THE CIRCLE METHOD, THE j FUNCTION, AND PARTITIONS

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1. Introduction

The circle method was first used by Hardy and Ramanujan [2] in 1918 to investigate the asymptotic growth of the partition function, which counts the number of ways to write n as a sum of positive integers. They exploited the fact that the generating function for the partition function is almost the Dedekind eta function, which is related to integral weight modular forms. The key step is using the transformation law for the eta function to rewrite their integral over a circle into something more tractable. In 1938, Rademacher published a strengthening of their method which allowed him to obtain a representation of $p(n)$ as a convergent series [4]. He also generalized it so it could deal with the coefficients of the $q-$expansions of other modular forms like the j function [3]. In 1943, he discovered a different contour of integration involving Ford circles that simplified the analysis of the partition function [5] and has become the standard account presented in textbooks like Apostol [1]. This paper will show two examples of using variants of the circle method to produce series representations for the partition function and for the coefficients $c(n)$ of the j function. I will first review background about the eta and j functions and then construct the required contours of integration in terms of the Farey sequence and Ford circles. Then I will review some analytic facts involving Bessel functions and Kloosterman sums. After that I will derive the series representations for $p(n)$ and $c(n)$.

Throughout, a summation $\sum_{h,k}{}'$ will denote a sum over relatively prime h and k.

2. Preliminaries on the eta and j Functions

First we define the Dedekind eta function. Background can be found in Apostol [1].

Definition 1. *The eta function is defined on the upper half plane $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ to be*

$$\eta(z) := e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i z n}). \tag{2.1}$$

It satisfies a transformational law very similar to that of an integral weight modular form.

Theorem 2. *For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$ and $z \in H$,*

$$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon(a,b,c,d)(-i(cz+d))^{1/2}\eta(z) \tag{2.2}$$

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where $\epsilon(a,b,c,d) := e^{\pi i(\frac{a+d}{12c}+s(-d,c))}$ *and* $s(h,k)$ *is the Dedekind sum* $\displaystyle\sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right)$.

The j function is defined in terms of Eisenstein series (again, see Apostol [1]).

Definition 3. *The j function (also called the Klein j-invariant) can be defined in terms of Eisenstein series E_k of weight k to be*

$$j(z) = 12^3 \frac{E_4(z)^3}{E_4^3(z) - 27E_6^2(z)} \tag{2.3}$$

for $z \in H$.

Remark 4. *This is normalized using the convention that the coefficient of q^{-1} in the $q-$expansion will be 1. In Apostol's book [1] and Rademacher's paper [3] the 12^3 is omitted from the definition of the j function but added in later formula.*

It is a modular function for $\mathrm{SL}_2(\mathbb{Z})$ in the sense that $j(z) = j(\frac{-1}{z})$ and $j(z) = j(z+1)$. This follows from the fact that the Eisenstein series of weight k is a modular forms for $\mathrm{SL}_2(\mathbb{Z})$. Because $E_4^3(z) - 27E_6^2(z)$ is a cusp form while E_4 is not, $j(z)$ has a simple pole at the cusp. In particular, letting $q = e^{2\pi iz}$, $j(z)$ can be written as

$$j(z) = \sum_{n=-1}^{\infty} c(n)q^n. \tag{2.4}$$

Remark 5. *The denominator $E_4^3(z) - 27E_6^2(z)$ is the modular discriminant $\Delta(z)$, which can also be expressed as $(2\pi)^{12}\eta(z)^{24}$. Combining this with the definition of the Eisenstein series shows the coefficients $c(n)$ are integers.*

Rademacher represented the $c(n)$ in terms of infinite series using a variant of the circle method. This will be proven in Section 5.

Theorem 6. *In the Fourier expansion for the modular function*

$$j(z) = c(-1)q^{-1} + c(0) + \sum_{n=1}^{\infty} c(n)q^n$$

where $q = e^{2\pi iz}$, the coefficient $c(n)$ for $n \geq 1$ is given by the convergent series

$$c(n) = \frac{2\pi}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_1\left(\frac{4\pi\sqrt{n}}{k}\right) \tag{2.5}$$

where I_1 is a Bessel function that will appear in Definition 26. $A_k(n)$ is the series

$$A_k(n) = \sum_{h \bmod k}{}^{'} e^{-\frac{2\pi i}{k}(nh+h')}$$

with h' defined to be an integer for which $hh' = -1 \mod k$.

