Flag Manifolds and Representation Theory

Winter School on Homogeneous Spaces and Geometric Representation Theory

Lecture III. Other Realizations of Representations

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Outline

- First, we'll look at geometric quantization in general for real reductive Lie groups, using negative bundles. This realizes Harish–Chandra modules as representations on nuclear Fréchet spaces but does not clarify unitarity.
- Second, we'll look at a simplification of cohomology realizations by using the cycle space and a double fibration transform.
- Third, if there is time, we'll look at some realizations corresponding to non–elliptic coadjoint orbits using a variation on Hodge–Kodaira theory. This gives unitarity but its application is limited.

Harish–Chandra Module & Globalization

- If $\pi \in \widehat{G}$ then $\pi|_K = \sum_{\kappa \in \widehat{K}} m(\kappa, \pi)\kappa$ where the multiplicities
$$m(\kappa, \pi) \leqq \deg \kappa < \infty$$
- A (strongly continuous) representation π of G on a locally convex top. vector space V_π is *admissible* if it is $\mathcal{U}(\mathfrak{g})$–finite, is K–semisimple, and $\pi|_K = \sum_{\kappa \in \widehat{K}} m(\kappa, \pi)\kappa$ where mult. $m(\kappa, \pi) < \infty$. Then V_π^K (K–finite vectors) is a $(\mathfrak{g}_{\mathbb{C}}, K)$–module, the *Harish–Chandra module* of π
- $\{$ Harish–Chandra modules $\} \leftrightarrow \{$admissible reps$\}$
- A *globalization* of a $(\mathfrak{g}_{\mathbb{C}}, K)$–module E is a G–module whose Harish-Chandra module is E.
- Globalizations of V_π^K: V_π^ω (analytic, minimal), V_π^∞ (C^∞), $V_\pi^{-\infty}$ (distribution), $V_\pi^{-\omega}$ (hyperfunction, maximal)
- Maximal globalization is an exact functor

Quantization Context

- Representation categories:
  - Harish–Chandra (H–C) modules
  - $(\mathcal{D}, K_{\mathbb{C}})$ modules on a flag manifold $Z = G/Q$
  - $K_{\mathbb{C}}$–equivariant constructible sheaves on Z
  - G–equivariant constructible sheaves on Z
  - {maximal globalizations}
- Objective: Dolbeault coho. $\leftrightarrow$ maximal globalization
- Relations:

$$\boxed{\begin{array}{ccccc}
\{\text{H–C modules}\} & & \leftrightarrow & & \{\text{maximal globalizations}\} \\
\updownarrow & & & & \updownarrow \\
\{(\mathcal{D}, K_{\mathbb{C}})\text{–modules}\} & \leftrightarrow & \{K_{\mathbb{C}}\text{–sheaves}\} & \leftrightarrow & \{G\text{–sheaves}\}
\end{array}}$$

- A very long history: the diagram commutes and the (derived) categories are equivalent

Full Flag I

- Look at the case $Q = B$ Borel and basic datum (H, B, χ):
- H CSG in G, $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{b}$ Borel in $\mathfrak{g}_{\mathbb{C}}$, χ finite dim rep of $(\mathfrak{b}, H)$
- Then $\mathfrak{b} = \mathfrak{h}_{\mathbb{C}} + \mathfrak{n}$ with $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$ and $\mathfrak{b}$ is a G–invariant polarization on the space G/H
- $\mathbb{E}_\chi \to G/H$ vector bundle associated to (H, B, χ)
- Consider cohomologies of $(C^\infty(G/H; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}^*), d_{\mathfrak{n}})$:
  - $r : (\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}})^* \to \mathfrak{n}^*$ restriction dual to $\mathfrak{n} \cong \mathfrak{b}/\mathfrak{h}_{\mathbb{C}} \hookrightarrow \mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$
  - $d_{\mathfrak{n}} : C^\infty(G/H; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*) \to C^\infty(G/H; \mathbb{E} \otimes \Lambda^{p+1} \mathbb{N}^*)$ unique first order G–invariant operator with symbol
  - $(\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}})^* \otimes E_\chi \otimes \Lambda^p \mathfrak{n}^* \to E_\chi \otimes \Lambda^{p+1} \mathfrak{n}^*$ given by
$$\phi \otimes e \otimes \omega \mapsto e \otimes (r(\phi) \wedge \omega)$$
- $\mathfrak{n} \cap \overline{\mathfrak{n}} = 0$ (so H is maximally compact) this leads to the C^∞ fundamental series representation s of G

