Bessel functions of kt and $kt^{1/2}$ (cont'd)

	$f(t)$	$g(p)=\int_0^\infty e^{-pt} f(t)\,dt$
(25)	$J_0(2a^{1/2}t^{1/2})$	$p^{-1}e^{-a/p}$ $\quad$ $\mathrm{Re}\,p>0$
(26)	$J_\nu(2a^{1/2}t^{1/2})$ $\quad$ $\mathrm{Re}\,\nu>-2$	$\frac{1}{2}a^{1/2}\pi^{1/2}p^{-3/2}e^{-\frac{1}{2}a/p}[I_{\frac{1}{2}\nu-\frac{1}{2}}(\frac{1}{2}a/p)$ $-I_{\frac{1}{2}\nu+\frac{1}{2}}(\frac{1}{2}a/p)]$ $\quad$ $\mathrm{Re}\,p>0$
(27)	$t^{1/2}J_1(2a^{1/2}t^{1/2})$	$a^{1/2}p^{-2}e^{-a/p}$ $\quad$ $\mathrm{Re}\,p>0$
(28)	$t^{n-1/2}J_1(2a^{1/2}t^{1/2})$	$(-1)^{n-1}a^{-1/2}n!\,p^{-n}e^{-\frac{1}{2}a/p}k_{2n}(\frac{1}{2}a/p)$ $\quad$ $\mathrm{Re}\,p>0$
(29)	$t^{-1/2}J_\nu(2a^{1/2}t^{1/2})$ $\quad$ $\mathrm{Re}\,\nu>-1$	$\pi^{1/2}p^{-1/2}e^{-\frac{1}{2}a/p}I_{\frac{1}{2}\nu}(\frac{1}{2}a/p)$ $\quad$ $\mathrm{Re}\,p>0$
(30)	$t^{\frac{1}{2}\nu}J_\nu(2a^{1/2}t^{1/2})$ $\quad$ $\mathrm{Re}\,\nu>-1$	$a^{\frac{1}{2}\nu}p^{-\nu-1}e^{-a/p}$ $\quad$ $\mathrm{Re}\,p>0$
(31)	$t^{-\frac{1}{2}\nu}J_\nu(2a^{1/2}t^{1/2})$	$\dfrac{e^{i\nu\pi}p^{\nu-1}}{a^{\frac{1}{2}\nu}\Gamma(\nu)}e^{-a/p}\,\gamma\left(\nu,\dfrac{a}{p}e^{-i\pi}\right)$ $\quad$ $\mathrm{Re}\,p>0$
(32)	$t^{\frac{1}{2}\nu-1}J_\nu(2a^{1/2}t^{1/2})$ $\quad$ $\mathrm{Re}\,\nu>0$	$a^{-\frac{1}{2}\nu}\gamma(\nu,\ a/p)$ $\quad$ $\mathrm{Re}\,p>0$
(33)	$t^{\frac{1}{2}\nu+n}J_\nu(2a^{1/2}t^{1/2})$ $\quad$ $\mathrm{Re}\,\nu+n>-1$	$n!\,a^{\frac{1}{2}\nu}p^{-n-\nu-1}e^{-a/p}L_n^\nu(a/p)$ $\quad$ $\mathrm{Re}\,p>0$
(34)	$t^{\mu-1/2}J_{2\nu}(2a^{1/2}t^{1/2})$ $\quad$ $\mathrm{Re}(\mu+\nu)>-\frac{1}{2}$	$\dfrac{\Gamma(\mu+\nu+\frac{1}{2})}{a^{1/2}\Gamma(2\nu+1)}p^{-\mu}e^{-\frac{1}{2}a/p}M_{\mu,\nu}(a/p)$ $\quad$ $\mathrm{Re}\,p>0$

Theorem 2.10. *If $f \not\equiv 0$ is a meromorphic modular form of weight $k \in \mathbb{Z}$, we have with $\rho := e^{\frac{2\pi i}{3}}$*

$$\operatorname{ord}(f;\infty) + \frac{1}{2}\operatorname{ord}(f;i) + \frac{1}{3}\operatorname{ord}(f;\rho) + \sum_{\substack{z \in \Gamma\backslash\mathbb{H} \\ z \not\equiv i,\rho \pmod{\Gamma}}} \operatorname{ord}(f;z) = \frac{k}{12},$$

where $\operatorname{ord}(f;\infty) = n_0$ *if* $f(\tau) = \sum_{n=n_0}^{\infty} a(n)q^n$ *with* $a(n_0) \neq 0$.

One can conclude from Theorem 2.10 the following dimension formulas, the proof is omitted here.

Corollary 2.11. *For $k \in \mathbb{N}_0$, we have*

$$\dim M_{2k} = \begin{cases} \left\lfloor \frac{k}{6} \right\rfloor + 1 & \text{if } k \not\equiv 1 \pmod{6}, \\ \left\lfloor \frac{k}{6} \right\rfloor & \text{if } k \equiv 1 \pmod{6}, \end{cases}$$

$$\dim S_k = \begin{cases} \left\lfloor \frac{k}{6} \right\rfloor & \text{if } k \not\equiv 1 \pmod{6}, \\ \left\lfloor \frac{k}{6} \right\rfloor - 1 & \text{if } k \equiv 1 \pmod{6}, k \geq 7, \\ 0 & \text{if } k = 1. \end{cases}$$

This easily gives the basis property of the Poincaré series.

Corollary 2.12. *(Completeness theorem) Let $k \geq 4$ even and $d_k := \dim(S_k)$. Then a basis of S_k is given by*

$$\{P_{k,n}; n = 1, \ldots, d_k\}.$$

In particular, M_k has a basis consisting of Eisenstein series and Poincaré series.