Remark 7. *Although the convergent series does not apply for $n = -1, 0$, by manipulating power series in the definition of the j function it follows that $c(-1) = 1$ and $c(0) = 744$.*

One can represent the coefficients of $\eta(z)$ in terms of a similar series. This will be proven in Section 6.

Theorem 8. *Write the $q-$series for $\frac{e^{2\pi iz/24}}{\eta(z)}$ with $q = e^{2\pi iz}$ as*

$$\frac{q^{\frac{1}{24}}}{\eta(z)} = \sum_{n=1}^{\infty} p(n)q^n.$$

Let $B_k(n) := \sum'_{0\le h<k} e^{\pi is(h,k)-2\pi inh/k}$. Then the coefficient $p(n)$ is given by the convergent series

$$p(n) = \frac{1}{\sqrt{2}\pi}\sum_{k=1}^{\infty} B_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(n-\frac{1}{24})}\right)}{\sqrt{n-\frac{1}{24}}}\right)$$

Remark 9. *The argument will actually give an error term for summing N terms of this series. As soon as the error term is less than $\frac{1}{2}$, the knowledge that $p(n)$ is an integer gives a way to compute $p(n)$. This is a significant improvement over trying to compute $p(n)$ by counting partitions. In fact, because the function $p(n)$ satisfies various congruence identities, it is usually possible to tolerate a much larger error and sum even fewer terms.*

3. The Farey Sequence and Rademacher's Path of Integration

The subdivision of the circle which Rademacher used in his early work can be expressed in terms of the Farey sequence, a sequence of fractions arising in elementary number theory. Later, Rademacher constructed his refined contour of integration in terms of Ford circles, which are also defined in terms of the Farey sequence.

3.1. **The Farey Sequence.**

Definition 10. *The Farey sequence of order n, denoted by F_n, is the increasing sequence of reduced fractions in $[0, 1]$ with denominator less than or equal to n.*

Example 11. *The Farey sequence of order 7 is*

$$0, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, 1$$

When passing from F_n to F_{n+m}, new terms are added to the Farey sequence. If $\frac{a}{b} < \frac{c}{d}$, then their mediant is defined to be $\frac{a+c}{b+d}$. Elementary algebra shows it lies between $\frac{a}{b}$ and $\frac{c}{d}$. It is also easy to tell when certain fractions are consecutive.

Proposition 12. *Suppose $0 \le \frac{a}{b} < \frac{c}{d} \le 1$ with $bc - ad = 1$. $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms in F_n when n satisfies*

$$\max(b, d) \le n \le b + d - 1.$$

Proof. Since $bc - ad = 1$, the fractions are in lowest terms. If $\max(b, d) \le n$, then $\frac{a}{c}$ and $\frac{b}{d}$ are both in F_n. We need to show they are consecutive if $n \le b + d - 1$. Suppose there is a reduced fraction $\frac{h}{k}$ lying strictly between them. But then we have

$$k = (bc - ad)k = b(ck - dh) + d(bh - ak) \tag{3.1}$$

and since $\frac{a}{c} < \frac{h}{k} < \frac{c}{d}$ implies that $ck - dh \ge 1$ and $bh - ak \ge 1$, we conclude $k \ge b + d$. Thus $\frac{a}{c}$ and $\frac{b}{d}$ are consecutive. □

Also observe that if $bd - ad = 1$ and $\frac{h}{k}$ is the mediant, it lies between them so $bh - ak \geq 1$ and $ck - dh \geq 1$. But using (3.1), $k = b + d$ if and only if $bh - ak = 1$ and $ck - dh = 1$. Following Apostol, say the fractions $\frac{a}{b} < \frac{h}{k}$ satisfy the uni-modular relation if $bh - ak = 1$.

This gives enough information to mechanically construct F_{n+1} from F_n.

