ON THE WORKS OF S. E. Cohn-Vossen
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In geometry, like in many other fields of mathematics, one can distinguish two kinds of problems: "in the small" (or "local") and "in the large" ("im Grossen", or "global"). Local problems are concerned with the study of curves, surfaces, functions, and so on, in sufficiently small domains. As examples one can give the theorems of Meusnier, Euler and Rodrigues on the curvature of a surface in a point, or Cauchy's theorem on the existence of solutions of a differential equation near the initial data. By contrast, in problems "in the large", the requirement that the "domain be sufficiently small" is removed, and the focus is on the study of curves, surfaces, functions, and so on, extended as far as possible. Thus, differential geometry "in the large" studies surfaces on all their given span and especially surfaces that do not admit further natural extension, such as, for example, closed surfaces or the pseudo-sphere, which cannot be extended beyond its singular curve. In much the same way, the theory of analytic functions "in the large" studies an analytic function in the whole domain to which it can be continued analytically, and the theory of differential equations "in the large" studies the solutions of those equations in their whole domain of existence.

Here are some examples demonstrating the essential differences between "local" and "global". Darboux has shown that any sufficiently small piece of analytic surface of positive (negative) curvature is bendable, while in 1899 Liebman has shown that the sphere is not bendable. Darboux has shown that if in a domain with coordinates u, v there is given a linear element $ds^2 = Edu^2 + 2F du\, dv + Gdv^2$, where E, F, G are analytic functions of u, v, then for each point (u_0, v_0) one can find an analytic surface $\bar{x} = \bar{x}(u, v)$, whose linear element is equal to the given ds^2 in some neighborhood (u_0, v_0), while in 1900 Hilbert proved that there exists no infinitely extendable regular surface of constant negative curvature.

In the first stage of its development, differential geometry was concerned almost exclusively with "local" problems, solving them by means of tools of classical analysis. Once these type of problems became sufficiently well studied, geometers became interested also in global problems. Thus at the dawn of our (20th) century global geometry emerged, and it now represents a rather wide independent direction of research.

In our opinion, the irresistible charm that problems in this domain have for the true geometer is due to three elements. First, the formulation of problems and the results of global differential provide a rich source of geometric intuition; they, as a rule, have a transparent qualitative content, in the absence of which geometry ceases altogether to be a genuine geometry.

Second, the material that this branch of geometry treats is qualitatively very diverse. Thus, in the theory of closed surfaces one immediately runs into their topological classification, and so we are faced from the very beginning with infinitely many distinct classes, not to mention other, metric differences concerning the structure of closed surfaces.

And third, the methods to which global differential geometry makes appeal are extremely diverse, ranging from elementary geometric considerations at one end to topology, the theory of partial differential equations, the theory of integral equations, and so on, at the other end.

A geometer working on problems of global differential geometry needs, without leaving, on the one hand, the real ground of geometric intuition and transparency, to penetrate the very core of modern general theories and methods of mathematics research. As an example we can mention Lyusternik and Shnirelʹman's solution of the problem of closed geodesics on closed surfaces.

Stefan (Stephan) Emanuilovich Cohn-Vossen (1902–1936) was a geometer in the precise meaning of this word; his mind blended the power of geometric intuition, simplicity and clarity of thought with in-depth mastering of various mathematical theories and methods. All his works, with a few exceptions, belong to the realm of global differential geometry, and they clearly display the characteristics about which we were just talking above. To become convinced of this, it suffices to read his splendid paper *Bending of surfaces in the large*, which appeared in the first issue of the journal Uspekhi Matematiceskikh Nauk.

Cohn-Vossen spent the last part of his life in Soviet Union, working as a professor at Leningrad State University and as a researcher at the Steklov Institute. His scientific and academic teaching activity have exerted a significant influence on the development of geometry in our country (see the obituary of Cohn-Vossen in the first issue of Uspekhi Matematiceskikh Nauk).

In Cohn-Vossen's work one can distinguish two main directions: the first years of his life in research (1926–1929) were devoted to problems concerned with bending of surfaces in the large; then, after an interruption in his work, he turned his attention to questions pertaining to the intrinsic geometry of surfaces, specifically, to the investigation of the total curvature and of geodesics on open surfaces.

The problem of bending of surfaces in the large was studied by Liebman, Hilbert, Blaschke, Weyl. The most important result of their works was the proof of rigidity of ovaloids.[1] Cohn-Vossen, in his 1927 paper *Two propositions on the rigidity of ovaloids* [2] [2] proved, first, that ovaloids in fact admit no isometric mappings except from motions (two isometric ovaloids are congruent) and, second, that every ovaloid becomes nonrigid if one removes any piece of it.

Next, in his 1929 paper *Nonrigid closed surfaces* [4], Cohn-Vossen established for the first time the existence of nonrigid closed surfaces,[3] and thus the existence of closed surfaces that admit nontrivial isometric mappings.

[1] An *ovaloid* is a closed convex surface whose curvature is positive everywhere. A surface is said to be *rigid* if it admits no infinitesimal bendings except from motions.

[2] The notation [·] refers to the list of Cohn-Vossen's works at the end of this paper.

[3] Except, of course, for surfaces containing flat pieces, which are always nonrigid.

In this way, in the realm of problems concerning bending of surfaces in the large Cohn-Vossen did contribute two major results.

In a somewhat different direction lies Cohn-Vossen's large 1928 work *Parabolic curve* [3], where he deals with the application of contact transformations to the Cauchy problem for second-order partial differential equations and the related problem of constructing a surface with a prescribed metric through a given curve. He solved these problems in singular cases in which they were not solvable from the earlier point of view. Thus, this work of Cohn-Vossen also treats bending problems, but this time locally. Nevertheless, its publication was a logical outcome if one views it as a step in the development of geometry in the large.

In the history of many mathematical theories one observes that mathematicians first study locally regular cases of the problems under consideration; only after that, in connection with global formulations of those problems, one starts looking at singularities, precisely because they very often play a determining role in such problems. For instance, the global—i.e., in the whole complex plane—investigation of an analytic function is before anything else connected with the investigation of its singular points. The same scenario is observed in geometry.

In what follows we survey in sufficient detail the aforementioned works of Cohn-Vossen.

In 1933 Cohn-Vossen publishes his first note [5] on the intrinsic geometry of surfaces. This was followed by his two large papers on the same topic, [6] and [10], and another two notes in Doklady Akad. Nauk SSSR, [7] and [8].[4] These works study the connection between the topological properties of a surface, its total curvature, and the behavior of the geodesic curves on the surface. The connection between the total curvature $C(\Phi)$ of a surface Φ and its Euler characteristic $\chi(\Phi)$ is expressed, in the case of closed surfaces, by the well-known Gauss-Bonnet[5] formula

$$C(\Phi) = 2\pi\chi(\Phi).$$

The connection between the topology of a surface and a metric that the surface may admit was studied in the case of metrics of constant curvature already by Klein and Killing, and then in a more general case by Hopf and Rinow. Hence, in this part of his investigations [6], Cohn-Vossen followed an already traced path. However, his investigations of geodesics [10] created an original direction and opened new horizons in global intrinsic geometry. Here, remarkable are not just the novelty of the formulation of the problem and the results

[4][5] is a preliminary announcement of the results proved in [6]; [9] reproduces the results announced in [7].

[5]See, e.g., W. Blaschke's *Differential′naya geometriya*, § 77 ONTI, 1935. The total curvature is defined as the integral of the Gaussian curvature over the entire surface. Now suppose that on the surface Φ there is given a triangulation with f triangles, k edges and e vertices; the number $f - k + e := \chi(\Phi)$ does not depend on the choice of the triangulation and is called the Euler characteristic of the surface Φ. In the general case, when the surface is open and hence does not admit a finite triangulation, the Euler characteristic must be defined in terms of the Betti numbers of the surface: $\chi(\Phi) = p^0 - p^1 + p^2$; see, e.g., H. Seifert and W. Threlfall *Lehrbuch der Topologie*, B. G. Teubner, Leipzig und Berlin, 1934.

obtained, but also—and in no lesser measure—the simplicity of the methods employed, which are geometric despite the remoteness of the formulation of the problem, and which dispense with any calculations that are so prevalent in the field of local geometry nowadays and that threaten to drown geometric thinking into tensor computations.

Cohn-Vossen considered finitely-connected open surfaces; each such surface Φ is homeomorphic to some closed surface Φ' with a finite collection of n points removed. Thus, the plane is homeomorphic with the sphere with one point removed, and the cylinder with the sphere with two points removed. The Euler characteristic $\chi(\Phi)$ of Φ can be calculated by the formula

$$\chi(\Phi) = \chi(\Phi') - n.$$

Now suppose that Φ is covered by domains homeomorphic with a square, and each such domain is parametrized by a pair u, v. Suppose further that in the intersection of any two such domains the passage from one set of parameters to the other is effected by a transformation that is sufficiently many times differentiable (three is enough) and whose Jacobian does not vanish. Finally, suppose that in each point there is given a positive quadratic form

$$ds^2 = E(u,v)du^2 + 2F(u,v)du\,dv + G(u,v)dv^2,$$

which is invariant under transformations of parameters, i.e., when one goes from parameters u_1, v_1 to u_2, v_2, the coefficients $E_1(u_1, v_1), \ldots$ transform into $E_2(u_2, v_2), \ldots$ in such a way that the quantity ds^2 does not change. In this case we say that on the surface Φ there is given a metric ds, or that Φ becomes a differential-geometric surface. To each smooth curve on Φ one assigns a length s, which does not depend on the choice of the parameters, given by

$$s = \int \sqrt{ds^2} = \int \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt.$$

The distance between two points is defined as the infimum of the lengths of the curves that connect the points. In this way the differential-geometric surface Φ becomes a metric space. Φ is said to be complete if any bounded sequence of points on Φ has an accumulation point. For example, a plane is complete, whereas a cone with the vertex removed is not a complete differential-algebraic surface.

If the coefficients E, F, G are twice differentiable, then by the well-known Gauss formula [6] we can express in terms of them the Gauss curvature K in each point of the surface Φ. If the surface is open, then its total curvature $C(\Phi)$ can

[6] See, e.g., W. Blaschke's *Differential'naya geometriya*, § 45. The Gaussian curvature can be also defined in alternative, geometric ways, for instance, as the limit of the ratio of the sum of the angles of a triangle minus π to the area of the triangle when the triangle shrinks to a point; or, upon denoting by L the length of the circle of radius r centered at the point O and lying on the surface Φ, the curvature K_0 in the point O is given by $K_0 = \lim_{r\to 0} \frac{3}{\pi} \frac{2\pi r - L}{r^3}$, cf. Blaschke [op.cit.], § 71.

be defined as an improper integral, i.e., as the limit of the integrals

$$\int K\,dS \quad \left(dS = \sqrt{EG - F^2}\,du\,dv\right),$$

taken over bounded domains in the surface. Generally speaking, for an open surface $C(\Phi)$ may not even exist.

Cohn-Vossen proved the following fundamental theorem [6]:

If the complete finitely-connected surface Φ has total curvature $C(\Phi)$, then

$$C(\Phi) \leq 2\pi\chi(\Phi).$$

From this he deduced the following statement:

Any complete surface with everywhere-positive Gaussian curvature is homeomorphic to the sphere, or to the projective plane, or to the Euclidean plane.

P r o o f. Any surface Φ that is not homeomorphic to one of the surfaces listed above has the plane E as an infinitely-sheeted covering space.[7] If Φ is a complete differential-geometric surface, then the metric ds lifts naturally from Φ to the covering E, and one can verify that in this way E also becomes a complete differential-geometric surface. If the curvature on Φ is positive, then $C(\Phi) > 0$, and then on E the curvature is also positive, and hence the total curvature $C(\Phi)$ exists. But since $C(E) > 0$ and E covers Φ infinitely many times, it follows that $C(E) = +\infty$, which contradicts (1).

From the proved theorem one readily obtains the following theorem:

If Φ is a complete differential-geometric surface with everywhere-positive Gaussian curvature and is homeomorphic to the plane, then on Φ there are no closed (without multiple points) geodesic curves.

