Flag Manifolds and Representation Theory

Winter School on Homogeneous Spaces and Geometric Representation Theory

Lecture I. Real Groups and Complex Flags

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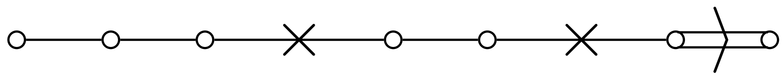
Parabolic Subalgebras

- G: conn. simply conn. complex semisimple Lie group
- $\mathfrak{g}$: the Lie algebra of G
- H: Cartan subgroup (centralizer of a Cartan subalg. $\mathfrak{h}$)
- $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{h})$: root system, so $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$
- $\Sigma^+ = \Sigma(\mathfrak{g}, \mathfrak{h})^+$ is a choice of positive root system
- $\Psi = \Psi(\mathfrak{g}, \mathfrak{h})$: simple system of roots for Σ^+
- Φ is an arbitrary subset of Ψ
- $\mathfrak{q} = \mathfrak{q}_\Phi := \mathfrak{q}^{nilp} + \mathfrak{q}^{red}$ *parabolic subalgebra* where
  - $\mathfrak{q}^{nilp} = \sum_{\Sigma^+ \setminus \langle \Phi \rangle} \mathfrak{g}_{-\alpha}$ is its nilradical and
  - $\mathfrak{q}^{red} = \mathfrak{h} + \sum_{\langle \Phi \rangle} \mathfrak{g}_\alpha$ is the Levi (reductive) component
  - Notation example: 

Parabolic Subgroups

- $Q = Q_\Phi = \{g \in G \mid Ad(g)\mathfrak{q} = \mathfrak{q}\}$: *parabolic subgroup* of G.
- A parabolic subgroup is its own normalizer
- Each G–conjugacy class of parabolic subgroups contains just one of the $2^{\#\Psi}$ groups Q_Φ
- Extremes: $Q_\emptyset$ is a Borel subgroup and $Q_\Psi = G$
- Let π_λ be an irreducible finite dimensional representation of G of highest weight λ. Let v_λ be a highest weight vector and $\Phi = \{\psi \in \Psi \mid \langle \lambda, \psi \rangle = 0\}$. Then Q_Φ is the G–stabilizer of the line $v_\lambda \mathbb{C}$. In the action of G on the assoc. complex projective space, Q_Φ is the isotropy subgroup of G at $[v_\lambda]$.
- Let G_0 be a real form of G. Then *parabolic subgroup* of G_0 means a subgroup that is a real form of a parabolic subgroup of G.

Complex Flag Manifolds

- the $Z = G/Q$, Q parabolic in G, are the *complex flag manifolds*
- Q_z and $\mathfrak{q}_z$: isotropies of G and $\mathfrak{g}$ at $z \in Z$
- G_u: compact real form of G
- can assume that $H_u := G_u \cap H$ is a maximal torus in G_u
- fact: $G_u \cap Q$ is the centralizer of a subtorus of H_u and is a compact real form L_u of the reductive part Q^{red} of Q.
- Complex flag manifolds $Z = G/Q = G_u/L_u$ are characterized by any of
  - $Z = G/Q$ is a complex homogeneous projective variety
  - $Z = G_u/L_u$ is a compact simply connected homogeneous Kähler manifold
  - $Z = G_u/L_u$ where L_u is the centralizer of a torus

Homogeneous Vector Bundles

- In general: let A/B be a homogeneous space and $\beta : B \to \mathrm{End}\,(E_\beta)$ a linear representation
- Define $\mathbb{E}_\beta = A \times_\beta E_\beta$ where $A \times_\beta E_\beta = (A \times E_\beta)/\{(ab, v) = (a, \beta(b)v)\}$ for $a \in A, b \in B$.
- Then $\mu : \mathbb{E}_\beta \to A/B$, by $\mu[a, v] = aB$, is an A–homogeneous vector bundle with typical fiber E_β on which B acts by β
- Sections are given by $f : A \to E_\beta$ with $f(ab) = \beta(b)^{-1} f(a)$.
- This works w/o change in the categories C^0, C^k and C^∞.
- For the holomorphic category, one must define a complex structure on $\mathbb{E}_\beta$ for which there are sufficiently many local holomorphic sections. This means local sections annihilated by antiholomorphic vector fields on A/B.

