Modular Representation Theory of Endomorphism Rings

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Alperin's Weight Conjecture

Assumptions: Let G be a finite group and k be an algebraically closed field with characteristic $p > 0$.

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Definition and Remark

Let X be an indecomposable kG-module.

1. Let $H \leq G$. Then X is called H-projective, if $X \mid (X_H)^G$.
2. Any minimal subgroup Q of G such that X is Q-projective is called a vertex of X.
3. Vertices are p-subgroups of G. The set of vertices of X is a conjugacy-class of G.

Alperin's Weight Conjecture

Theorem (Green Correspondence)

Let $Q \leq G$ be a p-subgroup and $N_G(Q) \leq H \leq G$. Then there is a one-to-one-correspondence between

$$\{X : X \text{ indecomposable } kG\text{-module with vertex } Q\}_{\cong}$$
$$g \uparrow \quad \downarrow f$$
$$\{Y : Y \text{ indecomposable } kH\text{-module with vertex } Q\}_{\cong}$$

given via

1. $X_H = f(X) \oplus Z$, *where each indecomposable direct summand of* Z *has a vertex in* $\{Q^g \cap H : g \in G \setminus H\}$,
2. $Y^G = g(Y) \oplus W$, *where each indecomposable direct summand of* W *has a vertex in* $\{Q^g \cap Q : g \in G \setminus H\}$.

Alperin's Weight Conjecture

Definition

A weight for G is a tupel (Q, S), where Q is a p-subgroup of G and S is a simple $kN_G(Q)$-module with vertex Q. In this case, we call S a weight module and the Green correspondent $g(S)$ of S in G a weight Green correspondent (WGC).

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- Weights are understood up to the natural equivalence $\simeq$ induced by conjugation of G.
- (Q, S) is a weight if and only if S is a simple and projective $kN_G(Q)/Q$-module.
- Any WGC is (isomorphic to) a direct summand of the permutation module k_P^G, where $P \in \mathrm{Syl}_p(G)$.

Alperin's Weight Conjecture

Conjecture (Alperin's Weight Conjecture, 1987)

The number of simple kG-modules (up to isomorphism) is equal to the number of weights for G (up to equivalence).

Endomorphism Ring of k_P^G

- Let $Y := k_P^G = \bigoplus_{i=1}^n Y_i$ be a decomposition of k_P^G into indecomposable direct summands and put $E := \mathrm{End}_{kG}(Y)$.
- For $\varphi \in E$, $\varphi(y)$ denotes the image of $y \in Y$ under φ.
- The Hom-Functor $\mathrm{Hom}_{kG}(Y, -) : \mathrm{mod}\text{-}kG \to \mathrm{mod}\text{-}E$ induces a decomposition of E_E into the PIMs $P_i := \mathrm{Hom}_{kG}(Y, Y_i)$ (Fitting Correspondence).
- Of course, $\mathrm{hd}(P_i)$ is a simple E-module, but $\mathrm{soc}(P_i)$ is in general not simple.

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- Construct $Y = k_P^G$, $Y = \oplus_{i=1}^n Y_i$, E_E, $E_E = \oplus_{i=1}^n P_i$.
- Analyze $\mathrm{hd}(Y_i)$, $\mathrm{soc}(Y_i)$, $\mathrm{hd}(P_i)$, $\mathrm{soc}(P_i)$.
- Determine character table, the Cartan- and decomposition matrix of E.

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Observation

For almost all groups analyzed so far:

$$|\{\text{simple constituents of } \mathrm{soc}(E_E)\}_{\cong}|$$

$$|\{\text{simple } kG\text{-modules}\}_{\cong}| \qquad |\{\text{weights for } G\}_{\sim}|$$

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Observation

For almost all groups analyzed so far:

$$|\{\text{ simple constituents of } \mathrm{soc}(E_E)\}_{/\cong}|$$

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$$|\{\text{simple } kG\text{-modules}\}_{/\cong}| \;=\; |\{\text{ weights for } G\}_{/\simeq}|$$

Exception: M_{11} in characteristic 3.

Computational Experiments

From this, the following questions and ideas arise:

- Generalize analysis to more general modules than k_P^G, e.g. $\bigoplus k_{Q_i}^G$ for certain p-subgroups Q_i of G.
- Which role does the Hom-functor, and especially its evaluation at a simple kG-module play?
- Characterize families of groups for which the observation hold.