(If the metric ds is given, then the geodesic curves are defined as extremals of the variational problem $\delta \int ds = 0$; on sufficiently short segments, geodesic curves are shortest curves.)

P r o o f. Since on Φ the Gaussian curvature is positive, $C(\Phi)$ exists and, by (1), $C(\Phi) \leq 2\pi$. Hence, for any finite domain the total curvature is strictly less than 2π. But from the well-known Gauss-Bonnet formula it follows that the total curvature of a domain bounded by a closed geodesic is equal to 2π. (In the plane such a domain is homeomorphic to a disc, and so the Gauss-Bonnet formula is applicable.) This completes the proof.

We presented here these simple arguments of Cohn-Vossen [6] because they already contain in embrio the idea of further, more profound results that he obtained concerning geodesics on complete surfaces homeomorphic to the plane [10]. Following Cohn-Vossen, this kind of surface will be here referred to as a Riemannian plane. A geodesic curve is said to be complete if it can be extended indefinitely in both directions. Finally, a complete geodesic curve, each segment of which is the shortest curve between its endpoints, will be referred to as a line. To provide a representation about the results obtained by Cohn-Vossen, we list here a few of them.

[7] See, e.g., Seifert and Threlfall, *Lehrbuch der Topologie*, § 53.

1) If in the Riemannian plane Φ there exist a line, then Φ cannot have everywhere-positive total curvature.

2) Two complete geodesics on a Riemannian plane with everywhere-positive curvature necessarily intersect.

3) Let Φ be a Riemannian plane with everywhere-positive curvature. Let L be a complete geodesic on Φ with a multiple point. Then L contains one and only one unigon E.[8] Moreover, $L - E$ lies entirely inside E and consists of two geodesic rays with no multiple points.

4) Let Φ be a Riemannian plane with everywhere-positive curvature and with total curvature larger than π. Then through every point of Φ there passes a complete geodesic with no multiple points. Moreover, for any bounded set M in Φ there exists a bounded set N containing M with the property that every point of the domain $\Phi - N$ is the vertex of a geodesic unigon that contains M inside it. At the same time, on every complete geodesic with no multiple points there is a point that is not the vertex of any geodesic unigon. Therefore, the set of such points is nonempty and bounded.

We will not stop to discuss in detail the last works of Cohn-Vossen ([6]–[12]), since we plan to publish in one of the future issues of Uspekhi Matematicheskikh Nauk a translation of the main parts of the papers [6] and [10]; the papers [7] and [8] were published in the Doklady Akad. Nauk SSSR in Russian and therefore are easily accessible.

In what follows we review the first works of Cohn-Vossen ([1]–[4]). In addition to a presentation of the results of these works and, whenever possible, of the proofs of these results, we will provide brief references to works of other authors in which Cohn-Vossen's ideas are considered or used.

1. Singularities of convex surfaces [1]

Among the surfaces of positive curvature, the most studies from the point of view of global geometry are the closed convex surfaces—the ovaloids. Already in 1855 Bonnet found an estimate for the diameter of an ovaloid in terms of the bounds of its Gaussian curvature. Later, ovaloids were studied in various directions by Blaschke in his book *Circle and sphere*[9] The problem that Cohn-Vossen decided to study was that of the global structure (Gesamterstrekung) of an arbitrary surface of positive curvature.

In what follows we consider a twice-differentiable[10] surface whose curvature is positive everywhere.

Take an arbitrary point P on such a surface and consider the tangent plane T_0 at P. Let T_x denote the plane parallel to T_0 drawn at distance x from T_0 in the direction of the inward normal at the point P. From the positiveness of

[8] A *unigon* is a closed curve with no multiple points that is smooth everywhere, except for one point, in which there exists only one-sided tangents that meet at some angle.

[9] W. Blaschke, *Kreis und Kugel*, Leipzig, 1916.

[10] This means that each point of the surface has a neighborhood U homeomorphic to a disk, and in U on can choose parameters u, v such that the vector $\bar{x}(u, v)$ whose tip sweeps out U is a twice-differentiable function of u, v.

the curvature it follows that, for x sufficiently small, the plane T_x intersects the surface along a closed convex curve L_x, which shrinks to P when x goes to zero. (All intersections of the surface with a plane may a priori contain also other curves and points).

For sufficiently small x the plane T_x cuts out from the surface, in a neighborhood of the point P, a small "cap", which together with the convex domains bounded by the curve L_x in the plane T_x bounds a convex body.

By *regular layer* (of the surface with respect to the point P) we mean the collection of all the planes T_x with the property that all T_ξ with $0 < \xi \leq x$ intersect the surface along closed convex curves L_ξ which vary continuously with ξ and which shrink to the point P when $\xi \to +0$.

Two cases are possible: 1) the regular layer extends to infinity, i.e., all planes T_x with $0 < x < \infty$ belong to the regular layer; 2) the regular layer has a boundary plane T_{x_0}, i.e. all planes T_x with $0 < x < x_0$ belong to the regular layer, but not T_{x_0}. In the first case, like in the case of an elliptic paraboloid with vertex P, the surface extends to infinity and has no singularity. In the second case, the following theorem holds:

If the regular layer of some point of the surface has a boundary plane T_{x_0} (in addition to T_0, of course), then either the surface is an ovaloid and the plane T_{x_0} is tangent to it, or T_{x_0} contains singular points of the surface.

A point of a surface is called a *singular point* if it does not lie on the surface, but one can approach it as close as one wants along a curve that lies on the surface and has bounded length. If, for instance, we have a surface with boundary, then the boundary is excluded from the surface, since its points have no disk neighborhoods on the surface; each point of the boundary is singular, since one can approach it as close as one wants along a curve of bounded length lying on the surface.

The theorem formulated above represents the main result of Cohn-Vossen's work that we are discussing here. Its proof is carried out by elementary geometric arguments. In the theory of surfaces of positive curvature this theorem is, needless to say, one of the fundamental results, since it determines the form of such surfaces.

Furthermore, Cohn-Vossen proved, again by elementary arguments, the following theorem:

If the Gaussian curvature of a surface is everywhere not smaller than some positive constant, then the regular layer relative to any of its points has finite width.

In fact, the width of a regular layer can be estimated from above. Thus, suppose that everywhere on the surface the Gaussian curvature $K \geq ac > 0$. Pick a point P on the surface and take around it a "small cap" bounded by a plane T_a that is parallel to the tangent plane to the surface at P and lies at distance a from P. Let b denote the diameter of the surface's section by the plane T_a. Then the width x_P of the regular layer at P obeys the estimate

$$x_P < a + \frac{4\pi}{cb}.$$

This result is interesting because here, with no analyticity or finite smoothness assumptions, local characteristics of the surface allow one to estimate its behavior in the large.

2. Two propositions on the rigidity of ovaloids [2]

1. As we already mentioned above, in this paper [2] Cohn-Vossen proved the following two theorems:

a) *Any two isometric ovaloids are congruent.*

b) *If from an ovaloid one cuts out any piece, then the resulting surface is nonrigid.*

Cohn-Vossen's proof of the first theorem is given in his paper "*Bending of surfaces in the large*", published in the first issue of "Uspekhi Matematicheskikh Nauk". I have nothing to add to the simple and transparent treatment therein, so I refer the reader to the paper. One needs to emphasize the deep and clear formulation of problems of the theory of bending of surfaces given by Cohn-Vossen in sections 7 and 8 of his "Uspekhi" paper.[11]

Cohn-Vossen's method was developed and applied to other problems by a series of authors. Thus, H. Hopf and H. Samelson[12] gave a detailed and simplified re-proof of Cohn-Vossen's theorem, O. K. Zhitomirskii[13], provided a new, extremely simple analytic method for calculating the indices of singular points of direction fields, and, finally, H. Lewy[14], using Cohn-Vossen's method, proved Minkowski's theorem that an ovaloid is (uniquely) determined by its Gaussian curvature, given as a function of the normal to the surface.

To demonstrate the elegance and power of Cohn-Vossen's method, I will give here the proof of a rather general uniqueness theorem for ovaloids.[15]

2. So, we will prove the following theorem.

Let $f(x, y; \bar{n})$ be an analytic function of two variables x, y, ranging in the domain $x \geq y > 0$, and a unit vector $\bar{n}$, such that

$$\frac{\partial f}{\partial x} > 0, \quad \frac{\partial f}{\partial y} > 0 \tag{1}$$

everywhere.

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[11] Some of the problems posed by Cohn-Vossen have been solved by S. P. Olovyanishnikov (Mat. Sbornik **18** (60), pp. 429–446) and A. D. Aleksandrov (Doklady Akad. Nauk **36** (1942), pp. 211–216).

[12] H. Hopf and H. Samelson, *Zum Beweis des Kongruezsatzes für Eiflächen*, Math. Zeitschr., vol. **43** (1938), pp. 749–766.

[13] O. K. Zhitomirskii, *On the rigidity of ovaloids*, Dokl. Akad. Nauk SSSR, vol. XXV (1939), pp. 347–349.

[14] H. Lewy, *On differential geometry in the large, I (Minkowski's problem)*, Trans. Amer. Math. Soc., vol. **43** (1938), pp. 258–270)

[15] A. D. Aleksandrov, *A general uniqueness theorem for closed surfaces*, Dokl. Akad. Nauk SSSR, vol. XIX (1938), no. 4. In this Doklady note the theorem is proved in a different way, without resorting to indices, which is simpler; but, as we will see here, the index method has a higher degree of generality.

Let H and H' be two piece-wise analytic ovaloids and let R_1, R_2 and R_1', R_2' be their principal radii of curvature, considered as functions of the normal $\bar{n}$; assume that $R_1 \geq R_2$ and $R_1' \geq R_2'$. Then, if

$$f(R_1, R_2; \bar{n}) = f(R_1', R_2'; \bar{n}) \tag{2}$$

for all directions of the normal $\bar{n}$ (R_1, R_2, R_1', R_2') are taken in points with one and the same normal, and hence $f(R_1, R_2; \bar{n})$ and $f(R_1', R_2'; \bar{n})$ are functions of the normal $\bar{n}$), then the ovaloids H and H' are identical and lie parallel to one another.

This theorem includes as a particular case Minkowski's theorem asserting that an ovaloid is determined by the Gaussian curvature K, as a function of the normal. In this case

$$f(R_1, R_2; \bar{n}) = R_1 R_2 = \frac{1}{K},$$

and since $R_1 > 0$, $R_2 > 0$, it follows that

$$\frac{\partial f}{\partial R_1} = R_2 > 0, \quad \frac{\partial f}{\partial R_2} = R_1 > 0.$$

Further, our theorem also includes a theorem of Christoffel[16] which asserts that an ovaloid is (uniquely) determined by the sum of its principal radii of curvature, given as a function of the outer normal. In this case, $f(R_1, R_2; \bar{n}) = R_1 + R_2$.

Equally well, one can take

$$f(R_1, R_2; \bar{n}) = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

In the special case when when the mean curvature is constant, we obtain as a corollary a theorem of Liebman which asserts that the sphere is the only surface of constant mean curvature.

Now let us turn to the proof of our theorem. Let H and H' be two piecewise analytic ovaloids and $H(\bar{u})$ and $H'(\bar{u})$ be their support functions.[17] Suppose

[16]See Blaschke, *Differential′naya geometriya*, §§ 92 and 95.