Theorem 13. *The sequence F_{n+1} includes the sequence F_n. Each fraction which is not in F_n is the mediant of consecutive terms of F_n. If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in any F_n, they satisfy the uni-modular relation.*

Proof. The proof proceeds by induction on n. For $n = 1$, the Farey sequence is $0, 1$ while F_2 is $0, \frac{1}{2}, 1$. The statement is certainly satisfied. Now suppose that $\frac{a}{b}$ and $\frac{c}{d}$ satisfy the uni-modular relation and are consecutive in F_n. They are consecutive whenever

$$\max(b, d) \leq m \leq b + d - 1$$

by applying Proposition 12. Consider their mediant $\frac{h}{k}$. We know $\frac{h}{k}$ satisfies the uni-modular relation with $\frac{a}{b}$ and with $\frac{c}{d}$. This implies that h and k are relatively prime. Furthermore, $\frac{a}{b}$ and $\frac{c}{d}$ are not consecutive in $F_k = F_{b+d}$, but $\frac{a}{b}$ and $\frac{h}{k}$ are consecutive (and likewise $\frac{h}{k}$ and $\frac{c}{d}$) because $\max(b, k) = k$. This shows that in passing from F_n to F_{n+1} each new fraction inserted is a mediant of a consecutive pair in F_n and each consecutive pair satisfies the uni-modular relation. This completes the induction. □

This is enough theory to see how the Farey sequence of order N allows the subdivision of the circle of radius $e^{-2\pi N^{-2}}$ as done in Rademacher's original work [3] and [4]. Both $\frac{1}{\eta(z)}$ and $j(z)$ tend to infinity as z approaches the cusps for $\mathrm{SL}_2(\mathbb{Z})$. These correspond to $q = e^{ir}$ for r a rational number, points on the boundary of the unit circle. The cusps that turn out to contribute the most to the behavior of these functions are those with small denominators, which the Farey sequence represents.

Definition 14. *In the Farey sequence of order N, let the fraction $\frac{h}{k}$ lie between $\frac{a}{b}$ and $\frac{c}{d}$. Let $\xi_{h,k}$ be the image of $(\frac{a+h}{b+k}, \frac{c+h}{d+k})$ under the map $\theta \to e^{-2\pi N^{-2} + 2\pi i \theta}$. (Let $\xi_{0,1}$ be the image of $(-\frac{1}{N+1}, \frac{1}{N+1})$.) Define $\vartheta'_{h,k} = \frac{1}{k(b+k)}$ and $\vartheta''_{h,k} = \frac{1}{k(d+k)}$.*

The arcs $\xi_{h,k}$ are a division of the circle of radius $e^{-2\pi N^{-2}}$ which will let us focus attention on the most important cusps. Note that $\frac{h}{k} - \frac{a+h}{b+k} = \frac{h(b+k) - ak - hk}{k(b+k)} = \vartheta'_{h,k}$ and $\vartheta''_{h,k} = \frac{c+h}{d+k} - \frac{h}{k}$. Thus $\vartheta'_{h,k}$ reflects the size of the arc between the mediant of $\frac{a}{b}$ and $\frac{h}{k}$.

3.2. Ford Circles.

The Ford circles can be defined in terms of the Farey sequence, and will lead to Rademacher's improved contour of integration for the partition function.

Definition 15. *Given a rational number $\frac{h}{k}$ with $(h, k) = 1$, the Ford circle $C(h, k)$ is the circle centered at $\frac{h}{k} + \frac{i}{2k^2}$ with radius $\frac{1}{2k^2}$.*

Proposition 16. *Two Ford circles $C(a, b)$ and $C(c, d)$ are either tangent or disjoint. They are tangent if and only if $bc - ad = \pm 1$.*

Corollary 17. *Ford circles of consecutive Farey fractions are tangent.*

Proof. By the Pythagorean theorem, the distance between the centers of the circles is

$$\left(\frac{a}{b} - \frac{c}{d}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2.$$

The square of the sum of the radii is $\left(\frac{1}{2b^2}+\frac{1}{2d^2}\right)^2$. The difference between these two quantities is

$$\begin{aligned}\left(\frac{a}{b}-\frac{c}{d}\right)^2+\left(\frac{1}{2b^2}-\frac{1}{2d^2}\right)^2-\left(\frac{1}{2b^2}+\frac{1}{2d^2}\right)^2&=\left(\frac{ad-bc}{bd}\right)^2-\frac{1}{b^2}\frac{1}{d^2}\\&=\frac{(ad-bc)^2-1}{b^2d^2}\geq 0.\end{aligned}$$

Equality holds if and only if $ad-bc=\pm 1$, and equality corresponds to the circles being tangent. □

Before proceeding, we need a lemma from plane geometry.