Full Flag II

- Given H choose $\mathfrak{b}$ maximally real ($\mathfrak{n} \cap \bar{\mathfrak{n}}$ maximal) for $\mathfrak{h} \subset \mathfrak{b}$
- $H = T \times A$ and $P = MAN_{\mathfrak{h}}$ associated cuspidal parabolic
- $\eta = \psi \otimes \eta^0$ discrete series rep of $M^\dagger := Z_M(M^0)M^0$ where η^0 has $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$–antidominant infinitesimal character ν
- $\sigma \in \mathfrak{a}^*$ and $\pi_{\psi,\nu,\sigma} = \mathrm{Ind}_{M^\dagger AN_{\mathfrak{h}}}^G (\eta \otimes e^{i\sigma})$ inf char $\nu + i\sigma$
- This gives the C^∞ H–series of tempered representations
- Problems when $\mathfrak{b}$ is not maximally real:
  - $(C^\infty(G/H; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}^*), d_{\mathfrak{n}})$ not acyclic so it only computes hypercohomology (not cohomology)
  - $d_{\mathfrak{n}}$ need not have closed range (problem for topology)
- Solution: (i) alg. (Zuckerman) version of cohomology, (ii) hyperfn. ($C^{-\omega}$) coef. instead of smooth (C^∞), (iii) use subcpx. of $(C^{-\omega}(G/H; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}^*), d_{\mathfrak{n}})$ w/ natural topology

Full Flag III

- Zuckerman derived functor modules
  - $\mathcal{M}(\mathfrak{g}_{\mathbb{C}}, K)_{(K)}$: category of K–finite $(\mathfrak{g}_{\mathbb{C}}, K)$–modules
  - $\Gamma : \mathcal{M}(\mathfrak{g}_{\mathbb{C}}, H \cap K)_{(H\cap K)} \to \mathcal{M}(\mathfrak{g}_{\mathbb{C}}, K)_{(K)}$ maximal K–finite K–semisimple submodule, $R^p\Gamma$ its derived functors
  - $A^p(G, H, \mathfrak{b}, \chi) = (R^p\Gamma)(\mathrm{Hom}_{\mathfrak{b}}(\mathcal{U}(\mathfrak{g}), E_\chi)_{(H\cap K)})$ Zuckerman
  - C^{for}: $(H \cap K)$–finite formal p.s. sec. at $1H$ in G/H
  - Fact: $A^p(G, H, \mathfrak{b}, \chi) \cong H^p(C^{for}(G/H; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}^*)_{(K)}, d_{\mathfrak{n}})$
- Cauchy–Riemann modules
  - $G/H \to S = G \cdot \mathfrak{b} \subset Z$, S is CR submanifold of Z
  - $\mathfrak{n}_S = \mathfrak{n}/(\mathfrak{n} \cap \overline{\mathfrak{n}})$: antiholo cotangent space of S
  - $\overline{\partial}_S : C^\infty(S; \Lambda^p \mathbb{N}_S^*) \to C^\infty(S; \Lambda^{p+1}\mathbb{N}_S^*)$ CR operator
  - the complex $(C^{-\omega}(S; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}_S^*), \overline{\partial}_S)$ is isomorphic to $(\{C^{-\omega}(G) \otimes \mathbb{E}_\chi \otimes \Lambda^\bullet(\mathfrak{n}/(\mathfrak{n} \cap \overline{\mathfrak{n}}))^*\}^{\mathfrak{n}\cap\overline{\mathfrak{n}}, H}, \delta_{\mathfrak{n}\cap\overline{\mathfrak{n}}})$