[17]If $\bar{u}$ is the outer normal vector to the ovaloid H, then the right-hand side of the equation $\bar{u}\bar{x} = H(\bar{u})$ of the tangent plane to H, regarded as a function of $\bar{u}$, is called the support function. Since replacing $\bar{u}$ by $\lambda\bar{u}$ with any $\lambda > 0$ gives the same tangent plane, $H(\lambda\bar{u}) = \lambda H(\bar{u})$ for any $\lambda > 0$. The ovaloid itself is the envelope of the family of planes $\bar{u}\bar{x} = H(\bar{u})$; therefore, the coordinates of its points are expressed in terms of $H(\bar{u}) = H(u_1, u_2, u_3)$ as

$$x_i = \frac{\partial H(u_1, u_2, u_3)}{\partial u_i}, \quad i = 1, 2, 3.$$

As is known, the principal directions on the surface are characterized by the fact that along them the displacement $d\bar{x}$ is connected with the differential of the normal by the Rodrigues relation $d\bar{x} = Rd\bar{n}$. If we replace the unit normal by an arbitrary vector $\bar{u}$ and use the expressions of x_i in terms of $H(u_1, u_2, u_3)$, we obtain the system $H_{u_i u_1} du_1 + H_{u_i u_2} du_2 + H_{u_i u_3} du_3 = R du_i$ (i=1,2,3). Since $H(u_1, u_2, u_3)$ is homogeneous, one of the solutions of this system corresponds to $R = 0$. The other two solutions, R_1 and R_2, give the principal radii of curvature and the corresponding principal directions of the surface.

that for these ovaloids

$$f(R_1, R_2; \bar{n}) = f(R_1', R_2'; \bar{n}) = g(\bar{n}), \tag{2}$$

where $f(R_1, R_2; \bar{n})$ is the function our theorem is concerned with.

Consider $d^2H(\bar{u})$, which is a quadratic form in the variables du_1, du_2, du_3. As is known, its eigenvalues are R_1, R_2 and 0, and to this last eigenvalue there corresponds the principal direction along $\bar{n}$, i.e., along the normal to the ovaloid H. The principal directions corresponding to R_1 and R_2 are precisely the principal directions on the surface of the ovaloid.

3. Denote

$$H(\bar{u}) - H'(\bar{u}) = Z(\bar{u}). \tag{3}$$

Let us prove that $d^2Z(\bar{u})$ is, in each point, either a sign-changing or an identically null quadratic form.

From (1) and the equality

$$f(R_1, R_2; \bar{n}) = f(R_1', R_2'; \bar{n})$$

it follows that the differences $R_1 - R_1'$ and $R_2 - R_2'$ either have different signs, or are simultaneously equal to zero. Further, the definition of $Z(\bar{u})$ gives

$$d^2H(\bar{u}) = d^2H'(\bar{u}) + d^2Z(\bar{u}).$$

Now if the form $d^2Z(\bar{u})$ were, say, positive, then the eigenvalues of the form $d^2H(\bar{u})$ would be larger than those of the form $d^2H'(\bar{u})$, and consequently the differences $R_1 - R_1'$ and $R_2 - R_2'$ would have the same sign, which is impossible. Similarly, the form $d^2Z(\bar{u})$ cannot be negative.

4. Since R_1 and R_2 are the eigenvalues of the second differential of the support function, $f(R_1, R_2; \bar{n})$ is a second-order differential expression in $H(\bar{n})$ on the unit sphere (i.e., in the domain where unit vector $\bar{n}$ ranges).[18] Let us show that for any choice of the parameters u, v in a neighborhood of an arbitrary point of the unit sphere, the differential equation

$$f(R_1, R_2; \bar{n}) = g(\bar{n}), \tag{4}$$

i.e.,

$$F(H_{uu}, H_{uv}, H_{vv}, H_u, H_v, H, u, v) = g(u, v), \tag{5}$$

can be solved for H_{uu}.

To do this, we replace the components u_1, u_2, u_3 of the vector $\bar{u}$ by new variables: $r = |\bar{u}|$ and the parameters u, v on the unit sphere. Then, thanks to the positive homogeneity of $H(\bar{u})$,

$$H(\bar{u}) = rH(u, v). \tag{6}$$

[18] R_1 and R_2 are the roots of the characteristic equation of the matrix of second derivatives $\|H_{u_i u_k}\|$ and hence are expressible through those derivatives.

Hence, for $r = 1$ we get

$$d^2H(\bar{u}) = H_{uu}du^2 + 2H_{uv}du\,dv + H_{vv}dv^2 + 2(H_u du + H_v dv)dr + \cdots . \quad (7)$$

(Here the dots denote the further terms of the form $H_u du^2$ which are necessarily present, since $d^2H(\bar{u})$ is a quadratic form in the du_i. We used the standard rule of transformation of the second differential.) When one increases H_{uu}, the eigenvalues of the quadratic form (7) do not decrease and at least one of them increases. Let us assume that, say,

$$\frac{\partial R_1}{\partial H_{uu}} > 0, \quad \frac{\partial R_2}{\partial H_{uu}} \geq 0 \quad (8)$$

Then (8) and (1) yield

$$\frac{\partial F}{\partial H_{uu}} = \frac{\partial f}{\partial R_1} \cdot \frac{\partial R_1}{\partial H_{uu}} + \frac{\partial f}{\partial R_2} \cdot \frac{\partial R_2}{\partial H_{uu}} > 0. \quad (9)$$

By the well-known implicit function theorem, it follows that if in a given point u, v the values of H_{uu}, H_{uv}, H_{vv}, H_u, H_v, H are given so that (5) is satisfied, then in a neighborhood of that point equation (5) can be solved with respect to H_{uu}:

$$H_{uu} = \Phi(H_{uv}, H_{vv}, H_u, H_v, H, u, v). \quad (10)$$

The conclusions obtained above are specific to our theorem; the ensuing arguments follow exactly Cohn-Vossen's method.

5. The function $Z(\bar{u}) = H(\bar{u}) - H'(\bar{u})$ is piecewise analytic, being the difference of the support functions of two piece-wise analytic ovaloids. On the unit sphere take some domain G where $Z(\bar{u})$ is analytic. If $d^2Z(\bar{u}) = 0$ identically in some point of G, this means that all the second derivatives of $Z(\bar{u})$ vanish in that point. Hence, the set of points where $d^2Z(\bar{u}) \equiv 0$ is the set of zeroes of several analytic functions. By a well-known theorem of Weierstrass, we conclude that this set has one of the following four forms: (1) it is the entire domain G; (2) consists of analytic curves and, possibly, isolated points; (3) consists of isolated points; (4) is empty.

Accordingly, if M denotes the set of points on the unit sphere where $d^2Z(\bar{u}) \equiv 0$, then the following three situations are possible:

1) M contains a curve (in particular, it may also contain a whole domain);
2) M consists of only isolated points;
3) M is empty.

We will prove that only the first case is possible, and that then M is in fact the entire surface of the unit sphere. This means that $d^2Z(\bar{u})$ vanishes identically, and so $Z(\bar{u})$ is a linear function:

$$Z(\bar{u}) = \bar{a}\bar{u}$$

(there is no constant term, due to the homogeneity of $Z(\bar{u})$). As is known, if the support functions of two ovaloids H and H' differ by a linear term $\bar{a}\bar{u}$ (i.e.,

$H(\bar{u}) - H'(\bar{u}) = \bar{a}\bar{u}$), then one of the ovaloids is obtained from the other by translation by the vector $\bar{a}$. Thus we will conclude that our ovaloids H and H' are identical and parallel to one another.

6. Suppose the set M contains a curve L. Then on L one has

$$d^2H(\bar{u}) = d^2H'(\bar{u}). \tag{11}$$

Now transport H' in a parallel manner so that some point A' with normal $\bar{n}_0$, pointing to the curve L (if $\bar{n}_0$ emanates from the center of the unit sphere) will coincide with the corresponding point A of the ovaloid H. Then equality (11) remains valid, because the parallel transport results only in the addition of a linear term to the support function. Moreover, in order for the points A and A' to coincide, the following relations must hold for the normal $\bar{n}_0$:

$$H(\bar{n}_0) = H'(\bar{n}_0) \quad \text{and} \quad dH(\bar{n}_0) = dH'(\bar{n}_0). \tag{12}$$

Thanks to (11), relations (12) will hold on the entire curve L. Choosing the parameters u, v so that on the curve L we have $u = 0$, we reach the the following conclusion: on the line $u = 0$, $H(u, v)$ and $H'(u, v)$ coincide, and so do their first and second derivatives. By what we established in Subsection 4 above, in a neighborhood of the line $\bar{n}_0$, H and H' satisfy the same equation (10). Hence, by Cauchy's theorem, H and H' coincide in a neighborhood of the curve L.

This (local) coincidence of the functions $H(u, v)$ and $H'(u, v)$ extends to the entire unit sphere: inside a domain of analyticity thanks to the analyticity of the functions, and across the boundary of such a domain by virtue of the argument just presented: just take for L a segment of this boundary.

Since $H(u, v) = H'(u, v)$, it follows that $H(\bar{u}) = H'(\bar{u})$, and so the ovaloids coincide. Hence, before translation they were identical and parallel to one another.

7. Now suppose that M is empty or consists of isolated points.

In each point of the unit sphere F where the quadratic form $d^2Z(\bar{u})$ does not vanish identically, two of its principal directions lying on the sphere correspond to eigenvalues of opposite signs. Take in each point the direction corresponding to the positive eigenvalue. In this way we obtain on F a continuous direction field R_0, and the points where $d^2Z(\bar{u})$ vanishes identically are precisely the singular points of this field.

However, it is known that the sphere admits no direction field without singularities, so we conclude that the set M is not empty.

Furthermore, it is known that the sum of the indices of singular points of a direction field on the sphere must be equal to 4. We are going to show that the index of any singular point in our direction field is ≤ 0, which will imply that M cannot consists of isolated points, thus completing the proof of the theorem.

8. Let D denote the envelope surface of the family of planes

$$\bar{u}\bar{x} = Z(\bar{u}).$$

Here $Z(\bar{u})$ is, so to say, the support function of the surface D.

The surface D and the unit sphere F are in a correspondence via parallel normals, and the tangent planes to D and F in the corresponding points are parallel.

Since the form $d^2Z(\bar{u})$ is not definite, and its eigenvalues are the principal radii of curvature of the surface D, it follows that D has negative curvature. Hence, the correspondence between D and F reverses the direction of motion along circuits.

The principal axes of the quadratic form $d^2Z(\bar{u})$ give the principal directions on the surface. By Rodrigues's theorem, the principal directions are parallel to their images on the sphere. Thus, on D we have a field S_0 of directions that are parallel to the directions of the field R_0 on F.

Notice that here we have a complete analogy with the problem of bending of ovaloids. The role of the middle surface F is played here by the unit sphere F, and the role of the rotation diagram D — by the surface D. For this reason we will not prove explicitly that the index of any singular point of the field R_0 is ≤ 0. In the repeatedly aforementioned paper of Cohn-Vossen "*Bending of surfaces in the large*" (starting with the third row from below on its page 58) one can read his proof; it is not even needed to change notations.

Thus, verbatim repetition of Cohn-Vossen's arguments leads us to the proof of our theorem.

9. In the second part of his paper "*Two propositions on the rigidity of ovaloids*" Cohn-Vossen proved the following statement:

If from an ovaloid one removes any piece, one obtains a nonrigid surface.

Let us explain the idea of Cohn-Vossen's proof.

Let H be an ovaloid. Pick a point O inside H and introduce spherical coordinates r, θ, ϕ centered at O. Suppose that on H

$$ds^2 = Edu^2 + 2F du\, dv + Gdv^2. \tag{1}$$

We also have

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{2}$$

Hence,

$$\frac{1}{r^2}\left(ds^2 - dr^2\right) = d\theta^2 + \sin^2\theta\, d\phi^2. \tag{3}$$

This is precisely the length element of the unit sphere, and so the corresponding Gaussian curvature is equal to 1. On the other hand,

$$\frac{1}{r^2}(ds^2 - dr^2) = \frac{1}{r^2}\left\{\left(E - r_u^2\right)du^2 + 2\left(F - r_u r_v\right)du\, dv + \left(G - r_v^2\right)dv^2\right\}. \tag{4}$$

Therefore, this linear element is expressible through $r(u, v)$, its derivatives, and E, F, G. Calculating the expression for the Gaussian curvature of this linear element and setting it equal to 1, we obtain an equation for $r(u, v)$. In this equation the third-order derivatives nicely cancel out and one arrives at the Monge-Ampère type equation

$$A\left(\rho\tau - \sigma^2\right) + B\rho + C\sigma + D\tau + Q = 0,, \tag{5}$$

where $\rho = r_{uu}$, $\sigma = r_{uv}$, $\tau = r_{vv}$, and A, B, C, D, Q are polynomials in R, r_u, r_v, E, F, G and derivatives of E, F, G.

