SESQUILINEAR FORMS AND SYMMETRIC SPACES

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Dedicated to Professor Bang-Yen Chen on the occasion of his 75th birthday

Abstract. Let f be a sesquilinear form on $\mathbb{F}^n$ with positive Witt index r where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let $N_i(\mathbb{F}^n, f)$ denote the space of i-dimensional totally isotropic subspaces of $\mathbb{F}^n$ with respect to f where $i \leq r$. Then our main result will be the observation that $N_i(\mathbb{F}^n, f)$ is a symmetric space if and only if $n = 2i$. This gives us seven series of compact symmetric spaces. If we add the three series of Grassmannians $G_i(\mathbb{F}^n)$ over $\mathbb{F}$, we get all ten series of classical compact symmetric spaces.

1. Introduction

1.1. **Symmetric spaces.** A *symmetric space* is a Riemannian manifold M with the property that for every p in M there is an isometry σ_p of M to itself fixing p and reversing the orientation of the geodesics passing through p. It easily follows from the definition that a symmetric space M is homogeneous and can hence be written as a coset space K/L where K is the isometry group of M; see [4], Ch. IV, §3.

We will phrase our main result in terms of symmetric pairs. Let (K, L) be a pair consisting of a Lie group K and a compact subgroup L. We say that (K, L) is a *symmetric pair* if there is an involutive automorphism σ of K with

$$K_0^\sigma \subset L \subset K^\sigma$$

where K^σ is the fixed point group of σ and K_0^σ denotes the identity component of K^σ.

If (K, L) is a symmetric pair, then every K-invariant Riemannian metric on K/L makes it into a symmetric space. A K-invariant Riemannian metric on K/L can always be found by averaging.

If K is the isometry group of a symmetric space M and L the isotropy group of some point in M, then (K, L) is a symmetric pair.

We will say that $M = K/L$ is a *classical* symmetric space and (K, L) a *classical* symmetric pair if K and L are classical matrix groups or finite quotients or coverings of such groups.

1.2. **Sesquilinear forms.** We let $\mathbb{F}$ denote the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$. We will call $\mathbb{F}$ a field in spite of the noncommutativity of $\mathbb{H}$. We will consider the set $\mathbb{F}^n$ of n-tuples of elements in $\mathbb{F}$ (which we will call vectors) as a right vector space over $\mathbb{F}$, i.e., we will multiply a vector in $\mathbb{F}^n$ by a scalar from $\mathbb{F}$ on the right hand side. It is the noncommutativity of the quaternions that makes it necessary to decide on which side we multiply the scalars.

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We let $\sigma : \mathbb{F} \to \mathbb{F}$ denote an *anti-automorphism* of $\mathbb{F}$, i.e.,

$$\sigma(\alpha\beta) = \sigma(\beta)\sigma(\alpha),$$

which we will assume to be continuous.[1] These anti-automorphisms are the identities on $\mathbb{R}$ and $\mathbb{C}$, the conjugation on $\mathbb{C}$, and the conjugation composed with an inner automorphism on $\mathbb{H}$. For our purposes, we can disregard the inner automorphism in the case of $\mathbb{H}$ and assume that an anti-automorphism is the conjugation. We let ϵ denote either 1 or -1. A *sesquilinear form* is by definition a map

$$f : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$$

that is additive in both arguments and satisfies

$$f(x\alpha, y\beta) = \sigma(\alpha)f(x, y)\beta$$

for all x and y in $\mathbb{F}^n$ and all α and β in $\mathbb{F}$; see [1], §3 and especially [8], §8.1. A sesquilinear form f is said to be (σ, ϵ)-*Hermitian* if

$$f(x, y) = \epsilon\sigma(f(y, x))$$

for all x and y in $\mathbb{F}^n$.

A (σ, ϵ)-Hermitian form f is said to be *nondegenerate* if $f(x, y) = 0$ for all y in $\mathbb{F}^n$ implies $x = 0$. In the following, we will assume all forms to be nondegenerate.

1.3. **Totally isotropic subspaces.** Let f be a (σ, ϵ)-Hermitian form on $\mathbb{F}^n$. We will say that a subspace of $\mathbb{F}^n$ is *totally isotropic with respect to* f if the restriction of f to it is identically zero. The *Witt index of* f is defined to be the maximal dimension r of a totally isotropic subspace of $\mathbb{F}^n$ with respect to f. Note that r is at most half the dimension of $\mathbb{F}^n$, i.e., $2r \leq n$.

The automorphism group $\mathrm{Aut}(\mathbb{F}^n, f)$ is clearly a closed subgroup of $\mathrm{GL}(n, \mathbb{F})$ and hence a Lie group. Let W_1 and W_2 be subspaces of $\mathbb{F}^n$ that are isometric with respect to the restrictions of f. Then, by a theorem of Witt, there is an element of $\mathrm{Aut}(\mathbb{F}^n, f)$ that maps W_1 to W_2; see [1], p. 71.

