About the inductive McKay condition in the maximal split case

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Reminder

The McKay Conjecture

For every finite group H and every prime ℓ the equation

$$|\operatorname{Irr}_{\ell'}(H)| = |\operatorname{Irr}_{\ell'}(\mathrm{N}_H(P))|$$

holds, where P is a Sylow ℓ–subgroup of H, $\operatorname{Irr}(H)$ is the set of irreducible ordinary characters of H and

$$\operatorname{Irr}_{\ell'}(H) = \{\chi \in \operatorname{Irr}(H) \mid \ell \nmid \chi(1)\}.$$

The Reduction Theorem

Theorem (Isaacs–Malle–Navarro, 2006)

The McKay conjecture holds for all finite groups and a prime ℓ, if every non-abelian simple group X with $\ell \mid |X|$ satisfies the inductive McKay condition for ℓ.

The inductive McKay Condition for X and a prime ℓ

We have $\ell \mid |X|$ and a unique group $G = G(X)$ has the following properties:

- There exists a group $M \lneq G$ s.t.
  1. $\mathrm{N}_G(P) \leq M$ for some Sylow ℓ-subgroup P of G,
  2. For all $\sigma \in \mathrm{Aut}(G)$ with $\sigma(P) = P$: $\sigma(M) = M$.
- There exists a bijection $\Omega : \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(M)$ s.t.
  1. For every $\sigma \in \mathrm{Aut}(G)$ with $\sigma(M) = M$: $\Omega(\psi)^\sigma = \Omega(\psi^\sigma)$ (i.e., the bijection is $\mathrm{Stab}_{\mathrm{Aut}(G)}(M)$–equivariant.)
  2. For every group K with $G \lhd K$ the character $\psi \in \mathrm{Irr}_{\ell'}(G)$ induced to K **behaves as** $\Omega(\psi)$ induced to $\mathrm{N}_K(M)$.

Finite simple Groups:

- cyclic groups of prime order (abelian),
- 26 sporadic simple groups,
- simple alternating groups $\mathcal{A}_n$ ($n \geq 5$), and
- simple groups of Lie type.

Theorem (Malle, 2008)

The sporadic simple and the alternating simple groups satisfy the inductive McKay condition for all their prime divisors.

Simple groups of Lie type:

Examples

$\mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q)$, $\mathrm{PSp}_{2n}(q)$

Uniform Approach:

Every simple group X of Lie type can be constructed as follows:

p	prime
q	power of p
$\mathbf{G}$	simply–connected linear algebraic group over $\overline{\mathbb{F}}_q$,
$F : \mathbf{G} \to \mathbf{G}$	Frobenius endomorphism giving an $\mathbb{F}_q$–structure
$G = \mathbf{G}^F$	finite group of Lie type with $X \cong G/\mathrm{Z}(G)$

Example

For $X = \mathrm{PSL}_n(q)$, $\mathbf{G} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$, $F((a_{i,j})) = (a_{i,j}^q)$, $\mathbf{G}^F = \mathrm{SL}_n(q)$.

First Results:

In the following cases the inductive McKay condition is satisfied:

- $X \in \{\mathrm{PSL}_2(q), {}^2\mathrm{B}_2(q^2), {}^2\mathrm{G}_2(q^2)\}$ for all primes dividing $|X|$ (Isaacs–Malle–Navarro),
- $X = \mathrm{B}_n(2^?)$ for all primes dividing $|X|$ (Cabanes),
- $\ell = p$: root system of $\mathbf{G}$ is not of type A_n or D_n (Brunat, Brunat-Himstedt, Himstedt).

Theorem (Malle, 2007)

Let $\ell \neq p$ and P a Sylow ℓ–subgroup of G. Then there exists an F–stable torus $\mathbf{S} \leq \mathbf{G}$, s.t.

1. $\mathrm{N}_G(P) \leq \mathrm{N}_G(\mathbf{S})$,
2. *For all $\sigma \in \mathrm{Aut}(G)$ with $\sigma(P) = P$: $\sigma(\mathrm{N}_G(\mathbf{S})) = \mathrm{N}_G(\mathbf{S})$.*

Remark

Hence we can choose $\mathrm{N}_G(\mathbf{S})$ as our subgroup M.

Theorem (S., 2008/2009)

Let $L := \mathrm{C}_G(\mathbf{S})$ *and* $N := \mathrm{N}_G(\mathbf{S})$.

