Flag Manifolds and Representation Theory

Winter School on Homogeneous Spaces and Geometric Representation Theory

Lecture II. Real Group Orbits and Tempered Representations

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Harish–Chandra Class

- To minimize subscripts, in this lecture G will be the real group, $G_{\mathbb{C}}$ is its complexification, $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$.
- A real reductive Lie group G is of *Harish-Chandra class* or *class* $\mathcal{H}$ if
  - the component group G/G^0 is finite,
  - the derived group $[G^0, G^0]$ has finite center, and
  - if $g \in G$ then $\mathrm{Ad}(g)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$.
- The first two conditions can be weakened to allow infinite center. The third ensures that representations have infinitesimal characters.
- Now G is a real reductive Lie group of *class* $\mathcal{H}$ and $\widehat{G}$ is its unitary dual (equivalence classes of irreducible unitary representations).

Characters

- $\pi \in \widehat{G}$, $\mathcal{H}_\pi$ representation space
- $\dot{\pi} : L^1(G) \to \mathcal{B}(\mathcal{H}_\pi)$ by $\dot{\pi}(f)v = \int_G f(x)\pi(x)v dx$
- If $f \in C_c^\infty(G)$ then $\dot{\pi}(f)$ is trace class and
- $\Theta_\pi : C_c^\infty(G) \to \mathbb{C}$, by $f \mapsto \mathrm{trace}\ \dot{\pi}(f)$, is a distribution
- Θ_π is the *distribution character* of π. It determines π.
- $Z(\mathfrak{g}_\mathbb{C})$: center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{C})$.
- $Z(\mathfrak{g}_\mathbb{C})$ acts by scalars on $\mathcal{H}_\pi$, $d\pi(z)v = \chi_\pi(z)v$
- χ_π is the *infinitesimal character* of π
- $Z(\mathfrak{g}_\mathbb{C})$ = bi–invariant differential operators on G
- Θ_π is a joint eigendistribution of $Z(\mathfrak{g}_\mathbb{C})$: $z\Theta_\pi = \chi_\pi(z)\Theta_\pi$
- Elliptic regularity: Θ_π is C^ω on regular set G', finite jumps across singular set, so is integration against a C^ω funct.

Discrete Series I

- $\pi \in \widehat{G}$ is *discrete series* if it is a subrep. of the left regular rep of G on $L^2(G)$, i.e. if its matrix coefficients are $L^2(G)$.
- $\widehat{G}_{disc} \neq \emptyset$ iff G has a compact Cartan subgroup T.
- Let $\pi \in \widehat{G}_{disc}$. Then π is determined by $\Theta_\pi|_{T \cap G'}$. We parameterize $\widehat{G}_{disc}$ by parameterizing the $\Theta_\pi|_{T \cap G'}$.
- Define $G^\dagger := TG^0 = Z_G(G^0)G^0$. The Weyl group $W^\dagger := W(G^\dagger, T)$ equals $W^0 := W(G^0, T^0)$ and is a normal subgroup of $W = W(G, T)$.
- Let $\chi \in \widehat{T}$. Since T^0 is commutative, $d\chi(\xi) = \lambda(\xi)I$ where $\lambda \in i\mathfrak{t}_0^*$ and where $I = ident$ on the rep. space of χ.
- Then χ (modulo $W = W(G, T)$) determines π as follows

Discrete Series II

- Fact: λ is nonsingular: $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Sigma$.
- $\pi_\chi^0 \in \widehat{G^0}_{disc}$ has distribution character such that
  - $\Theta_{\pi_\chi^0}(x) = \pm \frac{\sum_{w \in W^0} sign(w) e^{w\lambda}}{\prod_{\alpha \in \Sigma^+}(e^{\alpha/2} - e^{-\alpha/2})}$ for $x \in T^0 \cap G'$
- $\pi_\chi^\dagger \in \widehat{G^\dagger}_{disc}$ has distribution character such that
  - $\Theta_{\pi_\chi^\dagger}(zx) = \chi(z)\Theta_{\pi_\chi^0}(x)$ for $x \in T^0 \cap G'$ and $z \in Z_G(G^0)G^0$
- $\pi_\chi \in \widehat{G}_{disc}$ has distribution character supported in $G^\dagger$ and such that
  - $\Theta_{\pi_\chi}|_{G^\dagger} = \sum_{gG^\dagger \in G/G^\dagger} \Theta_{\pi_\chi^\dagger} \cdot \mathrm{Ad}(g^{-1})$
- Conclusion: $\widehat{G}_{disc}$ is paramerized by pairs (λ, ζ) modulo $W(G,T)$ where $\lambda \in i\mathfrak{t}^*$ is nonsingular and $\zeta \in \widehat{Z_G(G^0)}$ equals $\exp(\lambda)$ on Z_{G^0}: $\pi_\chi \leftrightarrow (\lambda, \zeta)$ where $\chi = \zeta \otimes \exp(\lambda)$.