Let $N_i(\mathbb{F}^n, f)$ denote the space of i-dimensional totally isotropic subspaces of $(\mathbb{F}^n, f)$ for $i \leq r$. Then it is clear from the theorem of Witt we just quoted that $\mathrm{Aut}(\mathbb{F}^n, f)$ acts transitively on $N_i(\mathbb{F}^n, f)$. The isotropy group of the action of $\mathrm{Aut}(\mathbb{F}^n, f)$ on $N_i(\mathbb{F}^n, f)$ at some given element is a closed subgroup. It follows that we can write $N_i(\mathbb{F}^n, f)$ as a quotient space of $\mathrm{Aut}(\mathbb{F}^n, f)$ and a closed subgroup and hence that $N_i(\mathbb{F}^n, f)$ is a differentiable manifold, which is in fact a closed submanifold of the Grassmannian $G_i(\mathbb{F}^n)$ and hence compact.

In the following, we will simplify the notation by setting $G = \mathrm{Aut}(\mathbb{F}^n, f)$. We will say that a Lie group, which we do not assume to be connected, is *reductive* if its Lie algebra is the direct sum of an Abelian and a semisimple Lie algebra.

We will need the following lemma on the structure of G.

Lemma 1.1. *Assume the Witt index r of $(\mathbb{F}^n, f)$ is positive. Then G is a non-compact, possibly disconnected, reductive Lie group. If K is a maximal compact subgroup of G, then K acts transitively on $N_i(\mathbb{F}^n, f)$ for all $i \leq r$.*

We will prove the above lemma and the following theorem in Section 2 by looking at the different possibilities for a (σ, ϵ)-Hermitian form f on $\mathbb{F}^n$. The group G will in all cases turn out to be a well-known classical Lie group, which we know to be reductive. The center of G is in all cases either zero- or one-dimensional. It is also

[1](Anti)-automorphisms of $\mathbb{R}$ and $\mathbb{H}$ are continuous, but $\mathbb{C}$ has discontinuous ones.

well known that any two maximal compact subgroups in these reductive groups G are conjugate to each other.

We can now formulate our main result.

Theorem 1.2. *Assume the Witt index r of $(\mathbb{F}^n, f)$ is positive and that L is the isotropy group of the action of K on $N_i(\mathbb{F}^n, f)$ at some given totally isotropic i-plane. We assume furthermore that $r \geq 2$ if $\mathbb{F} = \mathbb{R}$ and $\epsilon = 1$.*

Then (K, L) is a symmetric pair if and only if i is equal to the Witt index r and $n = 2r$.

If $r = 1$ and $\epsilon = 1$, then K/L is diffeomorphic to a sphere for all three fields, but only for the real field is (K, L) a symmetric pair; see the remark at the end of 2.1.1 for details.

Finally, we would like to point out that it is assumed in the definition of symmetric pairs on p. 209 in [4] that the group here denoted by K is connected. We have decided not to follow this since the spaces $N_k(\mathbb{F}^{2k}, f)$ are not always connected. This is of course not essential for our results and could be modified; see also the remarks in Section 3.2 at the end of the paper.

2. The proofs of Lemma 1.1 and Theorem 1.2

It is an easy task to list the possible types of (σ, ϵ)-Hermitian forms by going through the possibilities for $\mathbb{F}$, σ, and ϵ . We bring the result in the following table. In the last column, we explain how the forms are usually referred to.

$\mathbb{R}$	$\sigma =$ identity	$\epsilon = +1$	symmetric forms
		$\epsilon = -1$	symplectic forms
$\mathbb{C}$	$\sigma =$ identity	$\epsilon = +1$	symmetric forms
		$\epsilon = -1$	symplectic forms
	$\sigma =$ conjugation	$\epsilon = +1$	Hermitian forms
		$\epsilon = -1$	skew-Hermitian forms
$\mathbb{H}$	$\sigma =$ conjugation	$\epsilon = +1$	Hermitian forms
		$\epsilon = -1$	skew-Hermitian forms

There are eight lines in the table. Note that a Hermitian form over $\mathbb{C}$ becomes skew-Hermitian if we multiply it by i and vice versa. This will make these two cases equivalent from our point of view since the automorphism groups and the spaces $N_i(\mathbb{F}^n, f)$ do not change when we multiply f by i. In the quaternionic case, the Hermitian and the skew-Hermitian forms are genuinely different.

We are therefore left with *seven different types of (σ, ϵ)-Hermitian forms.* It turns out that each type has a *normal form*, which will lead to a complete classification of these sesquilinear forms. These normal forms are well known from Linear Algebra with the possible exception of the skew-Hermitian forms on $\mathbb{H}^n$. All cases can be found in [2], §12. We will use some elementary theory of symmetric spaces and Lie groups in the proofs, but no use will be made of the classifications of symmetric spaces and simple Lie groups.

It will be convenient to group the seven types of normal forms as follows:

(1) The symmetric forms over $\mathbb{R}$ and the Hermitian forms over $\mathbb{C}$ and $\mathbb{H}$, which we will refer to as *inner products on $\mathbb{F}^n$*.
(2) The symplectic forms over $\mathbb{R}$ and $\mathbb{C}$
(3) The symmetric forms over $\mathbb{C}$ and the skew-Hermitian forms over $\mathbb{H}$.

We will consider the seven cases separately and prove the lemma and the theorem for each case.