1. *There exists a bijection*

$$\mathrm{Irr}\,(N) \to \mathcal{P} \text{ with } \mathcal{P} := \left\{ (\lambda, \eta) \;\middle|\; \begin{matrix} \lambda \in \mathrm{Irr}\,(L) \\ \eta \in \mathrm{Irr}\,(\mathrm{I}_N(\lambda)\,/L) \end{matrix} \right\}_{/\sim_N}.$$

2. *There exists a bijection*

$$\mathrm{Irr}_{\ell'}\,(N) \to \mathcal{P}_{\ell'} \text{ with } \mathcal{P}_{\ell'} := \left\{ (\lambda, \eta) \in \mathcal{P} \;\middle|\; \begin{matrix} \lambda \in \mathrm{Irr}_{\ell'}\,(L) \\ \eta \in \mathrm{Irr}_{\ell'}\,(\mathrm{I}_N(\lambda)/L) \\ \ell \nmid |N : \mathrm{I}_N(\lambda)| \end{matrix} \right\}_{/\sim_N}.$$

Theorem (Malle, 2007)

There exists a bijection $\mathrm{Irr}_{\ell'}\,(G) \to \mathcal{P}_{\ell'}$.

Corollary

There exists a bijection $\Omega : \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N)$.

About $\Omega : \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N)$**:**

1. Bijection $\mathrm{Irr}_{\ell'}(N) \to \mathcal{P}_{\ell'}$ depends on some choices.
2. Bijection $\mathrm{Irr}_{\ell'}(G) \to \mathcal{P}_{\ell'}$ uses
    - Jordan decomposition for groups with non–connected centre, and
    - d–Harish Chandra theory for disconnected groups.

Upshot

Let $\ell \neq p$ with $\ell \mid |X|$ and $G = G(X)$.

Theorem

The group G has the following properties:

- There exists a group $M \lneq G$ such that
  1. $\mathrm{N}_G(P) \leq M$ for some Sylow ℓ-subgroup P of G,
  2. For all $\sigma \in \mathrm{Aut}(G)$ with $\sigma(P) = P$: $\sigma(M) = M$.
- There exists a bijection $\Omega : \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(M)$.

Missing

1. For every $\sigma \in \mathrm{Aut}(G)$ with $\sigma(M) = M$: $\Omega(\psi)^\sigma = \Omega(\psi^\sigma)$
2. For every group K with $G \lhd K$ the character $\psi \in \mathrm{Irr}_{\ell'}(G)$ induced to K **behaves as** $\Omega(\psi)$ induced to $\mathrm{N}_K(M)$.

Corollary

If $\mathrm{Out}(G)$ *is trivial the group* $G/\mathrm{Z}(G)$ *satisfies the inductive McKay condition.*

Corollary (S.)

Let $\ell \neq p$. *For the group* G *the McKay conjecture is true for* ℓ

$$|\mathrm{Irr}_{\ell'}(G)| = |\mathrm{Irr}_{\ell'}(\mathrm{N}_G(P))|.$$

Aim:

- Prove that Ω has the desired properties.
- Specify ℓ such that Ω can be better controlled.

Idea:

- Assume ℓ to be a prime with

$$\begin{aligned} \ell &\mid (q-1) \text{ for } \ell \neq 2 \text{ or} \\ 4 &\mid (q-1) \text{ for } \ell = 2. \end{aligned}$$

- Equivalently, assume $C_G(\mathbf{S})$ is a maximal split torus T, (i.e., $T = \mathbf{T}^F$ for some maximal F–stable torus $\mathbf{T}$ and there exists an F–stable Borel $\mathbf{B}$ of $\mathbf{G}$ with $\mathbf{T} \leq \mathbf{B}$.)
(For $G = \mathrm{SL}_n(q)$ then $T = \{\text{diagonal matrices in } G\}$.)

From now on we assume $\ell \mid (q-1)$ and $\ell \neq 2$. Approach works also for $\ell = 2$ if $4 \mid (q-1)$.