Proof. Set $S := \operatorname{span}\{P_{k,1}, \ldots, P_{k,d_k}\} \subset S_k$ and $f \in S_k$ such that $f \perp S$ with respect to the Petersson inner product. Then f has a Fourier expansion of the shape $f(\tau) = \sum_{m \geq d_k+1} a(m)e^{2\pi i m\tau}$. From Corollary 2.11 we know the precise shape of d_k, yielding a contradiction to Theorem 2.10. To be more precise, we have for $k \not\equiv 2 \pmod{12}$

$$d_k + 1 = \left\lfloor \frac{k}{12} \right\rfloor + 1 > \frac{k}{12} = \operatorname{ord}(f;\infty) + \sum_{z \in \Gamma\backslash\mathbb{H}} \operatorname{ord}(f;z) \geq d_k + 1.$$

For $k \equiv 2 \pmod{12}$, we get

$$\frac{k}{12} = d_k + 1 + \frac{1}{6}.$$

Thus $\mathrm{ord}(f;\infty)=d_k+1$ and

$$\sum_{z\in\Gamma\backslash\mathbb{H}}\mathrm{ord}(f;z)=\frac{1}{6}$$

which is impossible. $\square$

We next compute the Fourier coefficients $a_n(m)$ of the Poincaré series $P_{k,n}$. For this, we require some special functions. Define the *Kloosterman sums*

$$S(m,n;c):=\sum_{a \pmod{c}^\star} e^{2\pi i\frac{am+\overline{a}n}{c}},$$

where the sum runs over all $a \pmod{c}$ that are coprime to c and $\overline{a}$ denotes the multiplicative inverse of $a \pmod{c}$. Moreover, we let J_r be the J-Bessel function of order r, defined by

$$J_r(x):=\sum_{\ell\geq 0}\frac{(-1)^\ell}{\ell!\Gamma(\ell+1+r)}\left(\frac{x}{2}\right)^{r+2\ell},$$

where $\Gamma(x)$ denotes the usual gamma-function.

Theorem 2.13. *We have for $n\in\mathbb{N}$*

$$a_n(m)=\left(\frac{m}{n}\right)^{\frac{k-1}{2}}\left(\delta_{m,n}+2\pi i^{-k}\sum_{c\geq 1}c^{-1}S(n,m;c)\,J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right), \tag{2.5}$$

where

$$\delta_{m,n}:=\begin{cases}1 & \text{if } m=n,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We again use that a set of representatives of $\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\left\{\begin{pmatrix}\star & \star\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z});(c,d)=1\right\}.$$

The contribution for $c=0$ is easily seen to give the first summand in (2.5). For $c\neq 0$ we use the identity

$$\frac{a\tau+b}{c\tau+d}=\frac{a}{c}-\frac{1}{c^2\left(\tau+\frac{d}{c}\right)}$$

and change $d\mapsto d+mc$, where d runs $\pmod{c}^\star$ and $m\in\mathbb{Z}$. This gives

$$P_{k,n}(\tau)=e^{2\pi i n\tau}+2\sum_{c\geq 1}c^{-k}\sum_{d \pmod{c}^\star}e^{\frac{2\pi i n a}{c}}\mathcal{F}\left(\tau+\frac{d}{c}\right),$$

where a is defined by $ad \equiv 1 \pmod c$ and

$$\mathcal{F}(\tau) := \sum_{m\in\mathbb{Z}} e^{-\frac{2\pi i n}{c^2(\tau+m)}} (\tau+m)^{-k}.$$

Now the classical Poisson summation formula yields that

$$\mathcal{F}(\tau) = \sum_{m\in\mathbb{Z}} a(m) e^{2\pi i m\tau}$$

with

$$a(m) = \int_{\mathrm{Im}(\tau)=\mathcal{C}} \tau^{-k} e^{-\frac{2\pi i n}{c^2\tau} - 2\pi i m\tau} d\tau$$

with $\mathcal{C} > 0$ arbitrary. For $m \leq 0$ we can deform the path of integration up to infinity yielding that $a(m) = 0$ in this case. For $m > 0$ we make the substitution $\tau = ic^{-1}(n/m)^{\frac{1}{2}} w$ to get

$$a(m) = i^{-k-1} c^{k-1} \left(\frac{m}{n}\right)^{\frac{k}{2}-\frac{1}{2}} \int_{\mathcal{C}-i\infty}^{\mathcal{C}+i\infty} w^{-k} e^{\frac{2\pi}{c}\sqrt{mn}\left(w-w^{-1}\right)} dw.$$

The claim follows using that for μ, $\kappa > 0$ the functions

$$t \mapsto (t/\kappa)^{\frac{\mu-1}{2}} J_{\mu-1}\left(2\sqrt{\kappa t}\right), \quad (t > 0)$$

and

$$w \mapsto w^{-\mu} e^{-\frac{\kappa}{w}}, \quad (\mathrm{Re}(w) > 0)$$

are inverses of each other with respect to the usual Laplace transform (8.412.2 of [23]). □

3. Weakly holomorphic modular forms

We next turn to *weakly holomorphic* modular forms which are still holomorphic on $\mathbb{H}$ but allow poles at the "cusps". The Fourier coefficients of such forms are growing much faster than those of holomorphic forms. Let us in particular describe this in the situation of the partition function.

Recall that a *partition* of a positive integer n is a nondecreasing sequence of positive integers (the *parts* of the partition) whose sum is n. Let $p(n)$ denote the number of partitions of n. For example, the partitions of 4 are

$$4 \qquad 3+1 \qquad 2+2 \qquad 2+1+1 \qquad 1+1+1+1$$