2.1. **Inner products.** Let f be a form on $\mathbb{F}^n$ that is symmetric if $\mathbb{F} = \mathbb{R}$ and Hermitian if $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. Then there is a basis of $\mathbb{F}^n$ such that f can be written as

$$f(x, y) = \sum_{i=1}^{k} \bar{x}_i y_i - \sum_{i=k+1}^{n} \bar{x}_i y_i$$

where we disregard the bars in the formula in the real case. The pair $(k, n-k)$ is called the *signature of* f. It is easy to see that the Witt index r of f is equal to $\min\{k, n-k\}$. We will mostly assume that $k \leq n-k$ and hence that $r = k$.

2.1.1. *Inner products on* $\mathbb{R}^n$. The automorphism group $G = \mathrm{Aut}(\mathbb{R}^n, f)$ is denoted by $\mathrm{O}(k, n-k)$ and called the *orthogonal group of type* $(k, n-k)$. The group $G = \mathrm{O}(k, n-k)$ has four connected components when $k \geq 1$. The group $G = \mathrm{O}(k, n-k)$ is simple except in some exceptional cases where it is semi-simple.[2] We set $\mathrm{O}(n) = \mathrm{O}(n, 0)$. The group $\mathrm{O}(n)$ is compact and has two components. The group $\mathrm{O}(k) \times \mathrm{O}(n-k)$ is a maximal compact subgroup in $\mathrm{O}(k, n-k)$. It is easy to see that the action of $\mathrm{O}(k) \times \mathrm{O}(n-k)$ on $N_i(\mathbb{R}^n, f)$ for all $i \leq r = k$ is transitive.

We would now like to prove Theorem 1.2 in this case. Let P be the i-plane in $N_i(\mathbb{R}^n, f)$ spanned by $e_1 + e_{k+1}, \ldots, e_i + e_{k+i}$ where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. We denote by L the isotropy group of P under the action of $K = \mathrm{O}(k) \times \mathrm{O}(n-k)$. Our goal is to show that (K, L) is a symmetric pair if and only if $n = 2i$. Now it is easy to see that a block matrix (A, B) in $K = \mathrm{O}(k) \times \mathrm{O}(n-k)$ is contained in L if and only if A is a block matrix of the form (C, D) in $\mathrm{O}(k)$ and B is a block matrix of the form (C, E) in $\mathrm{O}(n-k)$ where $C \in \mathrm{O}(i)$, $D \in \mathrm{O}(k-i)$, and $E \in \mathrm{O}(n-k-i)$. In particular, L is equal to the diagonal group $\Delta(\mathrm{O}(k))$ in K if $i = k = n - i$. This proves that (K, L) is a symmetric pair when $i = k$ and $2k = n$ since then $\Delta(\mathrm{O}(k))$ is the fixed point group of the involution of $K = \mathrm{O}(k) \times \mathrm{O}(k)$ that switches the two factors. The corresponding symmetric space $\mathrm{O}(k) \times \mathrm{O}(k)/\Delta(\mathrm{O}(k))$ can easily be seen to coincide with

$$\mathrm{O}(k).$$

Note that the symmetric space $N_k(\mathbb{R}^{2k}, f)$ has two components.

It is left to prove that (K, L) is not a symmetric pair when $1 < i < k$ or $2k < n$. We give an indirect proof and assume that $i < k$ or $2k < n$ and that L is open and closed in the fixed point group of an involution σ on K. Note that (the identity component of) K can consist of two, three, or four simple factors. We first assume that K has two factors. Now we know from [4], Theorem 5.3 on p. 379, that the involution σ either leaves the simple factors of K invariant or interchanges two such factors. The fact that L cannot be written as a product of subgroups of $\mathrm{O}(k)$ and $\mathrm{O}(n-k)$ shows that σ must interchange the factors which is not possible if $2k < n$, and L would have smaller dimension than the fixed point group if $i < k$ and $2k = n$. Similar arguments can be used when K has more than two simple factors. This finishes the proof of both Lemma 1.1 and Theorem 1.2 in this case.

It is easy to see that (K, L) is a symmetric pair and K/L a sphere when $i = r = 1$. In fact, if f is an inner product with signature $(n, 1)$ on $\mathbb{R}^{n+1}$, then $G = \mathrm{O}(n, 1)$

[2] It will not be important for us to know when $\mathrm{O}(k, n-k)$ is simple, but one finds necessary and sufficient conditions for this in [5] on p. 362.

and (K, L) is the symmetric pair $(\mathrm{O}(n) \times \mathrm{O}(1), \mathrm{O}(n-1) \times \mathrm{O}(1))$ which has the more familiar symmetric pair $(\mathrm{SO}(n), \mathrm{SO}(n-1))$ as its identity component.[3]

2.1.2. *Inner products on* $\mathbb{C}^n$. This case is very similar to the case of inner products on $\mathbb{R}^n$, see 2.1.1, so we only sketch it.