Lemma 18. *A point C lies on the circle with diameter $\overline{AB}$ if and only if $CD^2 = AD\cdot DB$, where D is the intersection of $\overline{AB}$ with the perpendicular to $\overline{AB}$ through C.*

Proof. Pick coordinates so the center of the circle is $(0,0)$ and $A=(-r,0)$ and $B=(r,0)$. Then $C=(x,y)$ lies on the circle if and only if $x^2+y^2=r^2$. But $D=(x,0)$, so $CD=y$, $AD=(x+r)$ and $DB=(r-x)$, so $CD^2=AD\cdot DB$ if and only if $y^2=r^2-x^2$ □

Next we consider how the Ford circles corresponding to three consecutive Farey fractions fit together.

Theorem 19. *Let $\frac{a}{b}<\frac{h}{k}<\frac{c}{d}$ be three consecutive Farey fractions. The points of tangency of $C(h,k)$ with $C(a,b)$ and $C(c,d)$ are given the points*

$$\alpha_{h,k}=\frac{h}{k}-\frac{b}{k(k^2+b^2)}+\frac{i}{k^2+b^2}$$

and

$$\alpha'_{h,k}=\frac{h}{k}+\frac{d}{k(k^2+d^2)}+\frac{i}{k^2+d^2}.$$

Furthermore, $\alpha_{h,k}$ lies on the semicircle whose diameter is the interval $[\frac{a}{b},\frac{h}{k}]$ on the real line.

Proof. This is an exercise in plane geometry. $\alpha_{h,k}$ divides the line segment L joining the center z_1 of $C(a,b)$ with the center z_2 of $C(h,k)$ into segments of length $\frac{1}{2b^2}$ and $\frac{1}{2k^2}$. Thus

$$\alpha_{h,k}=\frac{\frac{1}{2k^2}}{\frac{1}{2b^2}+\frac{1}{2k^2}}z_1+\frac{\frac{1}{2b^2}}{\frac{1}{2b^2}+\frac{1}{2k^2}}z_2$$

By definition $z_1=\frac{a}{b}+i\frac{1}{2b^2}$ and $z_2=\frac{h}{k}+i\frac{1}{2k^2}$. Thus

$$\begin{aligned}\alpha_{h,k}&=\frac{1}{\frac{1}{2b^2}+\frac{1}{2k^2}}\left(\frac{\frac{a}{b}+i\frac{1}{2b^2}}{2k^2}+\frac{\frac{h}{k}+i\frac{1}{2k^2}}{2b^2}\right)\\&=\frac{1}{\frac{1}{2b^2}+\frac{1}{2k^2}}\left(\frac{ab+\frac{i}{2}+hk+\frac{i}{2}}{2b^2k^2}\right)\\&=\frac{1}{b^2+k^2}\left(ab+hk+i\right)\\&=\frac{h}{k}-\frac{b}{k(k^2+b^2)}+\frac{i}{b^2+k^2}\end{aligned}$$

where the last step uses that $bh - ak = 1$. The formula for $\alpha'_{h,k}$ is obtained in a similar manner.