Full Flag IV

- Local cohomology complex
  - $\widetilde{S}$: germ of open nbhd. of S in Z, $u = \mathrm{codim}_{\mathbb{R}}(S \subset Z)$
  - $(C_S^{-\omega}(\widetilde{S}; \widetilde{\mathbb{E}_\chi} \otimes \Lambda^\bullet \mathbb{T}_Z^{0,1*}), \overline{\partial})$: $C^{-\omega}(\widetilde{S})$ coeff supported in S
  - $H^p(C_S^{-\omega}(\widetilde{S}; \widetilde{\mathbb{E}_\chi} \otimes \Lambda^\bullet \mathbb{T}_Z^{0,1*}), \overline{\partial}) \cong H_S^p(\widetilde{S}; \mathcal{O}(\widetilde{\mathbb{E}_\chi}))$ local coho
- **Theorem.** There are canonical isomorphisms

$$H^p(C^{-\omega}(G/H; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}^*), d_{\mathfrak{n}})$$
$$\cong H^p(C^{-\omega}(S; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}_S^*), \overline{\partial}_S)$$
$$\cong H_S^{p+u}(\widetilde{S}; \mathcal{O}(\widetilde{\mathbb{E}_\chi})).$$

These all have natural Fréchet topologies, the $\cong$ are topological, and the action of G is continuous. Resulting reps of G are canonically and topologically $\cong$ to the action of G on the maximal globalization of $A^p(G, H, \mathfrak{b}, \chi)$.

Extensions

This result (Schmid & W) was made more functorial by H.-W. Wong. He extended it first to more general flags and then to possibly infinite dimensional bundles. Main problems: functoriality and closed range for the differentials.

W. Schmid and W–: “Geometric Quantization and Derived Functor Modules for Semisimple Lie Groups”, J. Funct. Anal. 90 (1990), pp. 48-112

H-W. Wong, “Dolbeault Cohomological Realization of Zuckerman Modules Associated with Finite Rank Representations”, J. Funct. Anal. 129 (1995), pp. 428–454

H-W. Wong, “Cohomological Induction in Various Categories and the Maximal Globalization Conjecture”, Duke Math. J. 96 (1999), pp. 1–27

Convexity

- Fix a measurable open G_0–orbit $D = G_0(z) \subset Z$
- Let $\mathbb{E} \to D$ homogeneous holomorphic vector bundle
- Let $s = \dim_{\mathbb{C}} C$ and $n = \dim_{\mathbb{C}} D$
- There is an exhaustion function $\varphi : D \to \mathbb{R}$ whose Levi form has at least $n - s$ eigenvalues $\geqq 0$ at every point.
- So $H^q(D; \mathcal{O}(\mathbb{E})) = 0$ for $q > s$
- and if $\mathbb{E}$ is sufficiently negative in a sense to be made precise later, then $H^q(D; \mathcal{O}(\mathbb{E})) = 0$ for $q < s$
- so in the "sufficiently negative" case the cohomology of interest is $H^s(D; \mathcal{O}(\mathbb{E}))$.
- (If D is a bounded symmetric domain, then $s = 0$ and this leads to the holomorphic discrete series.)

Double Fibration

- $\mathfrak{X}_D$: incidence space $\{(z, C) \in D \times \mathcal{M}_D \mid z \in D\}$
- holomorphic double fibration:

$$\begin{tikzcd} & \mathfrak{X}_D \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ D & & \mathcal{M}_D \end{tikzcd}$$

  where μ is a holomorphic submersion and ν is a proper holomorphic map which is a locally trivial bundle
- $D = G_0(z)$: measurable open orbit, $L_0 = G \cap Q_z$
- $\mathbb{E}_\chi \to D$: G_0–homogeneous holomorphic vector bundle
- Theorem. If $q \geqq 0$ then the Leray derived sheaf for ν is given by $\nu_*^q(\mathcal{O}(\mu^*\mathbb{E}_\chi)) = \mathcal{O}(\mathbb{E}')$ where $\mathbb{E}' \to \mathcal{M}_D$ is the G_0–homogeneous, locally G–homogeneous, holomorphic vector bundle with fiber $H^q(C; \mathcal{O}(\mathbb{E}_\chi|_C))$ at $C \in \mathcal{M}_D$.