The automorphism group $G = \mathrm{Aut}(\mathbb{C}^n, f)$ is denoted by $\mathrm{U}(k, n-k)$ and called the *unitary group of type* $(k, n-k)$. Note that G has a one-dimensional center that consists of multiples of the identity by complex numbers with absolute value one. The group G modulo the center is isomorphic to the kernel of the map

$$\det : G = \mathrm{U}(k, n-k) \to \mathbb{C},$$

which is denoted by $\mathrm{SU}(k, n-k)$ and called the *special unitary group of type* $(k, n-k)$. The group $\mathrm{SU}(k, n-k)$ is simple which implies that G is reductive. Furthermore, $K = \mathrm{U}(k) \times \mathrm{U}(n-k)$ is a maximal compact subgroup of G where $\mathrm{U}(k) = \mathrm{U}(k, 0)$ and $\mathrm{U}(n-k) = \mathrm{U}(n-k, 0)$. It is easy to see that the action of $\mathrm{U}(k) \times \mathrm{U}(n-k)$ on $N_i(\mathbb{C}^n, f)$ for $i \leq \min\{k, n-k\}$ is transitive. The group K is connected and its center is two-dimensional.

The proof Theorem 1.2 in this case is very similar to the arguments in 2.1.1. The symmetric space $N_k(\mathbb{C}^{2k}, f)$ coincides with

$$\mathrm{U}(k).$$

Note that $N_k(\mathbb{C}^{2k}, f)$ has a one-dimensional Euclidean factor.

2.1.3. *Inner products on* $\mathbb{H}^n$. We will be very brief since this case is very similar to the previous two cases; see 2.1.1 and 2.1.2.

The automorphism group $G = \mathrm{Aut}(\mathbb{H}^n, f)$ is denoted by $\mathrm{Sp}(k, n-k)$ and is called the *quaternionic unitary group of type* $(k, n-k)$. The group $\mathrm{Sp}(k, n-k)$ is simple and connected and has $\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)$ as a maximal compact subgroup acting transitively on $N_i(\mathbb{H}^n, f)$ for $i \leq \min\{k, n-k\}$ where $\mathrm{Sp}(k) = \mathrm{Sp}(k, 0)$ and $\mathrm{Sp}(n-k) = \mathrm{Sp}(n-k, 0)$.

Again, we can argue as in 2.1.1 to prove Theorem 1.2 in this case. Here there is a slight simplification since K has precisely two simple factors. The symmetric space $N_k(\mathbb{H}^{2k}, f)$ coincides with

$$\mathrm{Sp}(k),$$

which is irreducible.

2.2. **The symplectic forms.** Let f be a symplectic form on $\mathbb{F}^n$. It follows that n must be even, i.e., $n = 2k$, and $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, and there is a basis of $\mathbb{F}^n$ such that f can be written as

$$f(x, y) = \sum_{i=1}^{k} (x_i y_{k+i} - x_{k+i} y_i).$$

The Witt index of f is equal to k, which is half the dimension n.

The maximal totally isotropic subspaces in $(\mathbb{F}^n, f)$ are usually called *Lagrangian subspaces* and $N_k(\mathbb{F}^n, f)$ is called a *Lagrangian Grassmannian (over* $\mathbb{F}$*)*.

[3]Putting an S in front of the symbol for a group will always mean passing to the subgroup of elements with determinant equal to one.

2.2.1. *Symplectic forms on* $\mathbb{R}^{2k}$. The automorphism group $G = \mathrm{Aut}(\mathbb{R}^{2k}, f)$ is denoted by $\mathrm{Sp}(2k, \mathbb{R})$ and called the *symplectic group over* $\mathbb{R}$.[4] It is well known that the group $\mathrm{Sp}(2k, \mathbb{R})$ is simple and connected. The form f is the imaginary part of the standard inner product on $\mathbb{C}^k$ that we assume to be identified with $\mathbb{R}^{2k}$. It now follows easily that $\mathrm{U}(k)$ is a maximal compact group in $\mathrm{Sp}(2k, \mathbb{R})$. To see that $\mathrm{U}(k)$ acts transitively on $N_i(\mathbb{R}^{2k}, f)$ for $i \le k$, let P be a plane in $N_i(\mathbb{R}^{2k}, f)$ with an orthonormal basis $\hat{e}_1, \dots \hat{e}_i$. Then $\hat{e}_1, \dots \hat{e}_i$ is clearly also orthonormal with respect to the unitary scalar product on $\mathbb{C}^k$ and it follows at once that $\mathrm{U}(k)$ acts transitively on $N_i(\mathbb{R}^{2k}, f)$.

We would now like to write $N_k(\mathbb{R}^{2k}, f)$ as a coset space. We let P be the subspace of $\mathbb{R}^{2k}$ spanned by the first k standard basis vectors. It is clear that P is a Lagrangian subspace and that an element of $\mathrm{U}(k)$ leaves it invariant if and only if it is contained in $\mathrm{O}(k)$. The involution σ on $\mathrm{U}(k)$ that sends A to the conjugated matrix $\bar{A}$ has $\mathrm{O}(k)$ as a fixed point group. It follows that $(\mathrm{U}(k), \mathrm{O}(k))$ is a symmetric pair and the symmetric space $N_k(\mathbb{R}^{2k}, f)$ coincides with

$$\mathrm{U}(k)/\mathrm{O}(k).$$

Note that $N_k(\mathbb{R}^{2k}, f)$ has a one-dimensional Euclidean factor.