Borel–Weil Theorem

- Again, G is a connected complex semisimple Lie group, Q is a parabolic subgroup and $Z = G/Q$ complex flag
- τ_λ is an irreducible representation of Q with highest weight λ and representation space E_λ
- $\mathbb{E}_\lambda \to Z = G/Q$ is the associated homogeneous holomorphic vector bundle and $\mathcal{O}(\mathbb{E}_\lambda) \to Z$ is the sheaf of germs of holomorphic sections.
- **Borel–Weil Theorem.** Let $H^0(Z; \mathcal{O}(\mathbb{E}_\lambda))$ denote the space of global holomorphic sections of $\mathbb{E}_\lambda \to Z$. If λ is Σ^+–dominant, i.e. $\langle \lambda, \psi \rangle \geqq 0$ for all $\psi \in \Psi$, then the action of G on the $H^0(Z; \mathcal{O}(\mathbb{E}_\lambda))$ is the irreducible representation of G with highest weight λ.

Proof of Borel–Weil

- Recall $\mathfrak{q} = \mathfrak{q}^{red} + \mathfrak{q}^{nil}$ with $\mathfrak{g}_u \cap \mathfrak{q} = \mathfrak{l}_u$ where $\mathfrak{l} = \mathfrak{q}^{red}$.
- $\Lambda := \left\{ \nu \in \mathfrak{h}^* \;\middle|\; \frac{2\langle \nu,\psi \rangle}{\langle \psi,\psi \rangle} \text{ integer } \geqq 0 \right\}$, highest weights ν of irreducible representations π_ν of G.
- $H^0(Z;\mathcal{O}(\mathbb{E}_\lambda)) = (L^2(G_u/L_u) \otimes E_\lambda)^{\mathfrak{q}^{nil}}$
- $= \sum_{\nu \in \Lambda}((V_\nu \otimes V_\nu^* \otimes E_\lambda)^{\mathfrak{l}})^{\mathfrak{q}^{nil}} = \sum_{\nu \in \Lambda}((V_\nu \otimes V_\nu^* \otimes E_\lambda)^{\mathfrak{q}^{nil}})^{\mathfrak{l}}$
  using the Peter–Weyl Theorem
  and because $\mathfrak{l} = \mathfrak{q}^{red}$ normalizes $\mathfrak{q}^{nil}$
- $= \sum_{\nu \in \Lambda}(V_\nu \otimes (V_\nu^* \otimes E_\lambda)^{\mathfrak{q}^{nil}})^{\mathfrak{l}} = \sum_{\nu \in \Lambda}(V_\nu \otimes E_\nu^* \otimes E_\lambda)^{\mathfrak{l}}$
  because $\mathfrak{q}^{nil}$ ignores E_λ and pushes V_ν^* to E_ν^*
- $= \sum_{\nu \in \Lambda} V_\nu \otimes (E_\nu^* \otimes E_\lambda)^{\mathfrak{l}} = V_\lambda$ as G–module.

Bott–Borel–Weil Theorem

- $W = W_{\mathfrak{g},\mathfrak{h}}$: Weyl group, generated by reflections in the roots, $w_\alpha : \nu \mapsto \nu - \frac{2\langle \nu,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$
- ρ: half the sum of the positive roots
- if $w \in W$ then length $\ell(w) = \#\{\alpha \in \Sigma^+ \mid -w(\alpha) \in \Sigma^+\}$
- if $\lambda \in \mathfrak{h}^*$ nonsingular ($\lambda \not\perp \alpha$ for $\alpha \in \Sigma$) then $q(\lambda) = \ell(w)$ where $w = w_\lambda \in W$ s.t. $\langle w(\lambda), \alpha\rangle > 0$ for every $\alpha \in \Sigma^+$
- **Bott–Borel–Weil Theorem** Let $\lambda \in \mathfrak{h}^*$, highest weight of a representation τ_λ of Q, defining $\mathbb{E}_\lambda \to Z$. If $\lambda + \rho$ is singular then every $H^q(Z; \mathcal{O}(\mathbb{E}_\lambda)) = 0$. If $\lambda + \rho$ is nonsingular then $H^q(Z; \mathcal{O}(\mathbb{E}_\lambda)) = 0$ for $q \neq q(\lambda + \rho)$, and $H^{q(\lambda+\rho)}(Z; \mathcal{O}(\mathbb{E}_\lambda))$ is the irreducible G–module of highest weight $w_{\lambda+\rho}(\lambda + \rho) - \rho$.

Proofs

- Bott's proof of the Bott–Borel–Weil Theorem combines the Borel–Weil Theorem, the Leray spectral sequence, and the Kodaira Vanishing Theorem. It is an analytic proof based on the theory of compact Kähler manifolds.
- Kostant's proof is based more on the structure of parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g}$ (new at the time) and his idea of looking at Lie algebra cohomology.
- More recent proofs are based on Lie algebra cohomology and/or the Bernstein–Gelfand–Gelfand resolution of certain modules
- Because of time constraints that's all I'll say about it.

