Composition algebras and Cartan's isoparametric hypersurfaces

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Abstract

We discuss Cartan's classification of isoparametric hypersurfaces with three different principal curvatures in spheres. In particular, we explain how the Theorem of Hurwitz on composition algebras relates to the Theorem of Cartan-Schouten on Riemannian manifolds with a flat metric connection having a skew-symmetric torsion.

1 Cartan's isoparametric hypersurfaces

In 1938, Cartan wrote the paper *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphèriques* ([5]) where he classified isoparametric hypersurfaces in spheres with three distinct principal curvatures. These families are now referred to as *Cartan's isoparametric hypersurfaces.* In modern terminology, he proved that a *compact isoparametric hypersurface in the sphere* $\mathbb{S}^{N+1}$ *with three distinct principal curvatures is a tube around the standard embedding of either the real, complex, quaternionic or Cayley projective plane. In particular,* $N = 3\nu$ *and the only possibilities for* ν *are* 1, 2, 4 *and* 8. Cartan already had proved that it is equivalent to classify the real homogeneous polynomials $F(x_1, \dots, x_{N+2})$ of degree three that satisfy the following two properties.

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(i) The squared norm of the gradient $\Delta_1(F) := \sum_i \left(\frac{\partial F}{\partial x_i}\right)^2$ is constant on the hypersphere $\mathbb{S}^{N+1}$, or, equivalently, there is a real number λ such that

$$\Delta_1 F = \lambda(x_1^2 + x_2^2 + ... + x_{N+2}^2)^2. \tag{1.1}$$

(ii) The polynomial F is harmonic:

$$\Delta_2(F) = \sum_i \frac{\partial^2 F}{\partial x_i^2} = 0. \tag{1.2}$$

Classically, $\Delta_1 F$ and $\Delta_2 F$ are called the first and the second differential parameters of F respectively. The first and the second differential parameters of F are therefore constant on the regular level sets of F intersected with $\mathbb{S}^{N+1}$, which explains why they are called *isoparametric* hypersurfaces. Geometrically, condition (i) implies that the regular level sets of F restricted to $\mathbb{S}^{N+1}$ are parallel and (ii) that they have constant mean curvature. One can now prove that their principal curvatures are constant and it also follows that they have three different values since F is of degree three.
Now Cartan uses the properties in (i) and (ii) to introduce new orthonormal coordinates in which the polynomial F can be written in a simplified form. It turns out that $N = 3\nu$ and that F can be reduced to the form

$$F = u^3 - 3uv^2 + \frac{3}{2}u\sum_i(x_i^2 + y_i^2) - 3u\sum_i z_i^2 + \frac{3\sqrt{3}}{2}v\sum_i(x_i^2 - y_i^2) + \sum_i z_i Q_i(x, y)\,,$$

where the first ν coordinates are denoted by $x_1, \ldots, x_\nu$, the next ν by $y_1, \ldots, y_\nu$, and then by $z_1, \ldots, z_\nu$. Finally one denotes the last two coordinates by u and v.
The polynomials Q_i in the above formula are quadratic since F is of degree three. They have the remarkable property that

$$\sum_{i=1}^{\nu}[Q_i(x, y)]^2 = 27(\sum_{i=1}^{\nu} x_i^2)(\sum_{i=1}^{\nu} y_i^2)\,. \tag{1.3}$$

It follows that the $Q_i(x, y)$ are bilinear. Setting $H_i = \frac{\sqrt{3}}{9}Q_i$, we get the identity

$$\sum_{i=1}^{\nu}[H_i(x, y)]^2 = (\sum_{i=1}^{\nu} x_i^2)(\sum_{i=1}^{\nu} y_i^2). \tag{1.4}$$

Cartan comments these equations as follows.

> La forme trilinéaire $\sum z_k Q_k$ aux trois séries de variables x_i, y_i, z_i jouit de propriétés remarquables. Si nous posons
>
> $$Q_k = 3\sqrt{3}H_k$$
>
> nous voyons que la première relation [here equation (1.3)] fournit une généralisation des formules bien connues de Lagrange et de Brioschi qui représentent le produit de deux sommes de ν carrés par une somme de ν carrés. On a en effet
>
> $$\sum_i [H_i(x, y)]^2 = \sum x_i^2 \sum y_i^2,$$
>
> les ν quantités $H_i(x, y)$ du premier membre étant bilinéaires par rapport aux x_i et aux y_i.