To check that $\alpha_{h,k}$ is on the semicircle, we simply use the previous lemma. The length of the perpendicular is the imaginary part of $\alpha_{h,k}$, $\frac{1}{b^2+k^2}$. It divides the segment from $\frac{a}{b}$ to $\frac{h}{k}$ into two segments of length $\frac{b}{k(k^2+b^2)}$ and $\frac{h}{k} - \frac{a}{b} - \frac{b}{k(k^2+b^2)}$. Then we have

$$\begin{aligned}
\frac{b}{k(k^2+b^2)}\left(\frac{h}{k} - \frac{a}{b} - \frac{b}{k(k^2+b^2)}\right) &= \frac{b}{k(k^2+b^2)}\left(\frac{1}{bk} - \frac{b}{k(k^2+b^2)}\right) \\
&= \frac{b}{k^2(k^2+b^2)}\left(\frac{(k^2+b^2)-b^2}{b(k^2+b^2)}\right) \\
&= \left(\frac{1}{b^2+k^2}\right)^2
\end{aligned}$$

as desired to show $\alpha_{h,k}$ lies on the semicircle. □

3.3. **Rademacher's Contour of Integration.** Given this description of the Ford circles, we can construct a path of integration for every integer N. It will be a path connecting the points i and $i+1$. [1]

Definition 20. *Consider the Ford cycles of the Farey sequence F_N. If $\frac{a}{b}$, $\frac{h}{k}$, and $\frac{c}{d}$ are consecutive the point of tangency between $C(a,b)$ and $C(h,k)$ and between $C(h,k)$ and $C(c,d)$ divide $C(h,k)$ into two arcs. $P(N)$ is the union of the arc with larger imaginary parts. For the circles $C(0,1)$ and $C(1,1)$ use only the part of the upper arc with real part between 0 and 1, and consider them as part of the same arc associated to the point $\frac{0}{1}$.*

Remark 21. *Like the Farey arcs on the circle of radius $e^{-2\pi N^{-2}}$, this contour is skirting the points $\frac{h}{k}$ in the Farey sequence of order N. They correspond to cusps on the boundary of the unit circle under the map $x = e^{2\pi i\tau}$. While Rademacher's contour is more complicated to describe, it greatly simplifies the analysis for the partition function.*

It is also useful to understand this path under the transformation $z = -ik^2(\tau - \frac{h}{k})$.

Proposition 22. *This sends the circle $C(h,k)$ to a circle K of radius $\frac{1}{2}$ around the point $z = \frac{1}{2}$. The points of contact $\alpha(h,k)$ and $\alpha'(h,k)$ from Proposition 19 are send to*

$$\begin{aligned}
z_1(h,k) &:= \frac{k^2}{k^2+b^2} + i\frac{kb}{k^2+b^2} \\
z_2(h,k) &:= \frac{k^2}{k^2+d^2} - i\frac{kd}{k^2+d^2}.
\end{aligned}$$

The upper arc joining $\alpha(h,k)$ and $\alpha'(h,k)$ corresponds to the arc not touching the imaginary $z-$axis.

Proof. The translation $\tau - \frac{h}{k}$ translates $C(h,k)$ so its center is at $\frac{i}{2k^2}$. Multiplication by $-ik^2$ expands the radius to $\frac{1}{2}$ and rotates the $z-$plane 90 degrees clockwise. This results in a circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}$. The formulas for the z_i are just substitution. □

It will be easier to analyze integrals over the upper arcs in $C(h,k)$ by converting them to integrals over this circle. To this end, the following bounds will be useful.

[1] There is an illuminating picture of the contour in the Wikipedia article on the Hardy-Littlewood circle method.

Proposition 23. *With the notation of the previous proposition,*

$$|z_1(h,k)| = \frac{k}{\sqrt{k^2+b^2}} \quad \text{and} \quad |z_2(h,k)| = \frac{k}{\sqrt{k^2+d^2}}$$

Moreover, all points z on the chord joining $z_1(h,k)$ and $z_2(h,k)$ satisfy

$$|z| < \frac{\sqrt{2}k}{N}$$

provided $\frac{a}{b}$, $\frac{h}{k}$, and $\frac{c}{d}$ are consecutive in F_N. The length of the chord is at most $2\sqrt{2}\frac{k}{N}$.