It is left to prove that $N_i(\mathbb{R}^{2k}, f)$ is not a symmetric space for $i < k$. Assume it is a symmetric space and let P be the plane in $N_i(\mathbb{R}^{2k}, f)$ spanned by $e_1, \dots, e_i$. Then the isotropy group at P of the action of $\mathrm{U}(k)$ on $N_i(\mathbb{R}^{2k}, f)$ is $\mathrm{O}(i) \times \mathrm{U}(k-i)$. Let σ be an involution of $\mathrm{U}(k)$ corresponding to the symmetric pair $(\mathrm{U}(k), \mathrm{O}(i) \times \mathrm{U}(k-i))$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$- and -1-eigenspaces of $d\sigma_e$, respectively. A straightforward calculation now shows that $[\mathfrak{p}, \mathfrak{p}]$ is not a subspace of $\mathfrak{k}$ as it should be since $d\sigma_e$ is a Lie algebra automorphism. This contradiction shows that $N_i(\mathbb{R}^{2k}, f)$ is not a symmetric space for $i < k$ and finishes the proof of Theorem 1.2 in this case.

2.2.2. *Symplectic forms on* $\mathbb{C}^{2k}$. The automorphism group $G = \mathrm{Aut}(\mathbb{C}^{2k}, f)$ is denoted by $\mathrm{Sp}(2k, \mathbb{C})$ and called the *symplectic group over* $\mathbb{C}$.[5] It is well known that the group $\mathrm{Sp}(2n, \mathbb{C})$ is simple and connected. Similar to the real case in 2.2.1, we identify $\mathbb{C}^{2k}$ with $\mathbb{H}^k$ and use the fact that f is the j-part of the standard quaternionic scalar product on $\mathbb{H}^k$; see [3], Ch. I, §VIII. It follows that $\mathrm{Sp}(k)$ is a maximal compact subgroup of $\mathrm{Sp}(2k, \mathbb{C})$. It now follows as in 2.2.1 that the action of $\mathrm{Sp}(k)$ on $N_i(\mathbb{R}^{2k}, f)$ is transitive for $i \le k$. We let P be the subspace of $\mathbb{C}^{2k}$ spanned by the first k standard basis vectors. Then we see with arguments as in 2.2.1 that $\mathrm{U}(k)$ is the isotropy subgroup at P under the action of $\mathrm{Sp}(k)$. We would now like to define an involution σ on $\mathrm{Sp}(k)$ having $\mathrm{U}(k)$ as a fixed point group. We write the quaternionic matrix $A \in \mathrm{Sp}(k)$ as $A = C + jD$, where C and D are matrices with complex entries. We set $\sigma(C + jD) = \bar{C} + j\bar{D}$ and calculate directly that σ is the inner group automorphism that we get by conjugating by $I_k j$. The fixed point group of σ consists of those elements of $\mathrm{Sp}(k)$ that leave $\mathbb{R}^k + j\mathbb{R}^k$ invariant. By the definition of $\mathrm{Sp}(k)$, the fixed point group can be described as the subgroup that leaves the inner product restricted to $\mathbb{R}^k + j\mathbb{R}^k$ invariant, i.e., it is $\mathrm{U}(k)$. It follows that $(\mathrm{Sp}(k), \mathrm{U}(k))$ is a symmetric pair and the symmetric space $N_k(\mathbb{C}^{2k}, f)$ coincides with

$$\mathrm{Sp}(k)/\mathrm{U}(k),$$

which is irreducible.

[4]This group is frequently denoted by $\mathrm{Sp}(k, \mathbb{R})$.

[5]This group is frequently denoted by $\mathrm{Sp}(k, \mathbb{C})$.

It is left to prove that $N_i(\mathbb{C}^{2k}, f)$ is not a symmetric space for $i < k$. Here we can use arguments similar to those in the previous case; see 2.2.1.

2.3. **Symmetric forms on $\mathbb{C}^n$ and skew-Hermitian forms on $\mathbb{H}^n$.** Here we are considering two cases that turn out to have rather similar normal forms.

2.3.1. *Symmetric forms on $\mathbb{C}^n$.* Let f be a symmetric form on $\mathbb{C}^n$. Then there is a basis of $\mathbb{C}^n$ such that f can be written as

$$f(x, y) = \sum_{i=1}^{n} x_i y_i.$$

The automorphism group of $(\mathbb{C}^n, f)$ is called the *complex orthogonal group* and is denoted by $\mathrm{O}(n, \mathbb{C})$. The group $\mathrm{O}(n, \mathbb{C})$ has two connected components that coincide with the preimages of 1 and -1 under the determinant. Furthermore, it has $\mathrm{O}(n)$ as a maximal compact subgroup. The Witt index r of f is equal to $\left[\frac{n}{2}\right]$.