The Sylow ℓ–subgroup of $\mathrm{SL}_n(q)$

- $|\,\mathrm{SL}_n(q)| = q^{\frac{n(n+1)}{2}}(q-1)^{n-1}\prod_{i=2}^{n}\Phi_i^{\lfloor \frac{n}{i} \rfloor}(q)$
- Let

$$T = \{\text{diagonal matrices in } G\} = \left\{ \begin{pmatrix} \star & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \star \end{pmatrix} \right\}.$$

- Let ℓ be an odd prime with $\ell \mid (q-1)$ and $\ell > n$. The Sylow ℓ–subgroup P of T is a Sylow ℓ–subgroup of G, and

$$\mathrm{N}_G(P) = N := \{\text{monomial matrices in } G\}.$$

- Let ℓ be an odd prime with $\ell \mid (q-1)$. Then

$$\mathrm{N}_G(P) \leq N := \{\text{monomial matrices in } G\}.$$

The Sylow ℓ–subgroup of G

$\mathbf{B}$	F–stable Borel of $\mathbf{G}$,
$\mathbf{T} \leq \mathbf{B}$	maximal torus,
$N := \mathrm{N}_G(\mathbf{T})$	
ℓ	odd prime with $\ell \mid (q-1)$.

Theorem

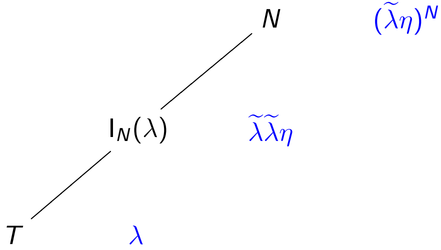
Let ℓ be an odd prime with $\ell \mid (q-1)$. Let $\mathbf{S}$ be the F–stable torus from Malle's theorem. Then the groups $N := \mathrm{N}_G(\mathbf{T})$ and $\mathrm{N}_G(\mathbf{S})$ coincide.

We can choose N as the group M for checking the inductive McKay condition.

The irreducible characters of N: Toy example

We have $T \lhd N$.
Assumption: N/T is cyclic.

- $T \lhd N \Rightarrow N$ acts on $\mathrm{Irr}\,(T)$
- Let $\lambda \in \mathrm{Irr}\,(T)$.
- $\mathrm{I}_N(\lambda)$ is the stabilizer of λ in N.
- Let $\widetilde{\lambda}$ is an extension of λ to $\mathrm{I}_N(\lambda)$ as irreducible character.
  ($\widetilde{\lambda}$ exists here, as N/T is cyclic.)
- For $\eta \in \mathrm{Irr}\,(\mathrm{I}_N(\lambda)\,/\,T)$:
  $(\widetilde{\lambda}\eta)^N \in \mathrm{Irr}\,(N)$.

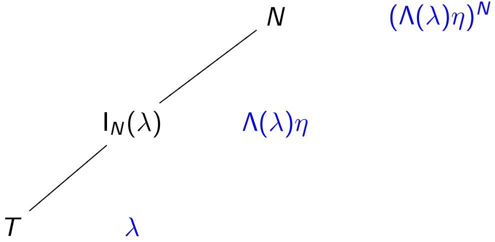


The general case

Theorem (S.)

For every $\lambda \in \mathrm{Irr}(T)$ there exists an extension $\Lambda(\lambda)$ of λ to $\mathrm{I}_N(\lambda)$ as irreducible character.

- Let $\lambda \in \mathrm{Irr}(T)$.
- Let $W(\lambda) = \mathrm{I}_N(\lambda)/T$.
- Let $\eta \in \mathrm{Irr}(W(\lambda))$
- $(\Lambda(\lambda)\eta)^N \in \mathrm{Irr}(N)$



$$\mathrm{Irr}(N) = \{ (\Lambda(\lambda)\eta)^N \mid \lambda \in \mathrm{Irr}(T),\ \eta \in \mathrm{Irr}(W(\lambda)) \}$$

Parametrizations of $\mathrm{Irr}(N)$

Proposition

1. $\mathrm{Irr}(N) = \left\{ (\Lambda(\lambda)\eta)^N \mid \lambda \in \mathrm{Irr}(T),\ \eta \in \mathrm{Irr}(W(\lambda)) \right\}$
2. $\mathrm{Irr}(N) \leftrightarrow \mathcal{P}$ with $\mathcal{P} := \left\{ (\lambda, \eta) \,\middle|\, \begin{array}{c} \lambda \in \mathrm{Irr}(T) \\ \eta \in \mathrm{Irr}(\mathrm{I}_N(\lambda)/T) \end{array} \right\}_{/\sim_N}$