Proof. Given the formulas for $z_1(h,k)$ and $z_2(h,k)$ in Proposition 22, the first assertion is clear. To prove the second, note that any point on the chord is $z = sz_1(h,k) + tz_2(h,k)$ for s and t non-negative real numbers with $s+t=1$. Thus $|z| \le \max(|z_1(h,k)|, |z_2(h,k)|)$. However, as

$$0 \le \frac{k^2+b^2}{2} - \left(\frac{k+b}{2}\right)^2$$

we know that

$$(k^2+b^2)^{\frac{1}{2}} \ge \frac{k+b}{\sqrt{2}} \ge \frac{N+1}{\sqrt{2}} > \frac{N}{\sqrt{2}}$$

using Proposition 12. Combined with the formula for $|z_1(h,k)|$ and the same argument for $z_2(h,k)$ this gives that $|z| \le \frac{\sqrt{2}k}{N}$ as desired. The length of the chord is $|z_1(h,k) - z_2(h,k)| \le |z_1(h,k)| + |z_2(h,k)| \le \frac{2\sqrt{2}k}{N}$. $\square$

4. Bessel Functions and Kloosterman Sums

The first order of business is to state the relevant facts about Bessel functions.

Definition 24. *The Bessel function of the first kind J_a is defined by the power series*

$$J_a(z) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+a+1)} \left(\frac{x}{2}\right)^{2m+a}.$$

Remark 25. *If a is a positive integer, $\Gamma(m+a+1)$ is just $(m+a)!$. This function arises as a solution to the differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-a^2)y = 0.$$

The Bessel functions appearing later are the imaginary (or modified) Bessel functions of the first kind.

Definition 26. *The imaginary Bessel function of the first kind I_a is defined to be $i^{-a}J_a(iz)$.*

This has series expansion

$$I_a(z) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+a+1)} \left(\frac{x}{2}\right)^{2m+a}. \tag{4.1}$$

It can also be represented in various ways as integrals. The relevant one is that

$$I_v(z) = \frac{(\frac{1}{2}v)^v}{2\pi i} \int_{c-\infty i}^{c+\infty i} t^{-v-1} e^{t+z^2/4t} dt \tag{4.2}$$

for $c > 0$, $\mathrm{Re}(v) > 0$.

Finally, there are sometimes elementary expressions for Bessel functions when a is a half integer. The relevant fact is that

$$I_{\frac{3}{2}}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz}\left(\frac{\sinh z}{z}\right). \tag{4.3}$$

All of these facts were culled from the literature on Bessel functions. Apostol used Watson's book on Bessel functions [6]. The only obscure fact is the integral formula, which Apostol found on page 181 with a slightly different but equivalent path of integration.

The other necessary preliminary is nontrivial bounds on Kloosterman sums.

Definition 27. *For integers a, b and m, define*

$$K_m(a,b) := \sum_{x \in (\mathbb{Z}/m\mathbb{Z})^\times} e^{2\pi i (\frac{ax+bx'}{m})}$$

where x' denotes a lift to $\mathbb{Z}$ of the inverse of x in $(\mathbb{Z}/m\mathbb{Z})^\times$.

There are different ways to obtain estimates on $K_m(a,b)$. The following special case is sufficient and was known before the Weil bound was proven. Rademacher attributes it to Salié and Davenport. Define $A_k(n) := K_k(-n, 1)$: then we have the bound

$$|A_k(n)| \leq C k^{\frac{2}{3}+\epsilon} (k,n)^{\frac{1}{3}}. \tag{4.4}$$

Rademacher also uses an extension of this to incomplete Kloosterman sums. The footnote on page 507 [3] explains where to look to derive the same estimate over shorter intervals.

5. The Convergent Series for the j Function

The proof of Theorem 6 will need a number of lemmas. They will be proven after they are used to deduce the main theorem. The starting point is to express the coefficients of the j function in terms of integrals.

Lemma 28. *Let $f(e^{2\pi i \tau}) = j(\tau)$. For $n > 0$, we have*

$$c(n) = \sum_{0 \leq h < k \leq N}{}' e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f(e^{\frac{2\pi i h}{k} - 2\pi(N^{-2} - i\phi)}) e^{2\pi n (N^{-2} - i\phi)} d\phi \tag{5.1}$$

where $\vartheta'_{h,k}$ and $\vartheta''_{h,k}$ are the boundary points of the Farey arcs defined in Definition 14. Also, recall that $\sum'$ denotes the sum over relatively prime h and k.

This can be split up into two pieces corresponding to the q^{-1} and $\sum_{n=0}^\infty c(n) q^n$ parts of the q-expansion of $j(z)$.

