Moreover, if one passes to other groups, then there are σ Eisenstein series of each type, where σ is the number of cusps, and, although they span the same vector space, they are not individually proportional. In fact, we will actually want to introduce a *third* normalization

$$\mathbb{G}_k(z) \;=\; \frac{(k-1)!}{(2\pi i)^k}\, G_k(z) \tag{12}$$

because, as we will see below, it has Fourier coefficients which are rational numbers (and even, with one exception, integers) and because it is a normalized eigenfunction for the Hecke operators discussed in §4.

As a first application, we can now determine the ring structure of $M_*(\Gamma_1)$

Proposition 4. *The ring $M_*(\Gamma_1)$ is freely generated by the modular forms E_4 and E_6.*

Corollary. *The inequality* (7) *for the dimension of $M_k(\Gamma_1)$ is an equality for all even $k \geq 0$.*

Proof. The essential point is to show that the modular forms $E_4(z)$ and $E_6(z)$ are algebraically independent. To see this, we first note that the forms $E_4(z)^3$ and $E_6(z)^2$ of weight 12 cannot be proportional. Indeed, if we had $E_6(z)^2 = \lambda E_4(z)^3$ for some (necessarily non-zero) constant λ, then the meromorphic modular form $f(z) = E_6(z)/E_4(z)$ of weight 2 would satisfy $f^2 = \lambda E_4$ (and also $f^3 = \lambda^{-1} E_6$) and would hence be holomorphic (a function whose square is holomorphic cannot have poles), contradicting the inequality $\dim M_2(\Gamma_1) \leq 0$ of Corollary 1 of Proposition 2. But *any* two modular forms f_1 and f_2 of the same weight which are not proportional are necessarily algebraically independent. Indeed, if $P(X,Y)$ is any polynomial in $\mathbb{C}[X,Y]$ such that $P(f_1(z), f_2(z)) \equiv 0$, then by considering the weights we see that $P_d(f_1, f_2)$ has to vanish identically for each homogeneous component P_d of P. But $P_d(f_1, f_2)/f_2^d = p(f_1/f_2)$ for some polynomial $p(t)$ in one variable, and since p has only finitely many roots we can only have $P_d(f_1, f_2) \equiv 0$ if f_1/f_2 is a constant. It follows that E_4^3 and E_6^2, and hence also E_4 and E_6, are algebraically independent. But then an easy calculation shows that the dimension of the weight k part of the subring of $M_*(\Gamma_1)$ which they generate equals the right-hand side of the inequality (7), so that the proposition and corollary follow from this inequality.

2.2 Fourier Expansions of Eisenstein Series

Recall from (3) that any modular form on Γ_1 has a Fourier expansion of the form $\sum_{n=0}^{\infty} a_n q^n$, where $q = e^{2\pi i z}$. The coefficients a_n often contain interesting arithmetic information, and it is this that makes modular forms important for classical number theory. For the Eisenstein series, normalized by (12), the coefficients are given by:

Proposition 5. *The Fourier expansion of the Eisenstein series* $\mathbb{G}_k(z)$ *(k even, $k>2$) is*

$$\mathbb{G}_k(z) \;=\; -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n\,, \tag{13}$$

where B_k *is the* k*th Bernoulli number and where* $\sigma_{k-1}(n)$ *for* $n \in \mathbb{N}$ *denotes the sum of the* $(k-1)$*st powers of the positive divisors of* n.

We recall that the Bernoulli numbers are defined by the generating function $\sum_{k=0}^{\infty} B_k x^k/k! = x/(e^x-1)$ and that the first values of B_k ($k>0$ even) are given by $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$, and $B_{14} = \frac{7}{6}$.