M_{11} in characteristic 3

$$p = 3,\ G = M_{11},\ P \in \mathrm{Syl}_3(G),\ |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$$

k_P^G	1	10	54	11^2	55_1	55_2	65_1^2	65_2^2	99^2
hd	1	10_1	10_1	5_2	5_1	10_3	10_2	$10_3 24$	24
soc	1	10_1	10_1	5_1	5_2	10_2	$10_2 24$	10_3	24

E_E	1	2	6	3^2	7_1	7_2	9_1^2	9_2^2	11^2
k_P^G	1	10	54	11	55_1	55_2	65_1	65_2	99
hd	1_1	1_2	1_3	2_1	1_4	1_5	2_2	2_3	2_4
soc	1_1	1_3	1_3	1_4	$1_3 1_4 2_1 2_4$	2_2	$2_2 2_4$	$1_5 2_3$	2_4

M_{11} in characteristic 3

Generalize k_P^G to $k_P^G \oplus k_Q^G$, where $|Q| = 3$.
$Y := (k_P^G \oplus k_Q^G)/$(Defect 0, multiplicities)

M_{11} in characteristic 3

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Y	12	1	55_1	11	66	126_1	120	126_2	54^*
hd	1	1	5_1	5_2	5_1	5_2	$5_2 10_3$	5_1	10_1
soc	1	1	5_2	5_1	5_2	5_2	$5_1 10_3$	5_1	10_1

Y	99^*	10	55_2	81_1	65_1	75	65_2	81_2
hd	24	10_1	10_3	10_2	10_2	$10_2 10_3$	$10_3 24$	24
soc	24	10_1	10_2	10_2	$10_2 24$	$10_2 10_3$	10_3	10_1

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soc	24	10_1	10_2	10_2	$10_2 24$	$10_2 10_3$	10_3	10_1

E_E	9_1	2	13	18	26_1	7	26_2	27	9_2
Y	12	1	55_1	66	126_2	11	126_1	120	54^*
hd	1_1	1_2	1_3	1_4	1_5	1_6	1_7	1_8	1_9
soc	$1_1 1_5$	1_1	1_5	1_5	1_5	1_7	1_7	$1_7 1_9$	1_9

E_E	3	17_1	11	16_1	14_1	17_2	14_2	16_2
Y	10	99^*	55_2	81_1	65_1	75	65_2	81_2
hd	1_{10}	1_{11}	1_{12}	1_{13}	1_{14}	1_{15}	1_{16}	1_{17}
soc	1_9	1_{11}	1_{17}	1_{13}	1_{13}	$1_{13} 1_{17}$	$1_{11} 1_{17}$	1_{17}

Hom-Functor

$\mathcal{F} := \mathrm{Hom}_{kG}(Y, -)$ evaluated at simple kG-module - what can happen?

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- In general: $\mathcal{F}(S)$ for a simple kG-module S not simple.
  For example: A_6 in characteristic 5: $\mathcal{F}(8) = \begin{array}{c} 1_a \\ 1_b \end{array}$.
- Even $\mathrm{soc}(\mathcal{F}(S))$ may not be simple.
  For example: M_{11} in characteristic 3. Then
  $\mathcal{F}(10_3) = 1_5 \oplus 2_3$.

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- Let $Y = \oplus_{i=1}^{n} Y_i$ be a decomposition into indecomposable direct summands.
- Assume: Head of Y_i is simple for all $1 \leq i \leq n$.
- $Y_i \cong Y_j \iff i = j$.
- Set $E := \mathrm{End}_{kG}(Y)$.

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Remark

There is a partial order '$\geq_*$' on $\{Y_i : 1 \leq i \leq n\}$, given via: $Y_i \geq_* Y_j$, if and only if there is a surjection $\varphi : Y_i \to Y_j$.

Hom-Functor

Notation:

- Fix $S = \mathrm{hd}(Y_i)$ for some $1 \leq i \leq n$.
- Then ${}^i\varphi_* : Y_i \to S$ may be considered as an element of $\mathcal{F}(S) = \mathrm{Hom}_{kG}(Y, S) \leq E$.