This equation must be satisfied also by the radius $r(u, v)$ for every ovaloid that is isometric to the given one, or for any piece of such an ovaloid, but then only in the corresponding domain of variation of u, v.

Once $r(u, v)$ is found from this equation, $\theta(u, v)$ and $\phi(u, v)$ are obtained by solving ordinary differential equations.

All that we discussed to this point goes back to Darboux and is now included in textbooks on differential geometry.

Now let us remove a piece from the ovaloid H; then we are left with a surface Φ with boundary L.

In order that Φ be nonrigid, it is necessary that there exist isometric (to first order) surfaces that are infinitesimally close to Φ (this is the meaning of infinitesimal bending). Consider such a surface and let the radius going to the point (u, v) be given by $r(u, v) + \delta r(u, v)$. Then $\delta r(u, v)$ satisfies the linear equation in variations resulting from (5). This is a linear second-order partial differential equation, and is in fact of elliptic type thanks to the positivity of the curvature of the surface Φ.

Therefore, if one prescribes $\delta r(u, v)$ on the boundary of Φ in arbitrary manner, the by a known existence theorem, $\delta r(u, v)$ is determined on the entire surface Φ. Now that $\delta r(u, v)$ is found, $\delta\theta(u, v)$ and $\delta\phi(u, v)$ are obtained by solving the ordinary differential equations that result from the equations for θ and ϕ. This establishes the existence of an infinitesimally close isometric surface $(r + \delta r, \theta + \delta\theta, \phi + \delta\phi)$, and hence the nonrigidity of Φ.

The arguments above show also that the vector of an infinitesimal bending depends on an arbitrary function given on the boundary of the surface Φ.

3. Parabolic curve [3]

1. A point on a surface is called *parabolic* if the Gaussian curvature in that point vanishes. A curve such that all its points are parabolic is called a *parabolic curve*. For example, the two circles that separate a convex set of the torus from a concave one are parabolic curves. If one subjects some surface F to a polar transformation with respect to a second-order surface then, generally speaking, one obtains again a surface F'. The points of F where the Gaussian curvature K is different from zero are transformed into regular points of F'; however, the points where $K = 0$ become singular points on F'. This elementary result is of the same nature as the fact that under a polar transformations in the plane, inflexion points of curves are transformed into turning points. A similar situation occurs also in the case of arbitrary contact transformations that are not simply point transformations.

Cohn-Vossen's work we are dealing with here is devoted to a multifaceted investigation of the singularities that arise in this way. It consists of three parts.

The first studies the geometric nature and the structure of the indicated singularities, as well as their relationship with contact transformations.

The second part explores the possibility of, conversely, removing a singularity of a surface by means of an appropriate contact transformation. This idea is applied to the Cauchy problem for second-order partial differential equations; specifically, by applying contact transformations one can, in Cohn-Vossen's words, "regularize" and solve the Cauchy problem in the case when this problem is not solvable in the usual meaning of the word. Thus, as Cohn-Vossen remarks, the notion of integral and the Cauchy-Kovalevskaya existence theorem are extended, similarly to the way in which, in the theory of functions of a complex variables, one passes, via regularization of poles, from the notion of regular function to that of meromorphic function.

In the third part of the work this method is applied to investigate singular boundary lines which, as a rule, arise when one bends surfaces. (It suffices to recall the turning edge of developable surfaces and the boundary and the apical curve of the known surfaces of constant Gaussian curvature.) In exactly the same manner one proves various transparent and interesting propositions concerning the formation and shape of such lines. The results obtained by Cohn-Vossen in the work under considerations are so numerous that it is impossible to present all of them here. To avoid complicating the exposition with differentiability and other assumptions on the functions involved, Cohn-Vossen assumes everywhere that the functions are analytic. We will proceed in the same manner.

2. Suppose there is given a 3-parameter family (termed here a complex) of surfaces (S), and some other surface F that does not belong to the complex. Then, as a simple counting of parameters shows, for each point P on F there exists, generally speaking, a surface S in the family (S) that is tangent to F in P (we ignore here the exceptions connected with the nonsolvability of the equations that are involved here).

When we pass along the surface F from the point P to a neighboring point P', the corresponding surface S of the complex also goes over into a neighboring surface S', which in general is different from S. However, from P can emanate a direction d such that S is tangent to F also in the points that are close to P in the direction d, i.e., when one moves from P in the direction d the surface of the complex (S) that is tangent to F does not change (is stationary). In this case the point P will be referred to as a *parabolic point of the surface* F *with respect to the complex* (S), and the direction d as a *contact direction in the point* P. (The case when the direction d is not uniquely determined, i.e., when S has conctact with F in more than one direction, will not be considered here.) A curve consisting of parabolic points of the surface F with respect to (S) will be referred to as a *parabolic curve* of the surface F with respect to the complex (S).

The ordinary parabolic points and curves are included in this definition—in their case (S) is the complex of *all* surfaces. Indeed, ordinary parabolic points are those points where the tangent plane in at least one direction is stationary. In this case d is the principal direction corresponding to the null curvature.

If for a parabolic curve the tangent at each of its points has contact direction d, then the entire curve lies on a surface of the complex; this is an obvious consequence of the above definitions.

An example of how this can happen is provided by the following assertion: a parabolic (in the usual sense) curve lies entirely in a single tangent plane if it is an envelope of asymptotic lines (indeed, in this case the curve has in any point an asymptotic direction, which is precisely the principal direction of null curvature).

Let's take for (S) the complex of all spheres of radius a. The parabolic points of this complex are the points in which one of the principal curvatures is equal to $1/a$; d will be the corresponding principal direction. Then the following theorem holds: *if along a curvature line L the principal curvature is constant and equal to $1/a$, then the surface is tangent along L to a sphere of radius a.*

3. By surface element one means, as is known, a pair consisting of a point and a plane passing through it. An element is specified by five numbers: the coordinates of the point, x, y, z, and two quantities that specify the direction of the plane, for which we one takes the coefficients p, q of its equation $z = px + y$.

Recall also that a strip is defined to be a one-parameter family of elements $(x(t), y(t), z(t), q(t))$ such that

$$\dot{z} = p\,\dot{x} + q\dot{y}, \tag{1}$$

i.e., the planes of the element are tangent to the curve formed by its points (note, however, that this curve is allowed to degenerate, so that, for example, a pencil of planes is also a strip).

Suppose we have a complex of surfaces (S), given in parametric form:

$$f(x, y, z; \xi, \eta, \zeta) = 0, \tag{2}$$

where x, y, z are the coordinates of points in space, and ξ, η, ζ are parameters specifying a surface in (S). Formula (2) defines a mapping of surfaces in (S) into points with coordinates ξ, η, ζ. In the terminology introduced by S. Lie, this mapping defines a contact transformation, as follows: if an element in the given "x-space" is specified by numbers x, y, z, p, q and in some auxiliary "ξ-space" we regard ξ, η, ζ as Cartesian coordinates and specify an element by numbers ξ, η, ζ, π, κ, then a contact transformation is given by the formulas

$$\left.\begin{array}{ll} 1. & f(x, y, z; \xi, \eta, \zeta) = 0, \\ 2. & f_x + p f_z = 0, \\ 3. & f_y + q f_z = 0, \\ 4. & f_\xi + \pi f_\zeta = 0, \\ 5. & f_\eta + \kappa f_\zeta = 0, \end{array}\right\} \tag{3}$$

where, as usual, $f_x = \partial f(x, y, z; \xi, \eta, \zeta)/\partial x$ and so on.[19]

[19]By definition, a contact transformation is an element-to-element transformation $(x, y, z, p, q) \to (\xi, \eta, \zeta, \pi, \kappa)$ which preserves strips, and hence preserves tangency of surfaces, since every surface can be regarded as a family of strips: the curve of the strip lies on the surface, while the planes of the strip are tangent planes to the surface along this curve. Analyt-

Under this transformation the surface F in x goes over into a surface Φ in ξ-space. Moreover, a parabolic curve ℓ on F with respect to (S) goes over into a singular line λ of Φ. Here by singular one means that the second derivatives of the functions that represent Φ become infinite on λ, whereas the surface F and the contact transformation are regular in a neighborhood of ℓ. To understand how such singularities arise it suffices to consider strips, since any surface can be assembled from strips.

Suppose the point P in x-space sweeps some strip ℓ. The contact transformation associated with the complex (S) maps the strip ℓ into a strip λ, which in turn is swept by the point Π corresponding to P. Let us take the arc length s of the curve ℓ as a parameter on ℓ and λ. Let $d\sigma$ be the differential of the arc λ and $d\tau$ be the differential of the angle of rotation of the plane of the strip λ. The derivatives $d\sigma/ds$ and $d\tau/ds$ are bounded and do not vanish simultaneously. This follows from the assumed regularity of the contact transformation which, as usual, requires that, first, the derivatives of the form $\partial\xi/\partial x$, ... $\partial\pi/\partial x$, ... be bounded (which implies the boundedness of $d\sigma/ds$ and $d\tau/ds$) and, second, that when dx, dy, dz, dp, dq are not simultaneously zero, the same holds true for $d\xi$, $d\eta$, $d\zeta$, $d\pi$, $d\kappa$ (which immediately implies that $d\sigma/ds$ and $d\tau/ds$ do not vanish simultaneously).

As long as $d\sigma/ds$ and $d\tau/ds$ are bounded and not simultaneously zero, the curvature of the strip λ can become infinite only when $d\sigma/ds = 0$, i.e., only when the point Π does not move, though P does. But by the very definition of the ξ-space, a point in this space is a surface of the complex (S) which passes through the element P. Therefore, the fact that P does not move means that the surface of (S) that is tangent to the strip is stationary. We thus proved the following statement: *the image of the strip ℓ under the contact transformation defined by the complex (S) has a singularity when ℓ has contact with a surface from (S).*

In the particular case when the whole strip ℓ lies on one of the surfaces of the complex, this strip is mapped into a pencil of elements that pass through a single point. For example, under a polar transformation parabolic circles on the torus are mapped into conical points.

In the case of higher-order contact, is is useful to note the following theorem:

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ically, the condition that the strips be preserved is expressed by the fact that from $\dot{z} = p\dot{x} + q\dot{y}$ must follow that $\dot{\zeta} = \pi\dot{\xi} + \kappa\dot{\eta}$. Formulas (3) do indeed give a contact transformation because, first, they connect one 5-tuple of variables with another 5-tuple via 5 equations and, second, in view of these formulas the relation $\dot{z} - p\dot{x} - q\dot{y} = 0$ implies $\dot{\zeta} - \pi\dot{\xi} - \kappa\dot{\eta} = 0$. Let us verify this last assertion. From formulas (3) and (1) it follows that if x, y, z, ξ, η, ζ are given as functions of t, then

$$\dot{f} = f_x\dot{x} + f_y\dot{y} + f_z\dot{z} + f_\xi\dot{\xi} + f_\eta\dot{\eta} + f_\zeta\dot{\zeta} = 0.$$

Now let us express f_x, f_y, f_ξ, f_η through f_z, p, q, f_ζ, pi, κ using the remaining formulas (3) and substitute the resulting expressions in the above equation. We get

$$f_z(\dot{z} - p\dot{x} - q\dot{y}) + f_\zeta(\dot{\zeta} - \pi\dot{\xi} - \kappa\dot{\eta}) = 0,$$

and so when $\dot{z} - p\dot{x} - q\dot{y} = 0$ we necessarily have $\dot{\zeta} - \pi\dot{\xi} - \kappa\dot{\eta} = 0$. (Here we have to assume that the partial derivatives f_x, ... f_ζ are different from zero, since otherwise the transformation would have singularities.)

Under a contact transformation of a strip ℓ, only contacts of odd order are mapped into turning points of the strip λ, while contacts of odd order are mapped into a "hidden singularity."