Proof. A well known and easily proved identity of Euler states that

$$\sum_{n\in\mathbb{Z}} \frac{1}{z+n} \;=\; \frac{\pi}{\tan \pi z} \qquad \bigl(z \in \mathbb{C}\setminus\mathbb{Z}\bigr)\,, \tag{14}$$

where the sum on the left, which is not absolutely convergent, is to be interpreted as a Cauchy principal value ($=\lim \sum_{-M}^{N}$ where M, N tend to infinity with $M-N$ bounded). The function on the right is periodic of period 1 and its Fourier expansion for $z \in \mathfrak{H}$ is given by

$$\frac{\pi}{\tan \pi z} \;=\; \pi\,\frac{\cos \pi z}{\sin \pi z} \;=\; \pi i\,\frac{e^{\pi i z}+e^{-\pi i z}}{e^{\pi i z}-e^{-\pi i z}} \;=\; -\pi i\,\frac{1+q}{1-q} \;=\; -2\pi i \left(\frac{1}{2}+\sum_{r=1}^{\infty} q^r\right),$$

where $q = e^{2\pi i z}$. Substitute this into (14), differentiate $k-1$ times and divide by $(-1)^{k-1}(k-1)!$ to get

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} \;=\; \frac{(-1)^{k-1}}{(k-1)!}\,\frac{d^{k-1}}{dz^{k-1}}\left(\frac{\pi}{\tan \pi z}\right) \;=\; \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1}\, q^r$$
$$(k \ge 2,\ z \in \mathfrak{H})\,,$$

an identity known as Lipschitz's formula. Now the Fourier expansion of G_k ($k>2$ even) is obtained immediately by splitting up the sum in (10) into the terms with $m=0$ and those with $m \neq 0$:

$$\begin{aligned} G_k(z) \;&=\; \frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n^k} + \frac{1}{2}\sum_{\substack{m,\,n\in\mathbb{Z}\\ m\neq 0}} \frac{1}{(mz+n)^k} \;=\; \sum_{n=1}^{\infty}\frac{1}{n^k} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &=\; \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty}\sum_{r=1}^{\infty} r^{k-1}\, q^{mr} \\ &=\; \frac{(2\pi i)^k}{(k-1)!}\left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n\right), \end{aligned}$$

where in the last line we have used Euler's evaluation of $\zeta(k)$ ($k>0$ even) in terms of Bernoulli numbers. The result follows.

The first three examples of Proposition 5 are the expansions

$$\begin{aligned}
\mathbb{G}_4(z) &= \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + 252q^6 + \cdots, \\
\mathbb{G}_6(z) &= -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + \cdots, \\
\mathbb{G}_8(z) &= \frac{1}{480} + q + 129q^2 + 2188q^3 + \cdots.
\end{aligned}$$

The other two normalizations of these functions are given by

$$\begin{aligned}
G_4(z) &= \frac{16\,\pi^4}{3!}\,\mathbb{G}_4(z) = \frac{\pi^4}{90}\,E_4(z), \qquad E_4(z) = 1 + 240q + 2160q^2 + \cdots, \\
G_6(z) &= -\frac{64\,\pi^6}{5!}\,\mathbb{G}_6(z) = \frac{\pi^6}{945}\,E_6(z), \quad E_6(z) = 1 - 504q - 16632q^2 - \cdots, \\
G_8(z) &= \frac{256\,\pi^8}{7!}\,\mathbb{G}_8(z) = \frac{\pi^8}{9450}\,E_8(z), \quad E_8(z) = 1 + 480q + 61920q^2 + \cdots.
\end{aligned}$$

Remark. We have discussed only Eisenstein series on the full modular group in detail, but there are also various kinds of Eisenstein series for subgroups $\Gamma \subset \Gamma_1$. We give one example. Recall that a *Dirichlet character* modulo $N \in \mathbb{N}$ is a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$, extended to a map $\chi : \mathbb{Z} \to \mathbb{C}$ (traditionally denoted by the same letter) by setting $\chi(n)$ equal to $\chi(n \bmod N)$ if $(n, N) = 1$ and to 0 otherwise. If χ is a non-trivial Dirichlet character and k a positive integer with $\chi(-1) = (-1)^k$, then there is an Eisenstein series having the Fourier expansion

$$\mathbb{G}_{k,\chi}(z) \;=\; c_k(\chi) + \sum_{n=1}^{\infty}\Bigl(\sum_{d|n}\chi(d)\,d^{k-1}\Bigr)\,q^n$$

which is a "modular form of weight k and character χ on $\Gamma_0(N)$." (This means that $\mathbb{G}_{k,\chi}(\frac{az+b}{cz+d}) = \chi(a)(cz+d)^k\mathbb{G}_{k,\chi}(z)$ for any $z \in \mathfrak{H}$ and any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ with $c \equiv 0 \pmod N$.) Here $c_k(\chi) \in \overline{\mathbb{Q}}$ is a suitable constant, given explicitly by $c_k(\chi) = \frac{1}{2}L(1-k,\chi)$, where $L(s,\chi)$ is the analytic continuation of the Dirichlet series $\sum_{n=1}^{\infty}\chi(n)n^{-s}$.