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Lemma

(a) Let $\langle \psi \rangle_k \leq \mathcal{F}(S)$ be simple E-submodule. Then ψ is of the form $\psi = {}^i\varphi_*$ for some $1 \leq i \leq n$ such that $\mathrm{hd}(Y_i) \cong S$.
(b) The following are equivalent.

(i) *Y_i is maximal w.r.t. $\geq_*$ for some $1 \leq i \leq n$.*
(ii) *$\langle {}^i\varphi_* \rangle_k$ is a simple E-submodule of $\mathcal{F}(S)$.*

Hom-Functor

Theorem

Let S be a simple socle constituent of Y and assume (possibly after renumbering) that $\mathrm{hd}(Y_k) \cong S$ for all $1 \leq k \leq l$ for some $l \leq n$ and that $\mathrm{hd}(Y_k) \ncong S$ for all $l+1 \leq k \leq n$.

(a) The following are equivalent:

- (i) *The E-submodule $\mathrm{soc}(\mathcal{F}(S))$ has exactly r non-isomorphic constituents.*
- (ii) *Among $\{Y_k : 1 \leq k \leq l\}$ there are exactly r maximal elements with respect to $\leq_*$.*

(b) Let $\langle\psi\rangle_k \leq E_E$ be simple E-submodule. Then there is $1 \leq i \leq n$ such that $\langle\psi\rangle \cong_E \langle {}^i\varphi_*\rangle$.

Corollary

Assume that for each simple module occurring in the head of Y *there is exactly one maximal element with respect to* $\leq_*$.

Hom-Functor

Corollary

Assume that for each simple module occurring in the head of Y *there is exactly one maximal element with respect to* $\leq_*$*. Then* $\mathrm{soc}(\mathcal{F}(S))$ *is a simple* E*-module for each simple constituent* S *of* $\mathrm{hd}(Y)$*. Moreover,*

$$\begin{array}{rcl} \{S : S \mid \mathrm{hd}(Y) \text{ simple}\}/_{\cong} & \rightarrow & \{T : T \mid \mathrm{soc}(E) \text{ simple}\}/_{\cong} \\ S & \mapsto & \mathrm{soc}(\mathcal{F}(S)) \end{array}$$

is a bijection.

For which families of groups does observation hold?

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Joint work with Gerhard Hiss.

Indecomposable Liftable Modules in Cyclic Blocks

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Joint work with Gerhard Hiss.
Assumptions: Let $\mathbf{B}$ be cyclic kG-block with defect group Q.
Question 3^*: Which indecomposable $\mathbf{B}$-modules are liftable?

Indecomposable Liftable Modules in Cyclic Blocks

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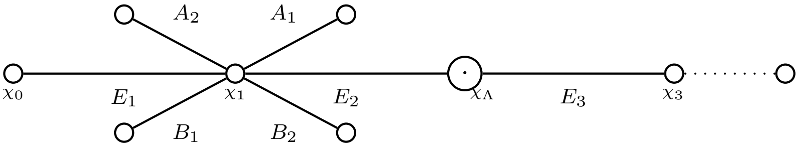
Question 3^*: Which indecomposable $\mathbf{B}$-modules are liftable?
Kupisch 1969: Description of indecomposable $\mathbf{B}$-modules in terms of paths on the Brauer tree:

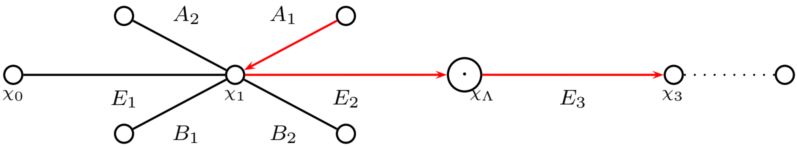
Paths of Type I:

Brauer tree, with multiplicity $m = 3$ of exceptional vertex:



Paths of Type I:

Choose path:



Paths of Type I:

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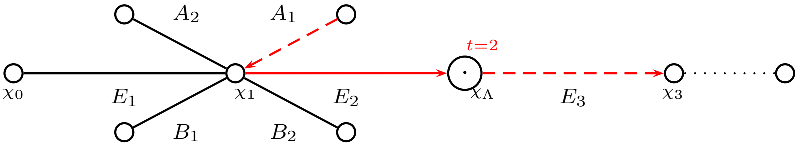