4. Let r, s, t denote, as usual, the second derivatives of z with respect to x and y, and let ρ, σ, τ denote the analogous derivatives of ζ with respect to ξ and η. We introduce new quantities by the formulas

$$\left.\begin{array}{llll} r = \dfrac{r_1}{r_5}, & s = \dfrac{r_2}{r_5}, & t = \dfrac{r_3}{r_5}, & rt - s^2 = \dfrac{r_4}{r_5}, \\ \rho = \dfrac{\rho_1}{\rho_5}, & \sigma = \dfrac{\rho_2}{\rho_5}, & \tau = \dfrac{\rho_3}{\rho_5}, & \rho\tau - \sigma^2 = \dfrac{\rho_4}{\rho_5}. \end{array}\right\} \tag{4}$$

Here the quantities $\rho_1, \dots, \rho_5$ and $r_1, \dots, r_5$ are of course related via

$$\rho_1\rho_3 - \rho_2^2 = \rho_4\rho_5, \qquad r_1 r_3 - r_2^2 = r_4 r_5. \tag{5}$$

One has the following result, due to Engel and Spitz: under contact transformations, the quantities r_i and ρ_i change according to the formulas

$$\rho_i = \sum_{k=1}^{5} c_{ik} r_k \quad (i = 1, \dots, 5) \tag{6}$$

(with $|c_{ik}| \neq 0$ if the transformation is not singular).

We will derive this result for a particular case that is important in the sequel, namely, for the so-called Legendre transformation, which is nothing else but the polar transformation with respect to a paraboloid.

In the general case formulas (6) are derived using the same idea.

For the Legendre transformation formulas (3) read

$$\left.\begin{array}{ll} 1. & f(x, y, z; \xi, \eta, \zeta) \equiv z + \zeta - x\xi - y\eta = 0, \\ 2. & -\xi + p = 0, \\ 3. & -\eta + q = 0, \\ 4. & -x + \pi = 0, \\ 5. & -y + \kappa = 0. \end{array}\right\} \tag{7}$$

Suppose that on the surface $z = z(x, y)$ we have a strip given by equations $x = x(t), \dots, q = q(t)$, with $\dot{z} = p\dot{x} + q\dot{y}$. Then

$$\dot{p} = r\dot{x} + s\dot{y}, \qquad \dot{q} = s\dot{x} + t\dot{y}. \tag{8}$$

But from equations (7) it follows that $\dot{\xi} = \dot{p}$, $\dot{\eta} = \dot{q}$ and $\dot{x} = \dot{\pi}$, $\dot{y} = \dot{\kappa}$, whence

$$\dot{\xi} = r\dot{\pi} + s\dot{\kappa}, \qquad \dot{\eta} = s\dot{\pi} + t\dot{\kappa}. \tag{9}$$

Solving these equations for $\dot{\pi}$ and $\dot{\kappa}$ we get

$$\dot{\pi} = \frac{t\dot{\xi} - s\dot{\eta}}{rt - s^2}, \qquad \dot{\kappa} = \frac{-s\dot{\xi} + r\dot{\eta}}{rt - s^2}. \tag{10}$$

Moreover,

$$\dot{\pi} = \rho\dot{\xi} + \sigma\dot{\eta}, \qquad \dot{\kappa} = \sigma\dot{\xi} + \tau\dot{\eta}. \tag{11}$$

Relations (10) and (11) completely independent of the choice of the strip, so the coefficients in them must be equal to one another. Therefore,

$$\rho = \frac{t}{rt - s^2}, \qquad \sigma = \frac{-s}{rt - s^2}, \qquad \tau = \frac{r}{rt - s^2}. \tag{12}$$

If we now introduce the quantities r_i and ρ_i via formulas (4), we obtain

$$\rho_1 = r_3, \quad \rho_2 = -r_3, \quad \rho_3 = r_1, \quad \rho_4 = r_5, \quad \rho_5 = r_4. \tag{13}$$

5. After all this formula preparation, we will address now the application of contact transformations to the Cauchy problem for second-order partial differential equations. This will allow us to solve (in a somewhat generalized sense) this problem in a case in which it was considered as unsolvable in the usual sense.

Suppose we are given a second-order partial differential equation

$$\Phi(\xi, \eta, \zeta; \pi, \kappa; \rho, \sigma, \tau) = 0 \tag{14}$$

and an analytic strip λ

$$\xi = \xi(t), \quad \eta = \eta(t), \quad \zeta = \zeta(t), \qquad \pi = \pi(t), \quad \kappa = \kappa(t);$$
$$\dot{\zeta} = \pi\dot{\xi} + \kappa\dot{\eta}.$$

The Cauchy problem is: construct a surface $\zeta = \zeta(\xi, \eta)$ which satisfies the given equation and passes through the strip λ.

The partial derivatives ρ, σ, τ must satisfy on the strip the three equations

$$\left.\begin{array}{ll} 1. & \Phi = 0, \\ 2. & \dot{\pi} = \rho\dot{\xi} + \sigma\dot{\eta}, \\ 3. & \dot{\kappa} = \sigma\dot{\xi} + \tau\dot{\eta}. \end{array}\right\} \tag{15}$$

Equations (15) for ρ, σ, τ may: 1) have a finite or countable number of solutions; or 2) have a continuum of solutions; or 3) have no solutions. In the first case to each solution there corresponds a solution to the Cauchy problem, and the derivatives ρ, σ, τ of resulting integral on the strip λ are precisely the solutions of equations (15). In the second case λ is a characteristic strip of equation (14) and the solutions of (14) are determined only up to an arbitrary function of a single variable.

In the third case the problem has no solutions in the ordinary sense. Using contact transformations, we reduce this third case to the first.

If, following formulas (4), we replace ρ, σ, τ by $\rho_1, \ldots, \rho_5$, then equations (15) become the homogeneous equations

$$\left.\begin{array}{ll} 1. & \Psi(\xi, \eta, \zeta; \pi, \kappa; \rho_1, \rho_2, \rho_3, \rho_4, \rho_5) = 0, \\ 2. & \rho_1\rho_3 - \rho_2^2 - \rho_4\rho_5 = 0, \\ 3. & \rho_5\dot{\pi} - \rho_1\dot{\xi} - \rho_2\dot{\eta} = 0, \\ 4. & \rho_5\dot{\kappa} - \rho_2\dot{\xi} - \rho_3\dot{\eta} = 0, \end{array}\right\} \tag{16}$$

where in the second equation we recognize relation (5).

Four homogeneous equations with five unknowns always have a system of solutions, in which not all unknowns are equal to zero. In the first two cases, 1) and 2), discussed for system (15), these solutions have $\rho_5 \neq 0$; in the third case, $\rho_5 = 0$.

6. Let us perform a contact transformation

$$(\xi, \eta, \zeta; \pi, \kappa) \to (x, y, z; p, q).$$

Under this transformation the strip λ is mapped into a strip ℓ such that in this new strip ℓ the equations obtained from (16) as a result of the transformation have on ℓ a solution with $r_5 \neq 0$. We can then pass (upon dividing by r_5 and eliminating r_4) to an ordinary system of the form (15). Thus, we have succeeded to transform the third case of the original Cauchy problem into the first case for the transformed equation. We cannot run into the second case, since that case, as one can verify, is invariant under contact transformations; hence, if it did not ocurr for the original system, it also cannot occur for the transformed system.

Thus, the transformed Cauchy problem has solutions $z = z_1(x, y)$,.... If we now use the inverse contact transformation to return to the original ξ-space, then the surfaces $z = z_1(x, y), \ldots$ go into surfaces $\zeta = \zeta_1(\xi, \eta), \ldots$. The latter satisfy the original equation $\Phi = 0$ and pass through the given strip λ. Because of these properties we take them as solutions of the given Cauchy problem, and we say that the contact transformation $\xi \to x$ "regularizes" the (Cauchy) problem.

The surfaces $\zeta = \zeta_1(\xi, \eta), \ldots$ have on the strip λ a singularity of precisely the type considered above in Subsection 3.

7. Let us prove that the method proposed above is always applicable. To this end we will look for a regularizing Legendre transformation. From formulas (13) it is seen that if $\rho_5 = 0$, but $\rho_4 \neq 0$, then this transformation gives $r_5 \neq 0$, $r_4 = 0$ and thus regularizes the Cauchy problem.

Now let us assume that $\rho_5 = 0$ and $\rho_4 = 0$. Then, since ρ_1, ρ_2, ρ_3 do not vanish simultaneously, the strip can be covered by a finite number of intervals, in each of which $\rho_i \neq 0$ $(i = 1, 2, 3)$ and, generally, $\alpha_1\rho_1 + 2\alpha_2\rho_2 + \alpha_3\rho_3 \neq 0$, where α_1, α_2, α_3 are some constants.

Let us make the point transformation

$$\bar{\xi} = \xi, \quad \bar{\eta} = \eta, \quad \bar{\zeta} = \zeta + \frac{1}{2}\left(\alpha_3\xi^2 - 2\alpha_2\xi\eta + \alpha_1\eta^2\right).$$

If in these new variables we differentiate $\bar{\zeta}$ with respect to $\bar{\xi}$ and $\bar{\eta}$, we get $\bar{\rho} = \rho + \alpha_3$, $\bar{\sigma} = \sigma - \alpha_2$, $\bar{\tau} = \tau + \alpha_1$, whence

$$\bar{\rho}\bar{\tau} - \bar{\sigma}^2 = \rho\tau - \sigma_2 + \alpha_1\rho + 2\alpha_2\sigma + \alpha_3\tau + \alpha_1\alpha_3 - \alpha_2^2$$

or, passing to the homogeneous variables ρ_i,

$$\bar{\rho}_4 = \rho_4 + \alpha_1\rho_1 + 2\alpha_2\rho_2 + \alpha_3\rho_3 + \left(\alpha_1\alpha_3 - \alpha_2^2\right)\rho_5.$$

Since $\rho_4 = \rho_5 = 0$, we finally get

$$\bar{\rho}_4 = \rho_4 + \alpha_1\rho_1 + 2\alpha_2\rho_2 + \alpha_3\rho^3 \neq 0,$$

thanks to the choice of the constants α_1, α_2, α_3.

Since now $\bar{\rho}_4 \neq 0$, the Legendre transformation regularizes the problem in the variables $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$, and so the problem in the original variables is regularized as well.[20]

Suppose that, upon regularizing the Cauchy problem, we found some surface F that solves the regularized problem. Then, under the inverse contact transformation from the x-space to the ξ-space, the set of elements that form surface F goes over into a set of elements Φ which, however, may turn out not to be a surface, but a pencil of strips with a common supporting curve λ. Obviously, in this case regularization gives nothing, and we say that the integral obtained through regularization degenerates. Cohn-Vossen gave a necessary and sufficient condition for this degeneracy to occur for any regularizing transformation. To explain this we confine ourselves here to the Legendre transformation.

Suppose that as the result of applying the Legendre transformation (inverse to the regularizing transformation) the surface F becomes a pencil Φ of strips with a common curve λ. Let us apply the inverse Legendre transformation from Φ to F. The Legendre transformation is a polar transformation. Consequently, the curve λ is mapped into a one-parameter family of planes, and the pencils of planes passing through the tangents to λ (such planes form precisely the strips emanating from Φ) are mapped into linear series of points. Therefore, the surface F is developable. Obviously, the converse also holds, i.e., if the surface F is developable, then Φ is a pencil of strips with a common curve λ, and not a surface.

We conclude that the integral obtained through regularization degenerates if and only if the problem regularized by means of the Legendre transformation admits a developable surface as solution.

If on the strip λ we have $\rho_5 = 0$, then after applying the Legendre transformation we obtain $r_4 = 0$, i.e., $rt - s^2 = 0$. This means that the curve λ is mapped into a parabolic curve, which under the inverse transformation goes over in a singular line. If we obtain a developable surface, then on that surface $r_4 = 0$ everywhere, and hence, after we apply the inverse to the regularizing transformation we obtain $\rho_5 = 0$ not only on the strip λ, but everywhere, i.e., the entire "surface" consists of singularities. This means precisely that the integral found via regularization degenerates.

8. Let us prove that the solution of the Cauchy problem obtained via regularization does not depend on the regularizing transformation.