The simplest example, for $N = 4$ and $\chi = \chi_{-4}$ the Dirichlet character modulo 4 given by

$$\chi_{-4}(n) \;=\; \begin{cases} +1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{15}$$

and $k = 1$, is the series

$$\mathbb{G}_{1,\chi_{-4}}(z) = c_1(\chi_{-4}) + \sum_{n=1}^{\infty}\Biggl(\sum_{d|n}\chi_{-4}(d)\Biggr)\,q^n = \frac{1}{4} + q + q^2 + q^4 + 2q^5 + q^8 + \cdots. \tag{16}$$

to be immeasurably deeper than the assertion about multiplicativity and was only proved in 1974 by Deligne as a consequence of his proof of the famous Weil conjectures (and of his previous, also very deep, proof that these conjectures implied Ramanujan's). However, the weaker inequality $|\tau(p)| \le Cp^6$ with some effective constant $C > 0$ is much easier and was proved in the 1930's by Hecke. We reproduce Hecke's proof, since it is simple. In fact, the proof applies to a much more general class of modular forms. Let us call a modular form on Γ_1 a *cusp form* if the constant term a_0 in the Fourier expansion (3) is zero. Since the constant term of the Eisenstein series $\mathbb{G}_k(z)$ is non-zero, any modular form can be written uniquely as a linear combination of an Eisenstein series and a cusp form of the same weight. For the former the Fourier coefficients are given by (13) and grow like n^{k-1} (since $n^{k-1} \le \sigma_{k-1}(n) < \zeta(k-1)n^{k-1}$). For the latter, we have:

Proposition 8. *Let $f(z)$ be a cusp form of weight k on Γ_1 with Fourier expansion $\sum_{n=1}^{\infty} a_n q^n$. Then $|a_n| \le Cn^{k/2}$ for all n, for some constant C depending only on f.*

Proof. From equations (1) and (2) we see that the function $z \mapsto y^{k/2}|f(z)|$ on $\mathfrak{H}$ is Γ_1-invariant. This function tends rapidly to 0 as $y = \Im(z) \to \infty$ (because $f(z) = \mathrm{O}(q)$ by assumption and $|q| = e^{-2\pi y}$), so from the form of the fundamental domain of Γ_1 as given in Proposition 1 it is clearly bounded. Thus we have the estimate

$$|f(z)| \;\le\; c\,y^{-k/2} \qquad (z \;=\; x+iy \in \mathfrak{H}) \tag{25}$$

for some $c > 0$ depending only on f. Now the integral representation

$$a_n \;=\; e^{2\pi n y} \int_0^1 f(x+iy)\, e^{-2\pi i n x}\, dx$$

for a_n, valid for any $y > 0$, show that $|a_n| \le cy^{-k/2}e^{2\pi n y}$. Taking $y = 1/n$ (or, optimally, $y = k/4\pi n$) gives the estimate of the proposition with $C = c\,e^{2\pi}$ (or, optimally, $C = c\,(4\pi e/k)^{k/2}$).

Remark. The definition of cusp forms given above is actually valid only for the full modular group Γ_1 or for other groups having only one cusp. In general one must require the vanishing of the constant term of the Fourier expansion of f, suitably defined, at every cusp of the group Γ, in which case it again follows that f can be estimated as in (25). Actually, it is easier to simply *define* cusp forms of weight k as modular forms for which $y^{k/2}f(x+iy)$ is bounded, a definition which is equivalent but does not require the explicit knowledge of the Fourier expansion of the form at every cusp.

♠ Congruences for $\tau(n)$

As a mini-application of the calculations of this and the preceding sections we prove two simple congruences for the Ramanujan tau-function defined by