Lemma 29. *Let w denote $N^{-2} - i\phi$, and define $D(x) = \sum_{m=0}^{\infty} c(m) x^m := f(x) - x^{-1}$. We have that $c(n) = Q(n) + R(n)$ with*

$$Q(n) := \sum_{0 \leq h < k \leq N}{}' e^{-\frac{2\pi i}{k}(nh + h')} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi}{k^2 w} + 2\pi n w} d\phi \tag{5.2}$$

$$R(n) := \sum_{0 \leq h < k \leq N}{}' e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} D(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{k^2 w}}) e^{2\pi n w} d\phi \tag{5.3}$$

where h' denotes an integer with $hh' = -1 \mod k$.

Now divide the contour of integration into three parts between the points

$$-\vartheta'_{h,k} = -\frac{1}{k(b+k)} \leq -\frac{1}{k(N+k)} < \frac{1}{k(N+k)} \leq \frac{1}{k(d+k)} = \vartheta''_{h,k}$$

where the $\frac{a}{b}$ and $\frac{c}{d}$ are the adjacent fractions in the Farey sequence of order N. Define

$$\begin{aligned} Q_0(n) &:= \sum_{k=1}^{N} \sum_{h \bmod k}{}' e^{-\frac{2\pi i}{k}(nh+h')} \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} e^{\frac{2\pi}{k^2 w}+2\pi n w} d\phi \\ Q_1(n) &:= \sum_{k=1}^{N} \sum_{h \bmod k}{}' e^{-\frac{2\pi i}{k}(nh+h')} \int_{-\frac{1}{k(b+k)}}^{-\frac{1}{k(N+k)}} e^{\frac{2\pi}{k^2 w}+2\pi n w} d\phi \\ Q_2(n) &:= \sum_{k=1}^{N} \sum_{h \bmod k}{}' e^{-\frac{2\pi i}{k}(nh+h')} \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(d+k)}} e^{\frac{2\pi}{k^2 w}+2\pi n w} d\phi \end{aligned} \tag{5.4}$$

Then $Q(n) = Q_0(n) + Q_1(n) + Q_2(n)$. The first step is to estimate $Q_0(n)$.

Lemma 30. *Letting $A_k(n) = \sum_{h \bmod k}{}' e^{-\frac{2\pi i}{k}(nh+h')}$ and I_1 denote the Bessel function of the first order with purely imaginary argument (equation 4.1). Then*

$$Q_0(n) = \frac{2\pi}{\sqrt{n}} \sum_{k=1}^{N} \frac{A_k(n)}{k} I_1\left(\frac{4\pi\sqrt{n}}{k}\right) + O(e^{2\pi n N^{-2}} n^{\frac{1}{3}} N^{-\frac{1}{3}+\epsilon}). \tag{5.5}$$

Next both $Q_1(n)$ and $Q_2(n)$ can be analyzed together.

Lemma 31. *We have that $Q_1(n)$ and $Q_2(n)$ are $O(e^{2\pi n N^{-2}} n^{\frac{1}{3}} N^{-\frac{1}{3}+\epsilon})$.*

The last step is to analyze $R(n)$.

Lemma 32. *We have that $R(n) = O(e^{2\pi n N^{-2}} n^{\frac{1}{3}} N^{-\frac{1}{3}+\epsilon})$.*

These estimates give the proof of Theorem 6.

Proof. Using Lemmas 30, 31, and 32 in the equation

$$c(n) = Q(n) + R(n) = Q_0(n) + Q_1(n) + Q_2(n) + R(n)$$

from Lemma 29 gives

$$c(n) = \frac{2\pi}{\sqrt{n}} \sum_{k=1}^{N} \frac{A_k(n)}{k} I_1\left(\frac{4\pi\sqrt{n}}{k}\right) + O(e^{2\pi n N^{-2}} n^{\frac{1}{3}} N^{-\frac{1}{3}+\epsilon}). \tag{5.6}$$

For a fixed $n > 0$, letting N go to infinity makes the error term goes to zero. This establishes Theorem 6. □

