceptional characters.
Thank you for listening!