The existence of isoparametric hypersurfaces with three different principal curvatures depends on the solubility of equation (1.4). One can reformulate (1.4) by saying that there is a nondegenerate bilinear form $H : \mathbb{R}^\nu \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ satisfying $|H(x,y)| = |x||y|$ for every x and y. Setting the i-th component of H equal to H_i, we get (1.4). The Euclidean space $(\mathbb{R}^\nu, \langle\,,\rangle)$ endowed with a product H satisfying $|H(x,y)| = |x||y|$ is called a *composition algebra*, and will be discussed in Section 2 below. A theorem of Hurwitz ([8]) states that a composition algebra with a unit is either the field of the real or the complex numbers, the skew field of the quaternions, or the division algebra of the Cayley numbers. The composition algebra defined by equation (1.4) does not necessarily have a unit, but that can be corrected with a simple change of coordinates as we will explain in Section 2, thus finishing the proof. In fact, Cartan never mentions in [5] the Theorem of Hurwitz although it is closely related to the results of Lagrange and Brioschi mentioned in the quote above. Instead he makes a detour and classifies himself composition algebras without saying so explicitly. He does this by introducing with help of the composition algebra an "*absolute parallelism*" on the unit sphere in $\mathbb{R}^\nu$, which is a flat metric connection having the great circles as geodesics. Now he can use a result of his with Schouten in [3] to finish the proof. We quote Cartan again.

> Il est connu [footnote referring to [3]] que les seuls espaces riemanniens admettant un parallélisme absolu isogonal sont les espaces représentatifs des groupes simples clos et l'espace elliptique à 7 dimensions, auxquels il faut ajouter l'espace repésentatif du groupe clos des rotations de la circonférence ($\nu = 2$) ainsi que les produits topologiques de deux ou plusieurs des espaces précédents. Parmi tous ces espaces ceux qui sont á courbure constante sont les espaces elliptiques à 1, 3 et 7 dimensions; aucun produit topologique ne peut convenir, car le ds^2 de l'espaces riemannien correspondant serait la somme de deux ds^2 portant sur des variables séparées u_i et v_j et le tenseur de Riemann ne pourrait pas être de la forme qui convient à un espace à courbure constante non nulle.
>
> Par conséquent le problème proposé n'admet de solution que pour $\nu = 2$, 4, 8, cas auxquels il faut naturellement ajouter $\nu = 1$!

By "parallélisme absolu isogonal", Cartan means a flat metric connection. One should stress that the Theorem of Hurwitz is very elementary and can be stated and proved with methods that do not go beyond high school mathematics; see e.g. the proof in [9], which was written by an interested layman. The Theorem of Cartan and Schouten is of course on a very different level. A modern proof can be found in [2].
After this detour, Cartan gives an explicit formula for the trilinear form $\mathcal{F} = \sum_k z_k H_k(x,y)$ in the four possible cases for ν. In fact, $\mathcal{F}$ has the form

$$\mathcal{F} = \frac{1}{2}((XY)Z + \bar{Z}(\bar{Y}\bar{X})),$$

where X, Y, and Z are understood to be real numbers if $\nu = 1$, complex numbers if $\nu = 2$, quaternions if $\nu = 4$, and Cayley numbers if $\nu = 8$. The bars over X, Y, and Z can be ignored in the case of the reals and are otherwise understood to be conjugation. With this Cartan can express the polynomial F in the following normal form

$$F = u^3 - 3uv^2 + \frac{3}{2}u(X\bar{X} + Y\bar{Y} - 2Z\bar{Z}) + \frac{3\sqrt{3}}{2}v(X\bar{X} - Y\bar{Y}) + \frac{3\sqrt{3}}{2}((XY)Z + \bar{Z}(\bar{Y}\bar{X})),$$

which finishes his classification of the polynomials satisfying (1.1) and (1.2).
The arguments of Cartan where he uses his result with Schouten instead of the more straightforward Theorem of Hurwitz have been considered rather obscure. In Section 3 of this note, we fill in details missing in Cartan's arguments and show how composition algebras can be classified with help of "absolute parallelism." His arguments turn out to be completely correct as was to be expected, although not very detailed.
The story here told is missing in the survey [10] on isoparametric hypersurfaces and their generalizations, which was written at the request of Franki Dillen. Saddened by his untimely death, we dedicate this note to his memory.