We will now show that the action of $\mathrm{O}(n)$ is transitive on $N_i(\mathbb{C}^n, f)$ for all $i \leq \left[\frac{n}{2}\right]$. To do this we split f into real and imaginary parts. Let $u + iv$ and $w + iz$ be elements in $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. Then

$$f(u + iv, w + iz) = \sum_{j=1}^{n} (u_j w_j - v_j z_j) + i \sum_{j=1}^{n} (u_j z_j + v_j w_j).$$

Let P be an element in $N_i(\mathbb{C}^n, f)$. We choose an orthogonal basis $z_1 = u_1 + iv_1, \ldots, z_i = u_i + iv_i$ in P with respect to the standard Hermitian scalar product in $\mathbb{C}^n$ where u_j and v_j are elements in $\mathbb{R}^n$ and assume that $\|z_j\|^2 = 2$ for all i. Then the equations $\|z_j\|^2 = 2$ and $f(z_j, z_j) = 0$ imply

$$\|u_j\|^2 = \|v_j\|^2 = 1 \text{ and } \langle u_j, v_j \rangle = 0.$$

Furthermore, the equations $\langle z_h, z_j \rangle = 0$ and $f(z_h, z_j) = 0$ for $h \neq j$ imply

$$\langle u_h, u_j \rangle = \langle v_h, v_j \rangle = 0 \text{ and } \langle u_h, v_j \rangle = 0.$$

As a consequence, we see that $u_1, \ldots, u_i, v_1, \ldots, v_i$ is an orthonormal set in $\mathbb{R}^n$. Conversely, we see that every such orthonormal set $u_1^*, \ldots, u_i^*, v_1^*, \ldots, v_i^*$ of $\mathbb{R}^n$ gives rise to a subspace in $\mathbb{C}^n$ spanned by $z_1^* = u_1^* + iv_1^*, \ldots, z_i^* = u_i^* + iv_i^*$ that is contained in $N_i(\mathbb{C}^n, f)$. Note that $2i \leq n$ since i is less than or equal to the Witt index, which is equal to $\left[\frac{n}{2}\right]$. It follows that the maximal compact subgroup $\mathrm{O}(n)$ of $\mathrm{O}(n, \mathbb{C})$ acts transitively on $N_i(\mathbb{C}^n, f)$ for all $i \leq \left[\frac{n}{2}\right]$.

Now assume that $n = 2k$ and let $P \in N_k(\mathbb{C}^{2k}, f)$. Then the above considerations make clear that the isotropy group of the action of $\mathrm{O}(2k)$ at P is $\mathrm{U}(k)$. Let σ be the involution of $\mathrm{O}(2k)$ defined as the conjugation by

$$J_k = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}.$$

Then $\mathrm{U}(k)$ is the fixed point group of σ and it follows that $(\mathrm{O}(2k), \mathrm{U}(k))$ is a symmetric pair. We have therefore proved that $N_k(\mathbb{C}^{2k}, f)$ is a symmetric space that coincides with

$$\mathrm{O}(2k)/\mathrm{U}(k).$$

We see that $N_k(\mathbb{C}^{2k}, f)$ has two connected components. One frequently refers to one to the components as an *orthogonal Grassmannian.*

It is left to prove that $N_i(\mathbb{C}^n, f)$ is not a symmetric space if $i < \left[\frac{n}{2}\right]$. To do this, we use methods analogous to those for the symplectic forms in 2.2; see the end of 2.2.1.

2.3.2. *Skew-Hermitian forms on* $\mathbb{H}^n$. Let f be a skew-Hermitian form on $\mathbb{H}^n$. Then there is a basis of $\mathbb{H}^n$ such that f can be written as

$$f(x,y) = \sum_{i=1}^{n} \bar{x}_i j y_i;$$

see [2], Satz on p. 434.[6] Another normal form will turn out to be more important for us. Analogous to the splitting of f into real and imaginary parts in 2.3.1, we will write $\mathbb{H}^n = \mathbb{C}^n + j\mathbb{C}^n$. We consider $u + jv$ and $w + jz$ in $\mathbb{H}^n = \mathbb{C}^n + j\mathbb{C}^n$. As in [7], we consider the form f on $\mathbb{H}^n$ defined by setting

$$f(u + jv, w + jz) = i \sum_{k=1}^{n} (\bar{u}_k w_k - \bar{v}_k z_k) + j \sum_{k=1}^{n} (u_k z_k + v_k w_k).$$

and verify that it is both nondegenerate and skew-Hermitian over $\mathbb{H}$. This form is not equal to the normal form above, but they are of course equal up to a basis change. The first sum in the definition of f is a nondegenerate Hermitian form with Witt index n on $\mathbb{C}^{2n} = \mathbb{C}^n + \mathbb{C}^n$ and the second sum is a nondegenerate symmetric form on $\mathbb{C}^{2n} = \mathbb{C}^n + \mathbb{C}^n$.

The automorphism group of $(\mathbb{H}^n, f)$ is called the *quaternionic anti-unitary group* and is denoted by $\mathrm{U}_\alpha(n, \mathbb{H})$.[7] The group $\mathrm{U}_\alpha(n, \mathbb{H})$ is connected and simple, and has $\mathrm{U}(n)$ as a maximal compact subgroup where we have embedded $\mathrm{U}(n)$ into $\mathrm{U}_\alpha(n, \mathbb{H})$ by letting $A \in \mathrm{U}(n)$ send $u + jv$ to $Au + j\bar{A}v$. The Witt index r of f is equal to $\left[\frac{n}{2}\right]$.