Proposition

1. $(\Lambda(\lambda)\eta)^N(1) = |N : \mathrm{I}_N(\lambda)|\eta(1)$
2. $\mathrm{Irr}_{\ell'}(N) \overset{1:1}{\leftrightarrow} \mathcal{P}_{\ell'}$ with

$$\mathcal{P}_{\ell'} := \left\{ (\lambda, \eta) \in \mathcal{P} \,\middle|\, \begin{array}{c} \lambda \in \mathrm{Irr}_{\ell'}(T) \\ \eta \in \mathrm{Irr}_{\ell'}(\mathrm{I}_N(\lambda)/T) \\ \ell \nmid |N : \mathrm{I}_N(\lambda)| \end{array} \right\}_{/\sim_N}$$

The irreducible ℓ'–characters of G

Notation:

$B := \mathbf{B}^F$	fixed points of an F-stable Borel subgroup of G,
$U := \mathbf{U}^F$	group of unipotent elements in B,
$T \cong B/U$	

For $G = \mathrm{SL}_n(q)$:

B	group of upper–triangular matrices in $\mathrm{SL}_n(q)$,
U	group of unipotent upper–triangular matrices in $\mathrm{SL}_n(q)$,
$T \cong B/U$	

Harish-Chandra induction

- Let $\lambda \in \mathrm{Irr}\,(T)$.
- $\overline{\lambda} \in \mathrm{Irr}\,(B)$ is the lift of λ to B using $T \cong B/U$.
- $\mathrm{R}_T^G(\lambda) = \overline{\lambda}^G$ character of G

Decomposition of $\mathrm{R}_T^G(\lambda)$

Let M be the $\mathbb{C}G$–module associated to $\mathrm{R}_T^G(\lambda) = \overline{\lambda}^G$.

$$
\begin{array}{ccc}
\mathrm{Irr}\left(G \,\middle|\, \mathrm{R}_T^G(\lambda)\right) \ni \mathrm{R}_T^G(\lambda)_\eta & \longleftarrow\!\shortmid & \eta \in \mathrm{Irr}\,(W(\lambda)) \\
\updownarrow & & \\
\{\text{ simple summands in M}\}/_{\cong} & & \updownarrow \\
\updownarrow & & \\
\mathrm{IRep}(\mathrm{End}_{\mathbb{C}G}(M)) & \longleftrightarrow & \mathrm{IRep}(\mathbb{C}W(\lambda))
\end{array}
$$

Explanations:

- Depending on $\Lambda(\lambda)$: we can choose a basis $\{\mathrm{T}_w \mid w \in W(\lambda)\}$ of $\mathrm{End}_{\mathbb{C}G}(M)$,
- The relations of the T_w's can be described explicitly
- Via Tits deformation theorem $\mathrm{IRep}(\mathrm{End}_{\mathbb{C}G}(M)) \leftrightarrow \mathrm{IRep}(\mathbb{C}W(\lambda))$

Lemma

$R_T^G(\lambda)_\eta \in \mathrm{Irr}_{\ell'}(G) \Leftrightarrow \eta \in \mathrm{Irr}_{\ell'}(W(\lambda))$ and $\ell \nmid |N : I_N(\lambda)|$.
(Conditions are the same as in the definition of $\mathcal{P}_{\ell'}$, which parametrized $\mathrm{Irr}_{\ell'}(N)$)

Remark

- $R_T^G(\lambda)_\eta$ depends on $\Lambda(\lambda)$
- Different pairs (λ, η) give the same character, i.e. $R_T^G(\lambda)_\eta = R_T^G(\lambda')_{\eta'}$ with $(\lambda, \eta) \neq (\lambda', \eta')$ possible

Bijection between $\mathrm{Irr}_{\ell'}(G)$ and $\mathrm{Irr}_{\ell'}(N)$

Theorem

Assume $\ell \mid (q-1)$ and $\ell \neq 2$.

1. *For an existing and suitably chosen set of extensions $\Lambda(\lambda)$ the map*

$$\Omega : \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N) \;\; \textit{with} \;\; \mathrm{R}_T^G(\lambda)_\eta \mapsto (\Lambda(\lambda)\eta)^N$$

   is well-defined.