2 Composition algebras

We will let $\langle\, ,\rangle$ denote the standard scalar product on $\mathbb{R}^n$ and $|\ |$ its norm. We will assume that $n \geq 1$ and let $\cdot$ denote a bilinear product on $\mathbb{R}^n$.
An algebra $\mathcal{A} = (\mathbb{R}^n, \langle\, ,\rangle, \cdot\,)$ is called a *composition algebra* if

$$|x \cdot y| = |x|\,|y|\,. \tag{2.1}$$

for every x and y in $\mathbb{R}^n$; see [6], Chapter 10. A *unit* in a composition algebra is an element e such that $e \cdot x = x \cdot e = x$ for every $x \in \mathbb{R}^n$. We will sometimes write $H(x, y)$ instead of $x \cdot y$ for the product in a composition algebra. We do this in particular in Section 3 where we explain Cartan's proof, since it is closer to his original notation.
The fields of the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$, the skew field of the quaternions $\mathbb{H}$, and the division algebra of the Cayley numbers (or octonions) $\mathbb{O}$ are examples of composition algebras with a unit. One can modify the product $\cdot$ in $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ by setting $H(x, y) = \bar{x} \cdot y$ for example, thus arriving at composition algebras without a unit.
Note that a composition algebra cannot have divisors of zero. Since the bilinear form $\cdot$ is clearly not degenerate, a composition algebra is also a division algebra, i.e., the two equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions in V for every a and b in V as long as $a \neq 0$.

Theorem 2.1 (Hurwitz [8])**.** *A composition algebra with a unit is isomorphic to* $\mathbb{R}$*,* $\mathbb{C}$*,* $\mathbb{H}$*, or* $\mathbb{O}$*.*

We can use the Theorem of Hurwitz to classify composition algebras in general; see [11], Prop. 1, p. 25. To see this let $\mathcal{A} = (\mathbb{R}^n, \langle\, ,\rangle, \cdot\,)$ denote a composition algebra that does not necessarily have a unit. We choose a unit vector x_0 in $\mathbb{R}^n$ and define orthogonal endomorphisms A and B of $\mathbb{R}^n$ by setting $A(x) = x_0 \cdot x$ and $B(x) = x \cdot x_0$. Now we define a new product H on $\mathbb{R}^n$ by setting $H(x, y) = B^{-1}(x) \cdot A^{-1}(y)$. Then $x_0^2 = x_0 \cdot x_0$ is a unit. Indeed,

$$H(x_0^2, x) = B^{-1}(x_0^2) \cdot A^{-1}(x) = x_0 \cdot A^{-1}(x) = A(A^{-1}(x)) = x$$

for all x in $\mathbb{R}^n$. Similarly $H(x, x_0^2) = x$ for all x in $\mathbb{R}^n$. Now we can use the Theorem of Hurwitz to classify the possibilities for H. In the proof of Cartan described in Section 1, we only need to know H up to an orthogonal transformation in each of the two entries. We can therefore assume that the composition algebra defined by (1.4) is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, thus finishing the classification of the polynomials F satisfying (1.1) and (1.2). These considerations now show that we have the following corollary of the Theorem of Hurwitz in which we do not assume the existence of a unit.

Corollary 2.1. *If* $\mathcal{A} = (\mathbb{R}^n, \langle\,,\rangle, \cdot\,)$ *is a composition algebra, then* $n = 1,\ 2,\ 4,$ *or* 8.