Tempered Series

- H Cartan subgroup of G; splits as $H = T \times A$
- Any $\chi_H \in \widehat{H}$ splits as $\chi_T \otimes e^{\gamma}$ with $\gamma \in i\mathfrak{a}^*$
- $Z_G(A) = M \times A$ and T Cartan subgroup of M
- Choices $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ and $\Sigma^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ determine $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$
- For any such choices $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is a *cuspidal* parabolic subalgebra of $\mathfrak{g}$ where $\mathfrak{n} = \sum_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha$
- $P = N_G(\mathfrak{p}) = MAN$: *cuspidal* parabolic subgroup of G
- If $\eta_{\chi_T} \in \widehat{M}_{disc}$, extend $\eta_{\chi_T} \otimes e^{\gamma}$ to P by $man \mapsto e^{\gamma}(a)\eta_{\chi_T}(m)$.
- Then $\pi_\chi := \mathrm{Ind}_P^G(\eta_{\chi_T} \otimes e^{\gamma})$ is independent of choice of positive root systems
- The π_χ associated to H form the "H–series" of G

Plancherel Formula

- If H is compact then the H–series is the discrete series
- If T is a Cartan subgroup of K, i.e. H is maximally compact, then the H–series is the "fundamental series"
- If A has maximal possible dimension then the H–series is the "principal series"
- The support of Plancherel measure on $\widehat{G}$ is the union (for $H \in Car(G)$) of the H–series.
- The Plancherel formula for G has form

$$f(x) = \sum_{Car(G)} \int_{\widehat{H}} \Theta_{\pi_{\chi_H}}(r_x f) d\pi_{\chi_H}$$

where $r_x f(y) = f(yx)$ and $d\pi_{\chi_H}$ is "Plancherel measure" for the H–series part of $\widehat{G}$.

Bundle over Open Orbit

- $Q_{\mathbb{C}}$: parabolic subgroup of $G_{\mathbb{C}}$ with $L = Q_{\mathbb{C}} \cap G$ compact.
- $Q_{\mathbb{C}}^{red} = L_{\mathbb{C}}$ and L contains a compact CSG T of G
- $T = Z_G(G^0)T^0$ and $L = Z_G(G^0)L^0$
- $Z = G_{\mathbb{C}}/Q_{\mathbb{C}}$ complex flag manifold
- $D = G(z) \cong G/L$: open G–orbit on Z
- $\chi \in \widehat{T}$, representation space E_χ:
  - $d\chi = \lambda \in i\mathfrak{t}^*$ and $\chi = \chi' \otimes e^\lambda$ where $\chi' \in \widehat{Z_G(G^0)}$,
  - $\tau_\chi^0 \in \widehat{L^0}$ with highest weight λ, and $\tau_\chi = \chi' \otimes \tau_\chi^0 \in \widehat{L}$,
  - τ_χ extends to holo. rep. of Q on E_χ
- $\mathbb{E}_\lambda \to Z$ associated $G_{\mathbb{C}}$–homogeneous holomorphic vector bundle; $\mathbb{E}_\lambda|_D \to D$ hermitian because L is compact

Square Integrable Dolbeault Cohomology

- $\mathcal{O}(\mathbb{E}_\chi) \to D$: sheaf of germs of holomorphic sections
- $H^q(D; \mathcal{O}(\mathbb{E}_\chi))$: (Dolbeault) sheaf cohomology
- $H^q_2(D; \mathcal{O}(\mathbb{E}_\chi))$: classes with $L^2(D)$ representatives
- $q(\beta)$: $\#\{\alpha \in \Sigma^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) \mid \langle \beta, \alpha \rangle < 0\} + \#\{\alpha \in \Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) \setminus \Sigma^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) \mid \langle \lambda, \alpha \rangle > 0\}$.
- **Theorem.** If $\lambda + \rho$ is singular then every $H^q_2(D; \mathcal{O}(\mathbb{E}_\chi)) = 0$
- **Theorem.** If $\lambda + \rho$ is regular and $q \neq q(\lambda + \rho)$ then $H^q_2(D; \mathcal{O}(\mathbb{E}_\chi)) = 0$
- **Theorem.** If $\lambda + \rho$ is regular and $w \in W(G, T)$ such that $\langle \lambda + \rho, \alpha \rangle > 0$ for all $\alpha \in \Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ then G acts irreducibly on $H^{q(\lambda+\rho)s}_2(D; \mathcal{O}(\mathbb{E}_\chi))$ by the discrete series representation with parameter $w \cdot \chi := w(\chi) \otimes e^{w(\lambda+\rho)}$