We will now show that the action of $\mathrm{U}(n)$ is transitive on $N_i(\mathbb{H}^n, f)$ for all $i \le \left[\frac{n}{2}\right]$. The arguments will be very similar to those we saw in 2.3.1, so we will only sketch them. Let P be an element in $N_i(\mathbb{H}^n, f)$ for some $i \le \left[\frac{n}{2}\right]$ and let $z_1 = u_1 + jv_1, \dots, z_i = u_n + jv_i$ be a basis in P such that

$$\langle z_h, z_j \rangle = 2\delta_{hj}$$

where $\langle z_h, z_j \rangle$ denotes the standard quaternionic inner product in $\mathbb{H}^n$. The equation $\langle z_h, z_h \rangle = 2$ is equivalent to

$$(u_h, u_h) + (v_h, v_h) = 2$$

where (u, v) is the standard Hermitian scalar product in $\mathbb{C}^n$.

On the other hand, $f(z_h, z_h) = 0$ is equivalent to

$$(u_h, u_h) - (v_h, v_h) = 0 \text{ and } \phi(u_h, v_h) = 0$$

[6] There is no proof of this normal form in [2]. A very short and elementary proof that any two nondegenerate skew-Hermitian forms on $\mathbb{H}^n$ are equivalent can be found in [6].

[7] Our notation is similar to the one on p. 435 in [2], but one should note that many authors denote this group by $\mathrm{SO}^*(2n)$. The reason for this notation is that $\mathrm{U}_\alpha(n, \mathbb{H})$ is isomorphic to $\mathrm{SO}(2n, \mathbb{C}) \cap \mathrm{SU}(n, n)$. One sees this with help of the second normal form above. First one observes that an automorphism of $\mathbb{C}^{2n}$ that comes from an automorphism of $\mathbb{H}^n$ has real determinant, which must be positive since $\mathrm{GL}(n, \mathbb{H})$ is connected. Such an endomorphism of $\mathbb{C}^{2n}$ preserving the first sum in the normal form belongs to $\mathrm{SU}(n, n)$ and to $\mathrm{SO}(2n, \mathbb{C})$ if it preserves the second sum. Finally one shows, that $\mathrm{SO}(2n, \mathbb{C}) \cap \mathrm{SU}(n, n)$ belongs to the image of $\mathrm{GL}(n, \mathbb{H})$ in $\mathrm{GL}(2n, \mathbb{C})$.

One can find a definition of a *special quaternionic anti-unitary group* $\mathrm{SU}_\alpha(n, \mathbb{H})$ in the literature, but our remarks show that this group actually coincides with $\mathrm{U}_\alpha(n, \mathbb{H})$.

where ϕ is the standard symmetric form $\phi(u,v) = \sum_{k=1}^{n} u_k v_k$ on $\mathbb{C}^n$. Note that $\phi(u,v) = 0$ is equivalent to $(u, \bar{v}) = 0$. Hence we get

$$(u_h, u_h) = (v_h, v_h) = 1 \text{ and } (u_h, \bar{v}_h) = 0.$$

Furthermore, $\langle z_h, z_j \rangle = 0$ für $h \neq j$ is equivalent to

$$(u_h, u_j) + (v_h, v_j) = 0 \text{ and } (u_h, \bar{v}_j) - (v_h, \bar{u}_j) = 0$$

and $f(z_h, z_j) = 0$ is equivalent to

$$(u_h, u_j) - (v_h, v_j) = 0 \text{ and } (u_h, \bar{v}_j) + (v_h, \bar{u}_j) = 0.$$

As a consequence of these considerations, we see that $u_1, \dots, u_i, \bar{v}_1, \dots, \bar{v}_i$ is an orthonormal set in $\mathbb{C}^n$. Note that $i \leq \left[\frac{n}{2}\right]$. Conversely, every such orthonormal set in $\mathbb{C}^n$ gives rise to a subspace P' in $N_i(\mathbb{H}^n, f)$. It is now clear that the action of $\mathrm{U}(n)$ is transitive on $N_i(\mathbb{H}^n, f)$ since we are assuming by definition that $A \in \mathrm{U}(n)$ sends $u + jv$ to $Au + j\bar{A}v$.

We now assume that $n = 2k$. We would like to show that $N_k(\mathbb{H}^{2k}, f)$ is a symmetric space. To see this we determine the isotropy group of the action of $\mathrm{U}(2k)$ on $N_k(\mathbb{H}^{2k}, f)$ at the k-plane P spanned by the quaternionic orthogonal basis $z_1 = e_1 + je_{k+1}, \dots, z_k = e_k + je_{2k}$, where the norms of the elements are all equal to $\sqrt{2}$. Assume that $A \in \mathrm{U}(2k)$ leaves P invariant. Then A sends the orthogonal basis into $Az_1 = Ae_1 + j\bar{A}e_{k+1}, \dots, Az_k = Ae_k + j\bar{A}e_{2k}$, which is an another quaternionic orthogonal basis of P. This shows that the isotropy group at P is isomorphic to $\mathrm{Sp}(k)$.