For later, we remark that polarizing (2.1) first in x and then in y, we get

$$\begin{aligned} \langle x\cdot y, x'\cdot y\rangle &= \langle x, x'\rangle\langle y, y\rangle\,, \\ \langle x\cdot y, x\cdot y'\rangle &= \langle x, x\rangle\langle y, y'\rangle\,. \end{aligned} \tag{2.2}$$

3 Cartan's arguments

We let H denote a product on $\mathbb{R}^n$ such that $\mathcal{A} = (\mathbb{R}^n, \langle\,,\rangle, H)$ is a composition algebra. Following Cartan we modify H as follows. If $H_1(x,y)$ is the first component of $H(x,y)$ in $\mathbb{R}^n$, we can write

$$\begin{aligned} H_1(x,y) &= a_1(x)y_1 + a_2(x)y_2 + \cdots a_n(x)y_n \\ &= \langle Ax, y\rangle, \end{aligned}$$

where A is an endomorphism of $\mathbb{R}^n$. Let us fix x. Using that left multiplication by x is a vector space isomorphism, we find a $y \in \mathbb{R}^n$ such that only the first component of $H(x,y)$ is nonzero. Hence

$$H_1(x,y)^2 = |H(x,y)|^2 = |x|^2\,|y|^2.$$

Now

$$H_1(x,y)^2 = \langle Ax, y\rangle^2 \le |Ax|^2\,|y|^2$$

implies $|x| \le |Ax|$.
Since

$$|x||Ax| = |H(x, Ax)| \ge |H_1(x, Ax)| = |\langle Ax, Ax\rangle| = |Ax|\,|Ax|,$$

we also get $|x| \ge |Ax|$ for every x in $\mathbb{R}^n$. It follows that $|x| = |Ax|$ for every $x \in \mathbb{R}^n$ and we have proved that A is an orthogonal endomorphism.
We modify the product H by setting

$$\tilde{H}(x,y) = H(A^{-1}x, y).$$

Clearly $(\mathbb{R}^n, \langle\,,\rangle, \tilde{H})$ is still a composition algebra. The first component $\tilde{H}_1(x,y)$ of $\tilde{H}(x,y)$ now clearly satisfies

$$\tilde{H}_1(x,y) = \langle x, y\rangle.$$

To simplify the notation, we will write $H(x,y)$ instead of $\tilde{H}(x,y)$. We will let $I(x,y)$ in $\mathbb{R}^{n-1}$ denote the image of $H(x,y)$ under the projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$ that we get by deleting the first component. Then

$$|H(x,y)|^2 = \langle x, y\rangle^2 + |I(x,y)|^2.$$

Now $|H(x,x)| = |x||x|$ implies that $I(x,x) = 0$ for all x in $\mathbb{R}^n$. Polarizing, we get

$$I(x,y) = -I(y,x)$$

for every x and y in $\mathbb{R}^n$. In particular,

$$H(x,y) = -H(y,x)$$

if $\langle x, y\rangle = 0$.

We now start defining the parallelism on $\mathbb{S}^{n-1}$ according to Cartan. Let two points x and x' in $\mathbb{S}^{n-1}$ be given. Let y be a vector in $T_x\mathbb{S}^{n-1}$. Then there is a unique element y' in $\mathbb{R}^n$ such that

$$H(x,y) = H(x',y').$$

First note that $\langle x, y\rangle = 0$ is equivalent to $\langle x', y'\rangle = 0$, i.e., y' is in $T_{x'}\mathbb{S}^{n-1}$. Also $|y| = |H(x,y)| = |H(x',y')| = |y'|$ shows that the map from $T_x\mathbb{S}^{n-1}$ to $T_{x'}\mathbb{S}^{n-1}$ that sends y to y' is an isometry. We say that y and y' are *parallel* if they correspond under this map. We now say that a vector field is *parallel* if its values are parallel in this sense. This defines a flat metric connection on $\mathbb{S}^{n-1}$ that we denote by ∇. We can easily describe the geodesics with respect to this connection.

Lemma 3.1. *The geodesics on $\mathbb{S}^{n-1}$ with respect to ∇ are the great circles.*

Proof. We consider the great circle

$$\gamma(t) = \cos tx + \sin ty,$$

where $\langle x, y\rangle = 0$. To prove that $\gamma'(t)$ is parallel along $\gamma(t)$, we need to show that $H(\gamma(t), \gamma'(t)) = H(x,y)$. This now follows immediately since $H(x,y) = -H(y,x)$. □

We now recall the Theorem of Cartan and Schouten in [3]; see also [2].