Other Cohomology Realizations

- One can replace $H^q(D; \mathcal{O}(\mathbb{E}_\chi))$ by the space of square integrable $\mathbb{E}_\chi$–valued harmonic $(0, q)$–forms on D; the result is the same.
- One can also replace $H^q(D; \mathcal{O}(\mathbb{E}_\chi))$ by the space of $\mathbb{E}_\chi$–valued harmonic spinors on D; the result is the same.
- Let L is reductive but not necessarily compact. Then D is s–convex (normalization: “Stein” is 0-convex) where $s = \dim C_0$. If $\mathbb{E} \to D$ is sufficiently negative then Dolbeault $H^q(D; \mathcal{O}(\mathbb{E}_\chi)) = 0$ for $q \neq s$ while $H^s(D; \mathcal{O}(\mathbb{E}_\chi))$ is a nuclear Fréchet space on which G acts irreducibly.
- Yet another approach is to consider the cases where L is reductive but noncompact, and use the resulting pseudo–Kähler metric on D to construct representations. This works under limited circumstances. More later.

Partially Complex Orbits

- Every orbit $X = G(x) \subset Z$ has constant CR dimension
- Holomorphic arc components of X are the equiv classes under: $x \sim x'$ if there is a chain of holomorphic maps $f_i : (|z| < 1) \to Z$ with image in X, where x in the first, where each meets the next, and where x' in the last.
- Let $H = T \times A \in Car(G)$ and $P = MAN$ an associated cuspidal parabolic in G. Then there are parabolics $Q_{\mathbb{C}}$ in $G_{\mathbb{C}}$ and orbits $X = G(x) \subset Z = G_{\mathbb{C}}/Q_{\mathbb{C}}$ such that
  - the holo arc components $S_{[x]}$ of X are complex in Z
  - the G–normalizer $N_{[x]}$ of $S_{[x]}$ is open in P
  - $U := M \cap Q_{x,\mathbb{C}}$ is compact and $U = Z_U(U^0)U^0$
- **Theorem** $G \cap Q_{x,\mathbb{C}} = UAN$, $N_{[x]} = M^{\dagger}AN$ and $S_{[x]} = M^{\dagger}/U$ is an open $M^{\dagger}$–orbit on the complex flag $M^{\dagger}_{\mathbb{C}}/(M^{\dagger}_{\mathbb{C}} \cap Q_{x,\mathbb{C}})$.

Partially Complex Bundles

- If $\mu \in \widehat{U}$ and $\sigma \in \mathfrak{a}^*$ then $\mu \otimes e^{\rho_{\mathfrak{a}}+i\sigma}$ defines a G–homogeneous vector bundle $\mathbb{E}_{\mu,\sigma} \to G/UAN = X$
- $\mathbb{E}_{\mu,\sigma} \to X$ is holomorphic over each holo arc component
- K is transitive on the space of holo arc components of X
- $\Lambda^{p,q} \to X$: bundle s.t. $\Lambda^{p,q}|_{S_{[kx]}}$ = (p,q)–form bundle on $S_{[kx]}$
- $H_2^{p,q}(X;\mathbb{E}_{\mu,\sigma})$: $\mathbb{E}_{\mu,\sigma}$–valued (p,q)–forms ω on X, in other words sections of $\mathbb{E}_{\mu,\sigma} \otimes \Lambda^{p,q}$, such that
  - $\omega|_{S_{[kx]}}$ harmonic with L_2 norm $\|\omega|_{S_{[kx]}}\| < \infty$ a.e. $k \in K$
  - global square norm $\|\omega\|^2 := \int_K \|\omega|_{S_{[kx]}}\|^2 dk < \infty$.
- $H_2^{p,q}(X;\mathbb{E}_{\mu,\sigma})$ is a Hilbert space with inner product

$$\langle \omega, \omega' \rangle = \int_K \langle \omega|_{S_{[kx]}}, \omega'|_{S_{[kx]}} \rangle_{L_2^{p,q}(S_{[kx]};\mathbb{E}_{\mu,\sigma}|_{S_{[kx]}})} dk$$

Partially Holomorphic Cohomology

- **Theorem.** If $d\mu$ is $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$–singular then every $H_2^{0,q}(X; \mathbb{E}_{\mu,\sigma}) = 0$
- **Theorem.** If $d\mu$ is $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$–regular and if $q \neq q_{\mathfrak{m}}(d\mu + \rho_{\mathfrak{m}})$ then $H_2^{0,q}(X; \mathbb{E}_{\mu,\sigma}) = 0$
- **Theorem.** If $d\mu$ is $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$–regular and if $q = q_{\mathfrak{m}}(d\mu + \rho_{\mathfrak{m}})$ then the natural action $\pi_{\mu,\sigma}$ of G on $H_2^{0,q}(X; \mathbb{E}_{\mu,\sigma})$ is the H–series representation defined by (μ, σ).