All the solutions found by means of some or another regularization satisfy on the strip λ the system (16) with $\rho_5 = 0$; the remaining ρ_is take determined

[20] This is true only for some piece of the given strip. But such pieces cover the entire strip, and since in the sequel we will prove that the resulting solution does not depend of the transformation used, the solutions for the pieces of the strip compose a solution for the entire strip.

values. Let F_1 and F_2 be two solutions that are obtained by different regularizations, but which give on the strip λ the same values of the ρ_is. Let us show that then F_1 and F_2 coincide. We perform a regularizing transformation; then the set of values ρ_i goes into a set of values r_i with $r_5 \neq 0$, which does not depend at all on F_1 and F_2. Under this regularization the surfaces F_1 and F_2 go into surfaces that solve a regularized Cauchy problem: they solve the transformed equation, pass through the given strip, and have on it the same values of the second derivatives r, s, t (obtained from the values r_i). Hence, by the Cauchy-Kovaleskaya theorem, they coincide.

In the case where the integral obtained via regularization does not degenerate, the method developed above yields a regular parametric representation of the integral, which justifies the name "regularization".

Under a contact transformation $\xi \to x$, an integral $\zeta = \zeta(\xi, \eta)$ that is singular on the strip λ goes into an integral $z = z(x, y)$ that is regular on the image of λ. The transformation $\xi \to x$ renders ξ, η, ζ as regular functions of x, y, z, p, q. If in these expressions for ξ, η, ζ we put $z = z(x,y)$, $p = \partial z(x,y)/\partial x$, $q = \partial z(x,y)/\partial y$, we get $\xi = \phi(x,y)$, $\eta = \psi(x,y)$, $\zeta = \theta(x,y)$. These functions are regular and provide a parametric representation of the surface, $\zeta = \zeta(\xi, \eta)$.

9. We now turn to applications of the preceding results to problems in the theory of surfaces.

A surface can be intrinsically regular, i.e, have a regular unboundedly continuable linear element, and at the same time have a singularity when regarded as an object (figure) in space. Examples are provided by the developable surfaces with their turning edges, or the known surfaces of revolution of constant curvature with their vertices and edges. It is probably useful to point out here that, as is known, the sphere, the plane and the cylinder are the only surfaces of constant curvature with no singularities.[21]

One of the fundamental problems in the theory of bending of surfaces is that of constructing in the (ξ, η, ζ)-space a surface with a prescribed metric. The precise formulation of this problem is as follows. Suppose that in the domain D where the parameters u, v range there are given an element $ds^2 = Edu^2 + 2F du\, dv + Gdv^2$ and a curve λ. Suppose also that in the space one is given a curve L. Construct a surface Φ that passes through L and has the given ds so that, in the sense of the metric on Φ, the curve L coincides with λ. (On λ and L one specifies beforehand points and directions, which must coincide, after which the curve λ must superpose the curve L.)

If we already have a surface Φ that solves the problem formulated above, then the curvature of L is not smaller than the geodesic curvature of λ in the corresponding points (as is known, the geodesic curvature can be defined solely in terms of ds^2, and is therefore given beforehand); this is because the geodesic curvature of a curve on the surface is equal to its ordinary curvature multiplied by the cosine of the angle made by the tangent plane to the surface and the

[21] According to the definition given in Section 1, "Singularities of convex surfaces", the presence of boundary points is also a kind of singular behavior.

osculating plane to the curve:

$$\frac{1}{\rho_{\mathrm{g}}} = \frac{\cos\theta}{\rho}. \tag{17}$$

Based on this, it is easy to list the various possible cases for the solution of the posed problem:

1) If $1/\rho_{\mathrm{g}} > 1/\rho$, i.e., the geodesic curvature of λ is larger than the curvature of L, the problem is not solvable.

2) If $1/\rho_{\mathrm{g}} < 1/\rho$, then, as is know, there are two surfaces that provide a solution of the problem.

3) $1/\rho_{\mathrm{g}} = 1/\rho$; in this case in formula (17) one has $\theta = 0$, i.e., the osculating plane to L must coincide with the tangent plane to Φ. This means that the curve L must be asymptotic on the surface Φ. And if the surface is regular, then on the asymptotic line L Enneper's relation between the torsion $1/T$ of L and the Gaussian curvature K of the surface holds:

$$K + \frac{1}{T^2} = 0.$$

If this relation is satisfied, then the posed problem is solvable, and in fact there exists an entire one-parameter family of surfaces that solve the problem, and of each of them the line L is asymptotic. (By a theorem of Gauss, K is expressed in terms of ds^2 and $1/T$ is given, since the curve L is given).

4) $1/\rho_{\mathrm{g}} = 1/\rho$, but Enneper's relation is not satisfied. In this case, even if a surface that solves the posed problem exists, the curve L on it must be a singular line (otherwise, since L is asymptotic, Enneper's relation would hold).

10. To deal with this last, fourth case, which earlier geometers were not able to handle, Cohn-Vossen did apply his method. This is possible because, according to Darboux, the problem one is trying to solve reduces to the Cauchy problem for an equation of Monge–Ampère type. This reduction is achieved as follows. If $\xi(u,v)$, $\eta(u,v)$, $\zeta(u,v)$ are the coordinates of a point on the sought-for surface, then

$$d\xi^2 + d\eta^2 + d\zeta^2 = E du^2 + 2F du\, dv + G dv^2,$$

or

$$d\xi^2 + d\eta^2 = \left(E - \zeta_u^2\right) du^2 + 2\left(F - \zeta_u\zeta_v\right) du\, dv + \left(G - \zeta_v^2\right) dv^2.$$

In this last equation, in the right-hand side we have the length element of a plane, and hence in the left-hand side we must have also such an element. It follows that the Gaussian curvature of our element must be equal to zero. Using the expression of the Gaussian curvature in terms of the coefficients of the linear element and setting it equal to zero we obtain an equation for $\zeta(u,v)$ which turns out to be an equation of Monge–Ampère type. We need, however, to make the surface pass through the given curve L so that the curve λ will coincide with L. As L and λ are given, $1/\rho_1$ and $1/\rho_{\mathrm{g}}$ are also given, and then from equation (17) one determines $\cos\theta$ and correspondingly two values for the angle θ, θ_1 and

θ_2, i.e., one has two possible positions of the tangent plane to the surface Φ.[22] If $\cos\theta_1 = \cos\theta_2$, then $\theta_1 = -\theta_2$. Therefore, if $\cos\theta$ is nowhere equal to 1 or 0, then the choice of one of the angles θ_1 or θ_2 in one point of the curve uniquely determines, by continuity, the choice of angle in all its other points. To these two possible choices of the angle, θ_1 or θ_2, correspond two surfaces that solve our problem in the case when $\cos\theta \neq 1$, $\cos\theta \neq 0$.[23]

If now $\cos\theta = 1$, i.e., $1/\rho_{\mathrm{g}} = 1/\rho$, then $\theta_1 = \theta_2 = 0$, and there is only one possible position for the tangent plane.

Thus, we have a curve L and tangent planes to Φ in the points of this curve. The curve L must be superposed by the curve λ. Hence, on λ there are given the coordinates ξ, η, ζ of the points of L. Next, since the linear element and the directions of the tangent planes are specified, the partial derivatives $\xi_u, \dots, \zeta_v$ on the curve λ are also determined.[24] It follows that on λ there are given ζ, ζ_u, ζ_v. Moreover, $\zeta(u, v)$ must satisfy an equation of Monge-Ampère type. Thus, the posed problem of constructing a surface with a given metric through a given curve reduces to a Cauchy problem. Once we have a solution of this Cauchy problem, the two remaining coordinates of the points of the surface, $\xi(u, v)$ and $\eta(u, v)$, can be obtained by solving ordinary differential equations (of Riccati type).

For the Cauchy problem at hand, the four possible cases indicated above that can arise when one solves our geometric problem have the following meaning

1) The Cauchy problem does not admit a formulation in real form ($\cos\theta > 1$).

2) The usual case of the Cauchy problem.

3) λ is a characteristic curve, and consequently the Cauchy problem has a continuum of solutions.

4) The Cauchy problem has no solutions in the ordinary sense.

This last case can be treated by the regularization method introduced by Cohn-Vossen. Hence, in the fourth case, too, there exists a surface (and in fact a unique one) that solves the Cauchy problem, but the curve L is singular on

[22] Clearly, we may consider that the angle θ between the tangent plane to Φ and the osculating plane to L ranges between $-\pi/2$ and $\pi/2$.

[23] If $\cos\theta$ takes the value 0 or 1 in isolated points on L, then the selected value of the angle θ can be uniquely continued to the whole L if one requires that when those points are crossed the derivatives of θ with respect to arclenght be continuous as well. If $\cos\theta = 0$ on the entire curve L, then $1/\rho_{\mathrm{g}} = 0$ and consequently $\lambda = L$ is a geodesic line. When $\cos\theta = 0$, the osculating plane to the curve L is perpendicular to the tangent plane to the surface, so that only one position is possible for it. However, in this case, too, there are two surfaces solving the posed problem. The reason is that, on the strip formed by the tangent planes to the sought-for surface along L one can distinguish two sides, so to say, right and left to L. The same distinction can be also made in the domain D in a neighborhood of λ. To one side of λ we can associate any of the two sides of L ($\lambda = L$ is a geodesic, and so both sides are equivalent), which leads to two surfaces. In special cases the two surfaces may coincide, if the metric ds is symmetric with respect to the curve λ.

[24] For instance, the vector of components ξ_u, η_u, ζ_u has length $\sqrt{E}$, lies in the tangent plane, and makes a specified angle with the curve L, since this is the angle between the line $v = \mathrm{const}$ and the curve λ, defined with respect to the given metric. The choice of the resulting two possible directions of the vector (ξ_u, η_u, ζ_u) is carried out based on its orientation with respect to the curvature vector of the curve λ, which must coincide with the projection of the curvature vector of the curve L onto the tangent plane.

it.

An exception is the case when the solution obtained via regularization degenerates. Cohn-Vossen has shown that this is possible only when λ is a geodesic line. Then $1/\rho_{\mathrm{g}} = 0$, and since $1/\rho_{\mathrm{g}} = 1/\rho$, we have $1/\rho = 0$, too. Consequently, the curve L is a line. Thus, the problem of realizing a geodesic as a line (along which Enneper's relation has no meaning, since the torsion of a line is not defined) plays a special role among the bending problems: it cannot be solved via regularization.

In all other cases regularization does achieve its goal. The problem turns out to be solvable, but the curve L is singular, and in fact is of the same nature as those discussed in Subsection 3.

11. Since regularization provides a regular parametric representation of the integral, on the surface Φ near the curve L we can choose parameters in such a way that ξ, η, ζ are represented by regular functions of the parameters, despite of the fact that L is singular. Here singularity reduces to the fact that in these new parameters $EG - F^2$ vanishes on L. Cohn-Vossen termed the corresponding net of parametric lines regularizing. He proved the following theorem.

1) *The curvature lines form a regularizing net on* Φ.

2) *The asymptotic lines form a regularizing net on* Φ *whenever the Gaussian curvature* $K < 0$ *everywhere on* Φ.

3) *The curvature lines and the asymptotic lines have* L *as an envelope.*

12. Further, Cohn-Vosses obtained another series of remarkable results on the curve $L = \lambda$ and the behavior of the surface Φ in its vicinity. Here we will confine to listing these results.

We assume that the geodesic curvature o λ does not change sign, i.e., in the domain D parametrized by u, v the curve λ is always concave in one side. Accordingly, the terms *inner* and *outer sides of* λ are meaningful.

On Φ the curve $L = \lambda$ is a turning edge; the two sheets of the surface that abut to L realize the metric on only one side relative to λ in the domain D, namely:

1) if $K \geq 0$, then in the outer side of λ;

2) if $K < 0$, but $(1/T^2) + K < 0$, then also in the inner side of λ.

3) if $K < 0$ and $(1/T^2) + K > 0$, then in the outer side of λ.