We would now like to prove that $(\mathrm{U}(2k), \mathrm{Sp}(k))$ is a symmetric pair. Note that the group $\mathrm{Sp}(k))$ is embedded into $\mathrm{U}(2k)$ as the subgroup of matrices of the form

$$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}.$$

where A and B are complex $k \times k$-matrices. This subgroup is the fixed point group of the automorphism σ of $\mathrm{U}(2k)$ defined by $\sigma(C) = J_k \bar{C} J_k^{-1}$ where J_k is defined as in 2.3.1. This shows that $(\mathrm{U}(2k), \mathrm{Sp}(k))$ is a symmetric pair and we have proved that $N_k(\mathbb{H}^{2k}, f)$ is a symmetric pair that coincides with

$$\mathrm{U}(2k)/\mathrm{Sp}(k),$$

which has a one-dimensional Euclidean factor. It is called the *quaternionic orthogonal Grassmannian.*

It is left to prove that $N_i(\mathbb{H}^n, f)$ is not a symmetric space if $i < \left[\frac{n}{2}\right]$. This we do with arguments analogous to those we used in 2.2.1.

3. Final Remarks

3.1. **The classification of compact symmetric spaces.** If we go through the seven cases of (σ, ϵ)-Hermitian forms in the proof in Section 2, then we see that the different $N_n(\mathbb{F}^{2n}, f)$ are the symmetric spaces in the following table.

inner product on $\mathbb{R}^{2n}$	$\mathrm{O}(n)$
inner product on $\mathbb{C}^{2n}$	$\mathrm{U}(n)$
inner product on $\mathbb{H}^{2n}$	$\mathrm{Sp}(n)$
symplectic form on $\mathbb{R}^{2n}$	$\mathrm{U}(n)/\mathrm{O}(n)$
symplectic form on $\mathbb{C}^{2n}$	$\mathrm{Sp}(n)/\mathrm{U}(n)$
symmetric form on $\mathbb{C}^{2n}$	$\mathrm{O}(2n)/\mathrm{U}(n)$
skew-Hermitian form on $\mathbb{H}^{2n}$	$\mathrm{U}(2n)/\mathrm{Sp}(n)$

Note that $\mathrm{O}(n)$ and $\mathrm{O}(2n)/\mathrm{U}(n)$ both have two connected components, and that $\mathrm{U}(n)$, $\mathrm{U}(n)/\mathrm{O}(n)$, and $\mathrm{U}(2n)/\mathrm{Sp}(n)$ all have a one-dimensional Euclidean factor. If we consider a connected component or split off a one-dimensional Euclidean factor where needed, the table above leads to the following seven irreducible compact symmetric spaces:

$$\mathrm{SO}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{SU}(n)/\mathrm{SO}(n), \mathrm{Sp}(n)/\mathrm{U}(n), \mathrm{SO}(2n)/\mathrm{U}(n), \mathrm{SU}(2n)/\mathrm{Sp}(n).$$

We would now like to compare these spaces with É. Cartan's classification of *classical* symmetric spaces, which can be found in [4], pp. 516 and 518. Cartan divides the class of irreducible compact symmetric spaces into type I and type II. The classical spaces in type I consist of the three series of Grassmann manifolds $G_k(\mathbb{F}^n)$ of k-planes in $\mathbb{F}^n$ and the four series

$$\mathrm{SU}(n)/\mathrm{SO}(n), \mathrm{Sp}(n)/\mathrm{U}(n), \mathrm{SO}(2n)/\mathrm{U}(n), \mathrm{SU}(2n)/\mathrm{Sp}(n),$$

which are all in our list. Type II consists of the classical compact groups

$$\mathrm{SO}(n), \mathrm{SU}(n), \mathrm{Sp}(n).$$

(Cartan divides the series $\mathrm{SO}(n)$ into two series according to their root systems, but that is not of importance to us).

Summarizing, the ten series of classical compact symmetric spaces in Cartan's classification are all related to the seven series $N_n(\mathbb{F}^{2n}, f)$ except the three series of Grassmannians $G_k(\mathbb{F}^n)$.

3.2. **Classical R-spaces.** The *connected components* of the spaces $N_i(\mathbb{F}^n, f)$ for $i \leq r$ that we have been considering are all examples of the R-spaces that were introduced by J. Tits in the 1950's in a study that would later lead to the notion of a spherical building; see [8].

Let G be a noncompact connected semisimple Lie group and P a parabolic subgroup. Then the quotient G/P is called an *R-space.* Let K be a maximal compact subgroup of G. Then the action of K on G/P is transitive and we can write $G/P = K/L$. If (K, L) is a symmetric pair, we call $G/P = K/L$ a *symmetric R-space.* It is now common to refer to R-spaces as *(real) generalized flag manifolds.*

The automorphism groups $\mathrm{Aut}(\mathbb{F}^n, f)$ we have been studying so far are in general neither connected nor semisimple, but reductive. If we take the identity components of these reductive groups, one can still define R-spaces and symmetric R-spaces as above, but it would not lead to anything new since these spaces coincide for a reductive group and that same reductive group modulo center. In the two cases $\mathrm{O}(k, n-k)$ and $\mathrm{O}(n, \mathbb{C})$, the automorphism groups are not connected. If we had considered instead the identity components $\mathrm{SO}_0(k, k)$ and $\mathrm{SO}(2k, \mathbb{C})$, we would have been led to one of the two components of $N_k(\mathbb{F}^{2k}, f)$.

It turns out that the classical symmetric R-spaces are precisely the three series of Grassmannians and the connected components of the spaces $N_n(\mathbb{F}^{2n}, f)$. We do not go into further detail since we have discussed these spaces in detail in [7].

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