deRham-Based Spectral Sequence

- Aim: interpret $H^r(D; \mathcal{O}(\mathbb{E}))$ as analytic object on $\mathcal{M}_D$.
- Step 1: $\mu^{(r)} : H^r(D; \mathcal{O}(\mathbb{E})) \cong H^r(\mathfrak{X}_D; \mu^{-1}(\mathcal{O}(\mathbb{E})))$.
- Step 2: Let $\Omega^p_\mu(\mathbb{E}) = \Omega^p_\mu \otimes_{\mathcal{O}} \mathcal{O}(\mu^*\mathbb{E})$, holomorphic p–forms along the fibers of μ, with the pull–back $\mu^*\mathbb{E}$ of $\mathbb{E}$ to $\mathfrak{X}_D$.
- Theorem. Then there is a spectral sequence

$$E_1^{p,q} = \Gamma(\mathcal{M}_D; \nu^q_* \Omega^p_\mu(\mathbb{E})) \implies H^{p+q}(D; \mathcal{O}(\mathbb{E})).$$

  The direct images $\nu^q_* \Omega^p_\mu(\mathbb{E}) = \mathcal{O}(\mathbb{V}^{(q)})$ for certain $(\mathfrak{g}, G_0)$–homogeneous vector bundles $\mathbb{V}^{(q)} \to \mathcal{M}_D$.
- If $\mathbb{E}_\chi \to D$ is sufficiently negative the $\nu^q_* \Omega^p_\mu(\mathbb{E}))$ are concentrated in degree $s = \dim_{\mathbb{C}} C$, and
- the spectral sequence collapses to $H^s(D; \mathcal{O}(\mathbb{E})) \cong \left(\ker : \Gamma(\mathcal{M}_D, \nu^s_* \Omega^0_\mu(\mathbb{E})) \xrightarrow{\nabla} \Gamma(\mathcal{M}_D, \nu^s_* \Omega^1_\mu(\mathbb{E}))\right)$.

Extensions

- All this is part of a joint project with Michael Eastwood.
- Given sufficient negativity, this identifies the range of the double fibration transform as the kernel of a particular system of PDE
- In the hermitian holomorphic case one can replace the deRham-based spectral sequence by a sequence based on the Bernstein–Gelfand–Gelfand (BGG) resolution. That allows one to just work with the entire flag instead of only the real group orbits, and then one can be completely explicit about “sufficient negativity”.
- In the hermitian nonholomorphic cases the BGG–based methods also seem to work, but this is not yet written down.

Indefinite Metric Quantization

- Let $X = G/H$ indefinite-hermitian symmetric space, e.g. the open orbit $D_i = U(p,q)/(U(p-i,i) \times U(i,q-i))$ in the flag manifold Z: q–dimensional subspaces of $\mathbb{C}^{p,q}$
- $\chi \in \widehat{H}$ (possibly ∞-dimensional)
- $\mathbb{E}_\chi \to X$ corresponding G–homogeneous holomorphic hermitian vector bundle over X
- An $\mathbb{E}_\chi$–valued (a,b)–form ω is *strongly harmonic* if $(\overline{\partial}\omega = 0$ and $\overline{\partial}^*\omega = 0)$
- $\mathcal{H}^{a,b}(X;\mathbb{E}_\chi)$: strongly harmonic $\mathbb{E}_\chi$–valued (a,b)–forms
- In an analog of the hermitian holomorphic case, if $\mathbb{E}_\chi \to X$ is sufficiently negative, then the global inner product on $\mathcal{H}^{a,b}(X;\mathbb{E}_\chi)$ is semidefinite and that unitarizes the ordinary Dolbeault cohomology $H^s(X;\mathcal{O}(\mathbb{E}_\chi))$.