Briefly speaking, if $(1/T^2) + K > 0$ (resp., $(1/T^2) + K < 0$), then the metric is realized in the outer (resp., inner) side of λ. In the intermediate case if $(1/T^2) + K = 0$, which is Enneper's relation, about which we assumed that it does not hold; hence, this case is excluded.

A well-known example is provided by the turning edge on a developable surface, where, as it should be, the metric is realized only in the outer side with respect to λ. In much the same way the theorem above is illustrated by the surfaces of revolution of constant curvature. On such surfaces there are flat turning edges, so that $1/T^2 = 0$, and consequently according to the theorem for $K > 0$ (resp. $K < 0$) the surface realizes the metric in the outer (resp., inner) side of the curve λ, which corresponds to the edge.

This distinction between outer and inner sides with respect to λ leads to an interesting phenomenon. Consider, for example, a plane curve λ with an inflection point. If we want to bend the plane so that λ goes over onto the edge L of a developable surface, then this can be done only with the part of the plane lying on the outer side of λ. However, in the inflection point the outer and inner sides switch places, and so in that point there arises a discontinuity.

This is a particular case of the following general proposition: a curve with a geodesic inflection point cannot be made into a singular line by means of a continuous bending — the surface would acquire a discontinuity which passes through this inflection point.

13. As one approaches a singular line, one of the principal radii of curvature tends to zero. Cohn-Vossen did obtain for it a remarkable formula.

Let s denote the geodesic distance from a given point P to the singular line. Let R_1 and R_2 denote the larger, respectively smaller radius of curvature in P. Suppose P converges to a point P_0 lying on the singular line $L = \lambda$. Further, let K denote the Gaussian curvature in the point P_0, and let $1/T$ and $1/\rho_{\mathrm{g}}$ denote the torsion and the geodesic curvature of the line $L = \lambda$. Then the following two relations hold:

$$\lim_{P\to P_0} \frac{2s}{R_1^2} = \rho_{\mathrm{g}} \left(\frac{1}{T^2} + K \right), \qquad \lim_{P\to P_0} \frac{1}{2sR_2^2} = \frac{1}{\rho_{\mathrm{g}}} \cdot \frac{K^2}{\dfrac{1}{T^2} + K}.$$

Further, Cohn-Vossen obtained some results about conical singular points, which also need to be treated as singular strips. However, he did not study such singular strips as completely as he did with the turning edge.

Finally, Cohn-Vossen turned his attention to isolated singular points (which are not strips). On a surface with everywhere regular metric one considers a point which is singular in the sense that although the tangent to the surface is continuous in that point, the radii of curvature acquire a singularity. Cohn-Vossen did prove that surfaces of negative curvature may have such isolated singular points, whereas surfaces of positive curvature may not.

4. Nonrigid closed surfaces [4]

1. A surface is called *rigid* if, as a whole (i.e., globally) it admits no infinitesimal bendings except for motions; in the opposite case the surface is called *nonrigid*.

Cohn-Vossen did first prove that there exist nonrigid closed surfaces (in addition to the trivial ones — a surface with a flat piece is always nonrigid, since this piece is rigid even for clamped boundaries),[25] namely, surfaces of revolution of genus 0 (of sphere type), as well as of genus 1 (of torus type). As it turns out, every such surface of revolution becomes nonrigid if one presses into it, so to say, an arbitrarily narrow and arbitrarily flat "ditch" that goes in the meridian's direction. However, the width and the depths of the ditch are not arbitrary, but must be selected from a certain set of values, which inclued

[25] If we displace the points of a piece of a plane along the normal to the plane, then, as it clearly follows from Pythagora's theorem, the strains will be small of second order with respect to the displacements.

arbitrarily small ones. Thus, every closed surface of revolution is the limit of a sequence of nonrigid surfaces of revolution.

We will sketch here the main steps of the reasoning that led Cohn-Vossen to this remarkable result. The surfaces under consideration are, needless to say, assumed to be two-times continuously differentiable. We confine ourselves to closed surfaces of revolution of genus 0, since essentially the same arguments work for surfaces of genus 1.

2. Let $\bar{x}(u,v)$ be a vector that describes the given closed surface of revolution, u be the height measured on the axis of the surface, v the longitude, and $r = r(u)$ the equation of the meridian of the surface.

Imagine that the surface deforms with time, so that we have a family $\bar{x}(u,v;t)$. Let the velocity (rate of deformation) at the initial moment be

$$\frac{\partial}{\partial t}\bar{x}(u,v;0) = \bar{z}(u,v).$$

The deformation will yield an infinitesimal bending if at the initial moment the lengths of curves on the surface are stationary, i.e.,

$$\left(\frac{\partial}{\partial t} ds^2\right)\bigg|_{t=0} = 0.$$

Since $ds^2 = d\bar{x}^2$, this gives

$$\frac{\partial}{\partial t} d\bar{x}^2 = 2d\bar{x}\,\frac{\partial}{\partial t} d\bar{x} = 2d\bar{x}\,d\bar{z} = 0,$$

i.e.,

$$d\bar{x}\,d\bar{z} = 0, \tag{1}$$

or, in expanded form,

$$\bar{x}_u\bar{z}_u = \bar{x}_u\bar{z}_v = \bar{x}_u\bar{z}_v + \bar{x}_v\bar{z}_u = 0. \tag{2}$$

This is the well-known textbook equation of infinitesimal bendings.

Let us choose the following coordinate vectors in each point of the surface: $\bar{e}$, parallel to the axis of the surface, $\bar{a}$, in the meridional plane and perpendicular to $\bar{e}$, and $\bar{a}' = d\bar{a}/dv$, which is directed along the tangent to the parallel.

Let us decompose, in each point of the surface, the vectors $\bar{x}$ and $\bar{z}$ with respect to the chosen unit vectors:

$$\begin{aligned}&\bar{x}(u,v) = \bar{e}u + \bar{(}v)r(u),\\ &\bar{z}(u,v) = \bar{e}\alpha(u,v) + \bar{a}(v)\beta(u,v) + \bar{a}'(v)\gamma(u,v).\end{aligned}$$

Rewriting equation (2) in components we get

$$\left.\begin{aligned}\alpha_u + r'\beta_u = 0,\\ \beta + \gamma_v = 0,\\ \alpha_v + r'(\beta_v - \gamma) + r\gamma_u = 0.\end{aligned}\right\} \tag{3}$$

Since $r'(u)$ becomes infinite in the poles, for the moment we exclude them from the surface, considering only a zone between two parallels. We shall also assume that $r'(u)$ does not become infinite elsewhere, i.e., the tangent plane to the surface is nowhere perpendicular to the axis.

3. Since α, β, γ must depend on v periodically with period 2π, we can expand them in Fourier series

$$\alpha(u,v) = \sum_{k=-\infty}^{\infty} e^{ikv}\phi_k(u),\ \beta(u,v) = \sum_{k=-\infty}^{\infty} e^{ikv}\psi_k(u),\ \gamma(u,v) = \sum_{k=-\infty}^{\infty} e^{ikv}\chi_k(u),$$

Here ϕ_k, ψ_k, χ_k are complex-valued functions of the variable u that, in order for α, β, γ to be real, must satisfy the conditions

$$\phi_{-k}(u) = \phi_k^*(u), \quad \psi_{-k}(u) = \psi_k^*(u), \quad \chi_{-k}(u) = \chi_k^*(u),$$

where the asterisk denotes complex conjugation.

Substituting these series expansions in equation (3) and setting the coefficients of e^{ikv} in the resulting expression equal to zero (since a Fourier series is equal to zero only when all its coefficients are equal to zero), we obtain a system that depends on u and the integer parameter k:

$$\left.\begin{array}{l} \text{a) } \phi_k'(u) + r'(u)\psi_k'(u) = 0, \\ \text{b) } ik\chi_k(u) + \psi_k(u) = 0, \\ \text{c) } ik\phi_k(u) + r'(u)\left[ik\psi_k(u) - \chi_k(u)\right] + r(u)\chi_k(u) = 0. \end{array}\right\} \tag{4}$$

Differentiating c), we eliminate ϕ_k and ψ_k from a) and b) and get

$$r\chi'' + \left(k^2 - 1\right) r''\chi_k = 0. \tag{5}$$

4. Each of the nonidentically vanishing integrals $\chi_k(u)$ of these equations for $k \geq 0$ yields, using equations b) and c) in (4) above, a real-valued, nonidentically vanishing velocity vector of an infinitesimal bending

$$\bar{z}_k(u,v) = \left(\bar{e}\phi_k + \bar{a}\psi_k + \bar{a}'\chi_k\right)e^{ikv} + \left(\bar{e}\phi_{-k} + \bar{a}\psi_{-k} + \bar{a}'\chi_{-k}\right)e^{-ikv}$$

(ϕ_k, ψ_k, χ_k and ϕ_{-k}, ψ_{-k}, χ_{-k} are complex conjugate). Since the problem is linear, a linear combination of such vectors with constant coefficients gives again a velocity vector of an infinitesimal bending. For $k = 0$ and $k = 1$ equations (4) and (5) are readily solved. In the case $k = 0$ one obtains an arbitrary screw motion around the axis of the surface, while for $k = 1$ one obtains an arbitrary screw motion around any of the lines perpendicular to the axis. (Particular cases of crew motions are one rotation or one translation.) Since any motion can be obtained as a sum of two motions of the previously indicated kind, the linear combinations $a_0 z_0(u,v) + a_1 z_1(u,v)$ exhaust all infinitesimal motions. Consequently, any nonidentically vanishing integral of equation (5) with $k \geq 2$ yields a nontrivial infinitesimal motion of the considered zone of our surface.

In what follows we shall always assume that $k \geq 2$.

5. The poles of the surface are singular points for equation (5). Since at the poles the bending vector cannot depend on v, it follows that there we must have $\phi_k = \psi_k = \chi_k = 0$ for all $k \geq 0$. To analyze equation (5) in the neighborhood of a pole it is convenient to take r as the independent variable. At the pole, i.e., for $r = 0$, we have $du/dr = 0$. For the sake of simplicity we may assume that $u''(r)|_{r=0} \neq 0$, i.e., the pole is not a parabolic point. Then, employing the usual method of Fuchsian theory,[26] we obtain two fundamental integrals of the equation for $\chi_k(r)$, of which one vanishes at the pole as r^{k+1}, while the other goes to infinity as r^{-k+1}. To the first integral there correspond (via formulas obtained from (4) when r is taken as independent variable) functions $\phi_k(r)$ and $\psi_k(r)$ that also vanish at the pole. Hence, this integral yields an infinitesimal bending that is regular at the pole as well.

However, in order to obtain an infinitesimal bending of the entire surface we need an integral that vanishes at both poles.

6. Let us assume, for the sake of definiteness, that the poles correspond to the values $u = \pm 1$. Then we have the equation

$$r(u)\chi_k''(u) + (k^2 - 1)r''(u)\chi_k(u) = 0. \tag{5}$$

For a given function $r(u)$ and not necessarily integer values of k, the problem of finding an integral of this equation that is regular in the interval $(-1, 1)$ and vanishes in its endpoints, and also finding the corresponding eigenvalue $k^2 - 1$, is a Sturm-Liouville problem. We in fact are dealing with the inverse problem of determining $r(u)$ so that, for integer $k \geq 2$, equation (5) has an integral that is regular in the interval $(-1, 1)$ and vanishes in its endpoints. In other words, we need to find a function $r(u)$ such that the above Sturm-Liouville problem has $k^2 - 1$ with an integer $k \geq 2$ as an eigenvalue.

Suppose $r(u)$ is given and let $\chi_k(u)$ denote the fundamental integral of equation (5) that is regular at $u = +1$. Then, generally speaking, when $u \to -1$, $\chi_k(u) \to \infty$, since the second fundamental integral becomes infinite at a pole.

Now we shall prove the following assertion: by means of an arbitrarily small and continuous up to the second derivative variation of $r(u)$ in an arbitrarily small part of the interval $(-1, 1)$ one can achieve that for the appropriately chosen fixed $k \geq 2$, $\chi_k(u)$ will pass into a function that is already regular for $u = -1$ as well. Hence, through such a change, corresponding to the function $r(u)$, the surface becomes nonrigid. This modification procedure is the most easily implemented for a convex surface, and we restrict our considerations to this, simplest case.