2. *If Ω is well–defined, it is a bijection.*

Equivalent: $\mathrm{Irr}_{\ell'}(N) \to \mathrm{Irr}_{\ell'}(G)$ with $(\Lambda(\lambda)\eta)^N \mapsto \mathrm{R}_T^G(\lambda)_\eta$ is a bijection

- this is well–defined, if the chosen extensions $\Lambda(\lambda)$ satisfy

$$\Lambda(\lambda^n) = \Lambda(\lambda)^n \text{ for every } \lambda \in \mathrm{Irr}(T) \text{ and } n \in N$$

- Ω is injective.
- $|\mathrm{Irr}_{\ell'}(G)| = |\mathrm{Irr}_{\ell'}(N)|$ from old results.

Properties of Ω

Theorem

For an existing and suitably chosen set of extensions $\Lambda(\lambda)$ the bijection Ω is well–defined and for $\sigma \in \mathrm{Aut}(G)$ with $\sigma(N) = N$: $\Omega(\psi)^\sigma = \Omega(\psi^\sigma)$ for every $\psi \in \mathrm{Irr}_{\ell'}(\psi)$.

Proof:

- Use the definition of $\mathrm{R}_T^G(\lambda)_\eta$ via $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{C}B} M_\lambda)$.
- Calculations show that equivariance of Ω depends on the control of

$$\Lambda(\lambda^\sigma)(\Lambda(\lambda)^\sigma)^{-1}$$

for any $\lambda \in \mathrm{Irr}(T)$ and $\sigma \in \mathrm{Aut}(G)$ with $\sigma(N) = N$.

Upshot for $\ell \mid (q-1)$

Theorem

The group G has the following properties:

- *There exists a group $M \lneq G$ such that.*
  1. $N_G(P) \leq M$ *for some Sylow ℓ-subgroup P of G,*
  2. *For all $\sigma \in \mathrm{Aut}(G)$ with $\sigma(P) = P$: $\sigma(M) = M$.*
- *There exists a bijection $\Omega : \mathrm{Irr}_{\ell'}(G) \longrightarrow \mathrm{Irr}_{\ell'}(M)$.*
  1. *For every $\sigma \in \mathrm{Aut}(G)$ with $\sigma(M) = M$: $\Omega(\psi)^\sigma = \Omega(\psi^\sigma)$*

Reformulation of last condition

For every group K with $G \lhd K$ the character $\psi \in \mathrm{Irr}_{\ell'}(G)$ induced to K **behaves as** $\Omega(\psi)$ induced to $\mathrm{N}_K(M)$.

This would be implied by

For every character $\psi \in \mathrm{Irr}_{\ell'}(G)$ there exists a group A with $G \lhd A$ such that

- ψ is A–invariant,
- $\mathrm{C}_A(G) = \mathrm{Z}(G)$
- $A/G \cong \mathrm{Out}(G)_\psi$.

Then

$$[\psi]_{A/G} = [\Omega(\psi)]_{\mathrm{N}_A(N)/N} \in \mathrm{H}^2(A/G, \mathbb{C}^\star)$$

Checking the last condition

Remark

If $\mathrm{Out}(G)$ is cyclic,

- the conditions on cocycles is automatically satisfied, and
- the group $G/\,\mathrm{Z}(G)$ satisfies the inductive McKay condition for ℓ.

Theorem

Assume G is not of type A_n, D_n or E_6. Then the last condition is satisfied for all $\psi \in \mathrm{Irr}_{\ell'}(G)$.

Idea of the proof:

- Let $\widetilde{G}$ be a group with $G \lhd \widetilde{G}$ coming from a reductive group with connected centre.
- Using Harish-Chandra induction there exists a bijection

$$\widetilde{\Omega} : \mathrm{Irr}_{\ell'}\left(\widetilde{G}\right) \longrightarrow \mathrm{Irr}_{\ell'}\left(\mathrm{N}_{\widetilde{G}}(N)\right).$$

- $\widetilde{\Omega}$ has some equivariance property and is compatible with Ω.

Remark

Idea of proof would work also for other primes if there is such bijections.

Theorem (S.)

- *Let X be a simple group of type*

$$^2\mathsf{A}_n, \mathsf{B}_n, \mathsf{C}_n, ^2\mathsf{D}_n, ^2\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2$$

  over $\mathbb{F}_q$.
- *Assume ℓ to be a odd prime with $\ell \mid (q-1)$ or $\ell = 2$ if $4 \mid (q-1)$.*

The simple group X satisfies the inductive McKay condition for ℓ.

Thank you!