Real Group Orbits

- G_0: real form of G
- $Q_z \cap \overline{Q_z}$ and $\mathfrak{q}_z \cap \overline{\mathfrak{q}_z}$: complexifications of real isotropies $G_0 \cap Q_z$ and $\mathfrak{g}_0 \cap \mathfrak{q}_z$
- fact: $\mathfrak{g}_0 \cap \mathfrak{q}_z$ contains a Cartan subalgebra of $\mathfrak{g}_0$
- consequence: only finitely many G_0–orbits on Z
- consequence: G_0 has open orbits on Z
- if $G_0(z)$ is open then Cartan $H_0 \subset G_0 \cap Q_z$ is fundamental (maximally compact for CSG of G_0)
- If $H_0 \subset G_0 \cap Q_z$ is a fundamental Cartan denote
$$W(\mathfrak{g}, \mathfrak{h})^{\mathfrak{h}_0} = \{w \in W(\mathfrak{g}, \mathfrak{h}) \mid w(\mathfrak{h}_0) = \mathfrak{h}_0\}$$
- open orbits are given by a double coset space of the Weyl group, $W(\mathfrak{g}_0, \mathfrak{h}_0) \backslash W(\mathfrak{g}, \mathfrak{h}) / W(\mathfrak{g}, \mathfrak{h})^{\mathfrak{h}_0}$

Examples

- $G_0 = SU(p,q)$ defined by $h(z,w) = \sum_1^p z_i\overline{w_i} - \sum_1^q z_{p+i}\overline{w_{p+i}}$
  - $Z = \{q$–dim subspaces of $\mathbb{C}^{p+q}\}$
  - G_0–orbits given by signature $(+,-)$ of h
  - Open: $D_i = \{z \in Z \mid \text{sign}\, h|_z = (i, q-i)\}$
  - $D_0 \cong \{W \in \mathbb{C}^{p\times q} \mid I - WW^* \gg 0\}$ bounded symmetric domain
  - Closed: $S = \{z \in Z \mid \text{null}\, h|_z = \min(p,q)\}$
    Bergman-Shilov boundary of D_0
- $Z = \mathbb{CP}^n$ and $G_0 = SL(n+1;\mathbb{R})$.
  - Orbits: $G_0(u+iv)$ with $u, v \in \mathbb{R}^{n+1}$
  - If u, v linearly independent: open orbit $G_0([e_1 + \sqrt{-1}e_2])$
  - If u, v linearly dependent: closed orbit $G_0([e_1])$

Compact Subvarieties

- $D = G_0(z)$: open G_0–orbit on $Z = G/Q$
- K_0: maximal compact subgroup of G_0
- C_0: unique K_0–orbit on D that is a complex submanifold
- Properties: Can assume $C_0 = K_0(z)$. Then the "base cycle" $C_0 = K(z) \cong K/(K \cap Q)$ is a flag manifold
- The "cycle space" $\mathcal{M}_D$ of D is the component of C_0 in $\{gC_0 \mid g \in G \text{ and } gC_0 \subset D\}$
- $\mathcal{M}_D$ is a contractible Stein manifold of known structure
- Suppose that G_0/K_0 is a bounded symmetric domain $\mathcal{B}$. Then $\mathcal{M}_D$ is $\mathcal{B}$ or $\overline{\mathcal{B}}$ (hermitian holomorphic case) or $\mathcal{B} \times \overline{\mathcal{B}}$ (hermitian non–holomorphic case)
- More precisely:

Hermitian Types

Let G_0 be simple and of hermitian type: the symmetric space G_0/K_0 is an irreducible bounded symmetric domain $\mathcal{B}$. If D is of holomorphic type in the sense that μ and ν in the double fibration

$$\begin{array}{ccc} & G_0/(Q_0^r \cap K_0) & \\ \mu \swarrow & & \searrow \nu \\ D = G_0/Q_0^r & & \mathcal{B} = G_0/K_0 \end{array}$$

can be made simultaneously holomorphic, then $\mathcal{M}_D$ is holomorphically equivalent to $\mathcal{B}$ or to $\overline{\mathcal{B}}$. If D is of nonholomorphic type in the sense that μ and ν cannot be made simultaneously holomorphic, then $\mathcal{M}_D \cong \mathcal{B} \times \overline{\mathcal{B}}$.