7. By the assumption that the meridian is convex, $r''(u) < 0$ for $-1 < u < +1$. Hence, from equation (5) it follows that

$$\frac{\chi_k''(u)}{\chi_k(u)} > 0 \quad \text{for}\, \chi_k(u) neq 0 \quad (-1 < u < +1). \tag{6}$$

This shows that $\chi_k(u)$ has no positive maxima and no negative minima in the interval $(-1, +1)$. Consequently, $\chi_k(u)$ cannot vanish at $u = -1$ (since

[26]See, e.g., Goursat's *A course in mathematical analysis*, Vol. II, Part II, *Differential equations* Ginn and Company, Boston, New York, 1917, §§ 409–412.

$\chi_k(1) = 0$, so assuming that $\chi_k(-1) = 0$ Rolle's theorem would lead to a contradiction). This completes the proof of the rigidity of convex surfaces of revolution.

8. Now let a be some number in the interval $(-1, +1)$. Let δ and ϵ be given, arbitrarily small positive numbers; in particular, assume that δ is so small that

$$-1 < a - \delta < a + \delta < 1.$$

Let us make an indentation in the meridian in the interval $(a - \delta, a + \delta)$ (i.e., of width 2δ), of depth and torsion no larger than ϵ. Rigorously speaking, we consider a function $r_0(u)$ with the following properties:

1) $r_0(u)$ is twice continuously differentiable in the interval $(a - \delta, a + \delta)$.

2) $r_0(u) - r(u) = 0$ outside the interval $(a - \delta, a + \delta)$.

3) $\max|r_0(u) - r(u)| \le \epsilon$ and $\max|r_0'(u) - r'(u)| \le \epsilon$.

4) $\frac{r_0''(a)}{r_0(a)} = 1$.

Clearly, such a function can be constructed, and we use it to represent the modified meridian. To $r_0(u)$ we associate a number η, $0 < \eta < \delta$, such that

$$\frac{r_0''(a)}{r_0(a)} \ge \frac{1}{2} \quad \text{for } a - \eta \le u \le a + \eta.$$

9. Consider the equation

$$r_0(u)\chi_{k_0}''(u) + (k^2 - 1)r_0''(u)\chi_{k_0}(u) = 0,$$

obtained from (5) upon replacing $r(u)$ by $r_0(u)$. Let χ_{k_0} be the integral of this equation that coincides with $\chi_k(u)$ for $a + \delta \le u \le 1$. Then in the interval $(a - \eta, a + \eta)$ one has

$$\frac{\chi_{k_0}''(u)}{\chi_{k_0}(u)} \le \frac{k^2 - 1}{2} \quad \text{for } \chi_{k_0}(u) \ne 0 \tag{6'}$$

The function $y(u) = \sin[(\sqrt{(k^2 - 1)/2})u]$ satisfies the equation

$$\frac{y''}{y} = -\frac{k^2 - 1}{2}.$$

By a well-known theorem of Sturm, the inequality (6') implies that the distance between successive zeros of the function $\chi_{k_0}(u)$ in the interval $(a - \eta, a + \eta)$ is not larger than the number of zeros of the function $y(u) = \sin[(\sqrt{(k^2 - 1)/2})u]$. Therefore, if k is chosen so large that $(\sqrt{(k^2 - 1)/2})\eta > \pi$, then $\chi_{k_0}(u)$ has at least one zero in the interval $(a - \eta, a + \eta)$. Let us choose k in this manner and keep it fixed in what follows.

10. Next, consider the function

$$r(u, t) = r_0(u) + t[r(u) - r_0(u)] \quad (0 \le t \le 1)$$

(in this way we gradually reduce the variation of $r(u)$). Let $\chi_k(u;t)$ be the integral of the equation

$$r(u;t)\chi''(u;t) + (k^2-1)r''(u;t)\chi(u;t) = 0 \tag{7}$$

that coincides with $\chi_k(u)$ for $a+\delta \le u \le 1$.[27] We have $\chi_k(u;1) = \chi_k(u)$, since $r(u;1) = r(u)$.

Thus, we obtained a family of equations (7) and a corresponding family of integrals, which as t grows continuously from 0 to 1 passes continuously from $\chi_{k_0}(u)$ to $k(u)$.

By the choice of k, the function $\chi_{k_0}(u)$ has zeros in the interval $(-1,+1)$; on the other hand, we have shown that the function $\chi_k(u)$ has no zeros in this interval. It follows that the integrals $\chi(u;t)$ vary in such a way that their zeros disappear from the interval $(-1,+1)$.

In view of the continuity of $\chi(u;t)$ with respect to t, its zeros can disappear in only two ways: 1) the zeros shift to the left, reach -1, and disappear from the interval (motion to the right is not allowed, since for $u \ge a+\delta$ we have $\chi(u;t) = \chi_k(u)$; or 2) the zeros approach one another and merge into a multiple zero, which afterwards disappears.

However, case 2) is not possible. Indeed, if $\chi(u;t)$ would have a multiple zero in the point u_0 (with $1 < u_0 < +1$)), then we would have $\chi(u_0;t) = \chi'(u_0;t) = 0$, and since $\chi(u;t)$ satisfies a second-order equation, it would follow that $\chi(u;t)$ vanishes on the whole interval (contradicting the fact that $\chi(u;t) = \chi_k(u)$ for $u \ge a+\delta$).

Thus, only case 1) remains. This means that there is a t_0 such that, for $t = t_0$, the function $\chi(u;t)$ vanishes also for $u = -1$ (and then the zero in the interval $(-1,+1)$ moves to -1). Consequently, the function $\chi(u;t_0)$ is a regular, in the whole closed interval $(-1,+1)$, integral of equation (7), and so

$$r_{t_0}(u) = r_0(u) + t_0\left[r(u) - r_0(u)\right]$$

represents the equation of the meridian of a nonrigid surface.

Here in the last part of the argument we reproduced only the idea of the proof, which still requires a rigorous setting. This setting, however, does no bring anything essentially new.

11. The fact that a surface is nonrigid does not yet imply that it is bendable (i.e., that it admits a finite continuous deformation that preserves length). However, the existence of a nonrigid surface does imply the existence of a pair of noncongruent isometric surfaces.

Let $\bar{(}u,v)$ be a nonrigid surface and $\bar{z}(u,v)$ be an infinitesimal deformation vector of this surface. By equation (1),

$$d\bar{x}\,d\bar{z} = 0.$$

Let us show that the surfaces $\bar{x}(u,v)+t\bar{z}(u,v)$ and $\bar{x}(u,v)-t\bar{z}(u,v)$ are isometric for any t. To this end we need to show that their linear elements are equal to

[27] In this interval $r(u;t) = r(u)$ and so equation (7) coincides there with (5).

one another. But

$$d(\bar{x}+t\bar{z})^2 = d\bar{x}^2 + 2td\bar{x}\,d\bar{z} + t^2 d\bar{z}^2 = d\bar{x}^2 + t^2 d\bar{z}^2$$

because $d\bar{x}\,d\bar{z} = 0$ by equation (1). The same conclusion holds true for $d(\bar{x}-t\bar{z})^2$, and so

$$d(x+t\bar{z})^2 = d(\bar{x}-t\bar{z})^2,$$

as we needed to prove.

If we now have a closed regular nonrigid surface $\bar{x}(u,v)$ and a bending vector $\bar{z}(u,v)$ of it, then, as one can readily see, for t sufficiently small the surfaces $\bar{x}(u,v)+t\bar{z}(u,v)$ and $\bar{x}(u,v)-t\bar{z}(u,v)$ will be closed, regular [28] and — in view of what we just proved — isometric.

Thus, from the existence, established by Cohn-Vossen, of nonrigid closed surfaces of revolution follows the existence of pairs of closed isometric (but not congruent) surfaces.[29]

12. Cohn-Vossen's work on nonrigid closed surfaces was continued and used in later studies of several authors. Thus, Rembs[30] and Lyukshin[31] applied Cohn-Vossen's method to study the rigidity of surfaces of revolution of negative curvature. They considered one-parameter families of such surfaces (of a special type) and, applying Cohn-Vossen's method described above, have shown that for a countable number of values of the parameter those surfaces are nonrigid, while for the remaining values they are rigid.

Deepening to some extent Cohn-Vossen's simple proof, discussed above, of the rigidity of convex surfaces of revolution, the author of the present paper found a necessary and sufficient condition for the rigidity of a convex surface of revolution that involves no regularity requirements.[32]

However, in our opinion, a substantial further development of Cohn-Vossen's work would be to provide answers to the following two questions:

1) Do there exist closed surfaces that admit a continuous finite bending?

2) Is every closed surface a limit of nontrivially-rigid closed surfaces? Can one estimate the density of nonrigid surfaces among all closed surfaces, i.e., determine which is the rule, and which is the exception: rigidity or nonrigidity?

The first problem formulated above was studied also by Cohn-Vossen in his work considered here. He proved that the surfaces he found admit also infinitesimal bendings of second order. This problem, it seems, is the most

[28] For sufficiently small t the surfaces obtained in this manner may, regardless of the twice differentiability of the vector $\bar{z}(u,v)$, have self-intersections, singular points, edges, etc.

[29] Of course, if in a sphere we make a small indentation with flat boundary, and then reflect it in the plane of the boundary, we obtain two isometric nonequal surfaces. This case, however, is trivial.

[30] E. Rembs, *Über die Verbiegung parabolisch berandeter Flächen negativer Krümmung*, Math. Zeit. Vol. **35** (1932) pp. 529–535

[31] V. S. Lyukshin (Lukchin), *Zur Theorie der Verbiegung der Rotationsflächen negativer Krümmung*, Rec. Math. [Mat. Sbornik] N.S., **2(44)**:3 (1937), 557–565; (Ljukschin, W.) *Über die Verbiegung von Rotationsflächen negativer Krümmung mit einem singulären Punkte*, Dokl. Acad. Sci. URSS, N. Ser. **17** (1937), 339–341.

[32] A. D. Alexandrov, *On infinitesimal bendings of irregular surfaces*, Mat. Sbornik **1(43)** (1936), 305–322.

important among the problems waiting to be solved in the global theory of bending of surfaces.

List of S. E. Cohn-Vossen's works

[1] *Singularitäten konvexer Flächen*, Math. Ann. **97** (1927), 377–386.

[2] *Zwei Sätze über die Starrheit der Eiflächen*, Göttinger Nachrichten (1927), 125–134.

[3] *Die parabolische Kurve. Beitrag zur Geometrie der Berührungstransformationen der partiellen Differentialgleichungen zweiter Ordnung und der Flächenverbiegung*, Math. Ann. **99** (1928), 273–308.

[4] *Unstarre geschlossene Flächen*, Math. Ann. **102** (1929), 10–29.

[5] *Sur la courbure totales des surfaces ouvertes*, C. R. Acad. Sci. Paris **197** (1933), 1165–1167.

[6] *Kürzeste Wege und Totalkrümmung auf Flächen*, Compositio Math. **2** (1935), 69–133.

[7] *On the existence of shortest paths*, Doklady Akad. Nauk. **III (VIII)**:8 (1935), 339–342.

[8] *Complete Riemannian spaces of positive curvature*, Doklady Akad. Nauk. **III (VIII)**:9 (1935), 387–389.

[9] *Existenz kürzester Wege*, Compositio Math. **3** (1936), 441–452.

[10] *Totalkrümmung und geodätische Linien auf einfachzusammenhängenden offenen vollständigen Flächenstücken*, Mat. Sbornik N. Ser. **1 (43)**:2 (1936), 139–164.

[11] *Der approximative Sinussatz für kleine Dreiecke auf krummen Flächen.* (*Auszug aus einem Brief an Prof. T. Levi-Civita*), Compositio Math. **8** (1936), 52–54.

[12] *Die Kollineationen des n-dimensionalen Raumes*, Math. Ann. **115** (1937), 80–86.