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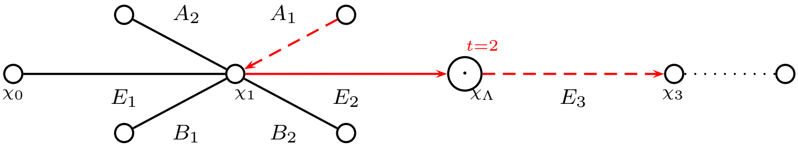

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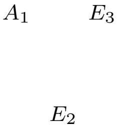

Indecomposable Liftable Modules in Cyclic Blocks

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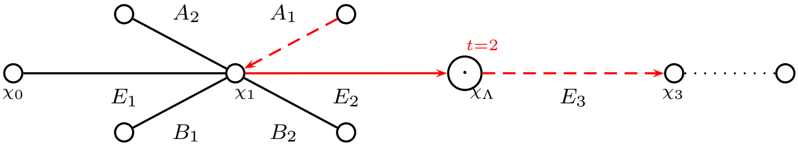

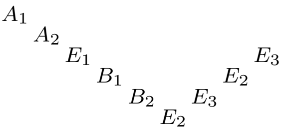

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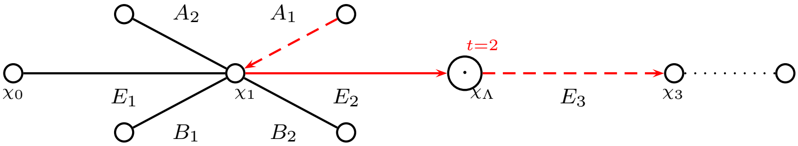
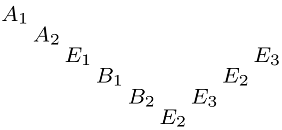
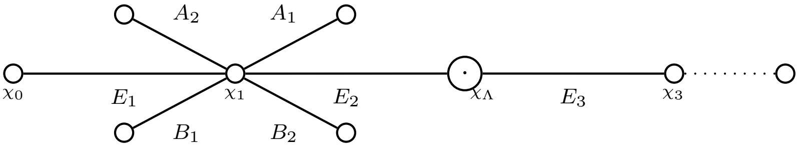




Note: The module Y is of Type I if and only if, $\mathrm{soc}(Y)$ and $\mathrm{hd}(Y)$ have no common constituents.

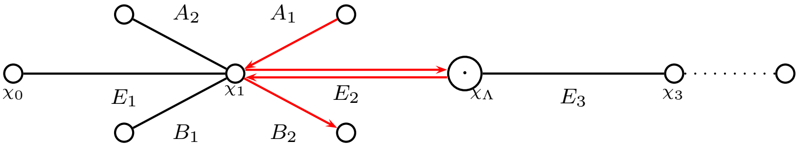
Paths of Type II:

Brauer tree, with multiplicity $m = 3$ of exceptional vertex:



Paths of Type II:

Choose path:



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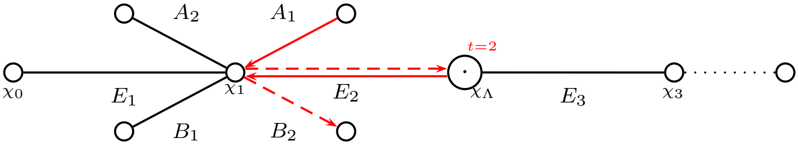

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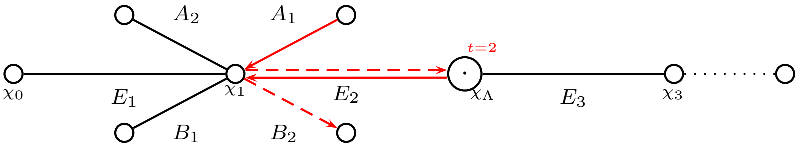

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$E_2 \quad B_2$

$A_1 \quad E_2$

Indecomposable Liftable Modules in Cyclic Blocks

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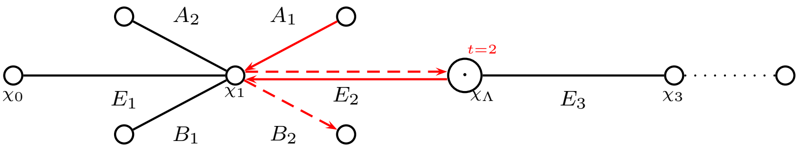