Theorem 3.1 (Cartan-Schouten). *Let (M, g, ∇) be a simply connected, complete and irreducible Riemannian manifold equipped with a flat metric connection ∇ that does not coincide with, but has the same geodesics as the Levi Civita connection of (M, g). Then M is either isometric to a compact simple Lie group with a bi-invariant metric or to a round sphere $\mathbb{S}^7$.*

Using the result of Cartan, see [4], p. 197, that the third Betti number of a compact simple Lie group never vanishes, we see that the only sphere having the structure of a simple Lie group is $\mathbb{S}^3$. Hence we see that the spheres $\mathbb{S}^{n-1}$ carrying Cartan's parallelism can only be $\mathbb{S}^3$ and $\mathbb{S}^7$ to which we have to add $\mathbb{S}^0$ and $\mathbb{S}^1$. This proves that the dimension n of the composition algebra can only be 1, 2, 4, and 8. The absolute parallelisms on the compact simple Lie groups are known, and Cartan and Schouten also classified those on $\mathbb{S}^7$ although it is not stated in the above formulation of their theorem. One can therefore use absolute parallelism to give a proof of the Hurwitz Theorem, albeit not a simple one.

Remark 3.1. In the case of the classical composition algebras, the endomorphism A is conjugation. Cartan's modified product is therefore $H(x,y) = \bar{x}y$.

4 Composition algebras and skew-symmetric torsion

We would now like to give an explicit formula for the torsion tensor of the flat connection that Cartan associated to a composition algebra; see Section 3. The torsion is skew-symmetric since the connection is metric and has the same geodesics as the Levi Civita

connection. Connections with skew-symmetric torsion were already studied by Cartan and are still today an active field of research; see e.g. [1].
We first review some elementary formulas. Suppose a connection ∇ is given on a Riemannian manifold (M, g). We let ∇^{LC} denote the Levi Civita connection of (M, g). The difference

$$A(X,Y) = \nabla_X Y - \nabla_X^{LC} Y$$

is a $(2,1)$-tensor which we use to define a $(3,0)$-tensor A by setting

$$\mathrm{A}(X,Y,Z) = \langle A(X,Y), Z\rangle.$$

It clearly follows that $\mathrm{A}(X,Y,Z) = -\mathrm{A}(X,Z,Y)$ if and only if ∇ is metric. It is also obvious that $\mathrm{A}(X,Y,Z) = -\mathrm{A}(Y,X,Z)$ if and only if the geodesics of ∇ and ∇^{LC} coincide. Note that a $(3,0)$-tensor is skew-symmetric with respect to all three transpositions if it is skew-symmetric with respect to two of them. Hence A is skew-symmetric if and only if ∇ is metric and has the same geodesics as ∇^{LC}. Applying the calculations in the proof of the uniqueness of the Levi Civita connection to a metric connection ∇, we get

$$\langle \nabla_X Y, Z\rangle = \langle \nabla_X^{LC} Y, Z\rangle + \frac{1}{2}\Big[\langle Z, T(X,Y)\rangle + \langle Y, T(Z,X)\rangle - \langle X, T(Y,Z)\rangle\Big],$$

where T is the torsion tensor of ∇; see [7], p. 87. It follows that

$$\mathrm{A}(X,Y,Z) = \frac{1}{2}\Big[\langle Z, T(X,Y)\rangle + \langle Y, T(Z,X)\rangle - \langle X, T(Y,Z)\rangle\Big]. \tag{4.1}$$

It follows from equation (4.1) that A is skew-symmetric if and only if the $(3,0)$-tensor

$$\mathrm{T}(X,Y,Z) = \langle T(X,Y), Z\rangle$$

is skew-symmetric. If the $(3,0)$-tensor T is skew-symmetric, (4.1) implies that

$$\mathrm{A}(X,Y,Z) = \frac{1}{2}\mathrm{T}(X,Y,Z); \tag{4.2}$$

see also [1]. We will refer to the $(3,0)$-tensor T as the *torsion* 3*-tensor*.

Next we turn to the connection defined by Cartan; see Section 3. We saw in Lemma 3.1 that it has the same geodesics as the Levi Civita connection. Thus A and T are both skew-symmetric. We now give an explicit formula for T in terms of the modified product H in Section 3.

Proposition 4.1. *We have* $\mathrm{T}_x(y,z,w) = 2\langle H(y,z), H(x,w)\rangle$ *for every* y*,* z *and* $w \in T_x\mathbb{S}^{n-1}$*.*

Proof. We will calculate A which by equation (4.2) suffices to determine T.
Let $x \in \mathbb{S}^{n-1}$ and y a unit vector in $T_x\mathbb{S}^{n-1}$. We consider the great circle $\gamma(t) = \cos t\, x + \sin t\, y$. A vector field $Z(t)$ along $\gamma(t)$ is by definition ∇-parallel if $H(\gamma(t), Z(t)) = H(x,z)$ for all t, where $z = Z(0)$. Taking derivative and setting $t = 0$, we get

$$H(x, Z'(0)) = -H(y,z). \tag{4.3}$$

Clearly,

$$\mathrm{A}_x(y,z,w) = -\langle \nabla_y^{LC} Z(0), w\rangle = -\langle Z'(0), w\rangle.$$

Using (2.2), we get $\mathrm{A}_x(y,z,w) = -\langle H(x, Z'(0)), H(x,w)\rangle$. By (4.3), we now get

$$\mathrm{A}_x(y,z,w) = \langle H(y,z), H(x,w)\rangle$$

which implies the formula we wanted to prove. □

It is very easy to use the formula for A in the proof of Proposition 4.1 to prove directly that A and hence also T is skew-symmetric.

Remark 4.1. Cartan's modified product applied to the classical composition algebras would give rise to the torsion 3-tensor

$$\mathrm{T}_x(y,z,w) = 2\langle \bar{y}z, \bar{x}w\rangle$$

on $\mathbb{S}^{n-1}$; see Remark 3.1.

Cartan's approach is not the only possibility to associate a flat metric connection with skew-symmetric torsion to a composition algebra. We sketch another way to do this.
Let a composition algebra $\mathcal{A} = (\mathbb{R}^n, \langle\,,\rangle, H)$ be given. We can then use both right and left multiplication in the composition algebra to define a natural parallelism on $\mathbb{S}^{n-1}$. We will use right multiplication.
We fix a point x in $\mathbb{S}^{n-1}$ and let z be an element of $T_x\mathbb{S}^{n-1}$. We now define a vector field Z on $\mathbb{S}^{n-1}$ by setting $Z_y = z \cdot a$ where a is the unique element of $\mathbb{R}^n$ such that $y = x \cdot a$. Notice that $Z_y \in T_y\mathbb{S}^{n-1}$ since $\langle Z_y, y\rangle = \langle z\cdot a, x \cdot a\rangle = \langle z, x\rangle = 0$ by (2.2). Similarly, we see that $\langle Z_y, W_y\rangle = \langle z, w\rangle$ where the vector field W with $W_x = w$ is defined as Z.
We now endow $\mathbb{S}^{n-1}$ with the flat metric connection ∇^R such that the vectors fields defined as Z and W are parallel. We will first calculate the difference tensor A.
Let x be a point in $\mathbb{S}^{n-1}$ and let $e_x \in \mathbb{R}^n$ the unique vector such that $x \cdot e_x = x$. The tangent space $T_x\mathbb{S}^{n-1}$ is the orthogonal complement of x. Therefore we can assume that a tangent vector to the unit sphere at x has the form $x \cdot y$, with $y \in < e_x >^{\perp}$.

Proposition 4.2. *The* $(3,0)$*-tensor* A *at a point* $x \in \mathbb{S}^{n-1}$ *is given by*

$$\mathrm{A}(x\cdot u, x\cdot v, x\cdot w) = -\langle (x\cdot v)\cdot u, x\cdot w\rangle,$$

for every $u, v, w \in < e_x >^{\perp}$.

We skip the proof of this proposition since it is similar to the one of Proposition 4.1. It follows easily from the formula in Proposition 4.2 that A is skew-symmetric. Hence the torsion is also skew-symmetric and satisfies the formula

$$\mathrm{T}(x\cdot u, x\cdot v, x\cdot w) = -2\langle (x\cdot v)\cdot u, x\cdot w\rangle$$

for $u, v, w \in < e_x >^{\perp}$.

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