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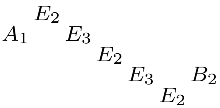

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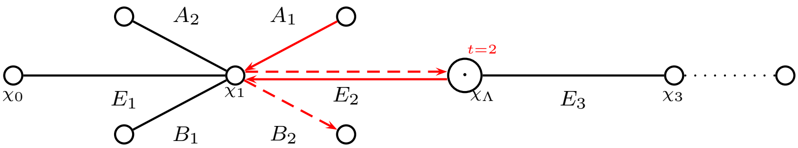

From this, we get the following indecomposable kG-module:

$$\begin{array}{ccccccc} & E_2 & & & & & \\ A_1 & & E_3 & & & & \\ & & & E_2 & & & \\ & & & & E_3 & & B_2 \\ & & & & & E_2 & \end{array}$$

Note: The module Y is of Type II if and only if, $\mathrm{soc}(Y)$ and $\mathrm{hd}(Y)$ have common constituents.

Indecomposable Liftable Modules in Cyclic Blocks

Theorem

Let τ be a path of Type I on the Brauer tree of $\mathbf{B}$. Then the indecomposable module X constructed from τ is liftable if and only if τ is as in the following figure and one of the following cases occurs:

(a) χ_1 is not the exceptional vertex. Then the character of a lift of X equals χ_1.

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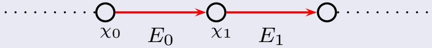
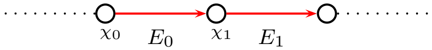


(a) χ_1 is not the exceptional vertex. Then the character of a lift of X equals χ_1.
(b) χ_1 is the exceptional vertex, which has multiplicity m. Each of E_0 and E_1 occurs t times as composition factor of X for some $1 \leq t \leq m$. The character of any lift of X is a sum of t distinct exceptional characters.

Indecomposable Liftable Modules in Cyclic Blocks



Note: If X of Type I, then

- X is uniserial in both cases.
- The head of X is either isomorphic to E_0 or E_1.
- If the head of X is isomorphic to E_0, the successor of E_1 around χ_1 equals E_0, and if the head of X is isomorphic to E_1, the successor of E_0 around χ_1 equals E_1.

Liftable Indecomposable Modules in Cyclic Blocks

Theorem

Let τ be a path of Type II on the Brauer tree. Then the indecomposable module X constructed from τ is liftable if and only if τ satisfies one the following three cases:
Case 1: The vertex χ_0 is a leaf of the Brauer tree.

Liftable Indecomposable Modules in Cyclic Blocks

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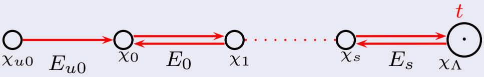
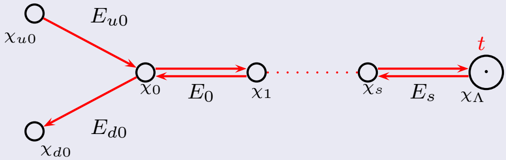
Case 1: The vertex χ_0 is a leaf of the Brauer tree.

Case 2: Either E_{u0} is in the head or in the socle of X. If E_{u0} is in the head of X, the successor of E_0 around χ_0 is E_{u0}. If E_{u0} is in the socle of X, the successor of E_{u0} around χ_0 is E_0.



Theorem

Let τ be a path of Type II on the Brauer tree. Then the indecomposable module X constructed from τ is liftable if and only if τ satisfies one the following three cases:
Case 3: The successor of E_{u0} around χ_0 is E_0 and E_{u0} is in the socle of X.



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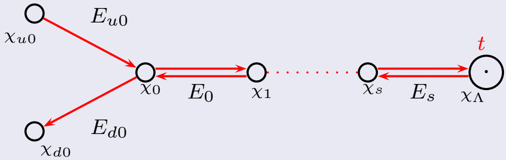

The character of any lift of X equals $\chi_0 + \chi_1 + \cdots \chi_s + \mu$, where μ is a sum of t distinct exceptional characters.

Thank you for listening!