5.1. **Proof of Lemma 28.** Cauchy's formula says that

$$c(n) = \frac{1}{2\pi i}\int_{C_N} \frac{f(x)}{x^{n+1}}dx$$

where C_N is the circle of radius $e^{-2\pi N^{-2}} < 1$ centered at the origin. The union of the disjoint Farey arcs $\xi_{h,k}$ for $\frac{h}{k}$ in the Farey sequence of order N is exactly this circle. Thus we have

$$c(n) = \sum_{0\leq h<k\leq N}{}' \frac{1}{2\pi i}\sum_{\xi_{h,k}} \frac{f(x)}{x^{n+1}}dx$$

In terms of the variable ϕ on $\xi_{h,k}$ defined by $x = e^{-2\pi N^{-2}+\frac{2\pi i h}{k}+2\pi i\phi}$ (arc length centered at $e^{2\pi i\frac{h}{k}}$), the integral becomes

$$c(n) = \sum_{0\leq h<k\leq N}{}' e^{-\frac{2\pi i n h}{k}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f\left(e^{\frac{2\pi i h}{k}-2\pi(N^{-2}-i\phi)}\right)e^{2\pi n(N^{-2}-i\phi)}d\phi \tag{5.7}$$

as $dx = x\cdot 2\pi i\cdot d\phi$.

5.2. **Proof of Lemma 29.** Since $j(\dfrac{a\tau+b}{c\tau+d}) = j(\tau)$ for τ in the upper half plane and $\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, taking

$$\tau = \frac{iz}{k}+\frac{h}{k} \quad\text{and}\quad \begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} h' & -\frac{hh'+1}{k}\\ k & -h\end{pmatrix}$$

gives $j(\dfrac{iz}{k}+\dfrac{h}{k}) = j(\dfrac{i}{kz}+\dfrac{h'}{k})$ and hence

$$f\left(e^{-\frac{2\pi z}{k}+\frac{2\pi i h}{k}}\right) = f\left(e^{-\frac{2\pi}{kz}+\frac{2\pi i h'}{k}}\right).$$

Now letting $w = N^{-2}-i\phi$ applying this to the representation (5.7) gives

$$c(n) = \sum_{0\leq h<k\leq N}{}' e^{-\frac{2\pi i n h}{k}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f\left(e^{\frac{2\pi i h'}{k}-\frac{2\pi}{k^2 w}}\right)e^{2\pi n w}d\phi.$$

This has the advantage of moving the contour in relation to the cusps to facilitate the analysis. Writing $f(x) = x^{-1}+D(x)$ with $D(x) = \sum_{m=0}^{\infty} c(m)x^m$ we can split the integral into two integrals. They are precisely the $R(n)$ and $Q(n)$ listed in Lemma 29, with $R(n)$ arising from x^{-1} and $Q(n)$ from $D(x)$.

5.3. **Proof of Lemma 30.** Splitting up the integral $Q(n)$ as in (5.4), we will first analyze $Q_0(n)$. The integral is independent of h, so

$$Q_0(n) = \sum_{k=1}^{N} A_k(n)\int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} e^{\frac{2\pi}{k^2 w}+2\pi n w}d\phi$$

where we define

$$A_k(n) = \sum_{h \bmod k}{}' e^{-\frac{2\pi i}{k}(nh+h')}$$

Note that this is an example of a Kloosterman sum. The strategy to evaluate the integral is to view it as one of the sides of a rectangle in the w plane with vertices $\pm N^{-2} \pm \frac{i}{k(N+k)}$. Since $w = N^{-2} - i\phi$,

$$Q_0(n) = \sum_{k=1}^{N} A_k(n) \frac{1}{i} \int_{N^{-2}-\frac{i}{k(N+k)}}^{N^{-2}+\frac{i}{k(N+k)}} e^{\frac{2\pi}{k^2 w}+2\pi n w} dw.$$

$Q_0(n)$ is the integral over the right side of the rectangle. Let $L_k(n)$ denote the integral over the entire rectangle, J_1 the integral from $N^{-2}+\frac{i}{k(N+k)}$ to $-N^{-2}+\frac{i}{k(N+k)}$, J_2 the integral from $-N^{-2}+\frac{i}{k(N+k)}$ to $-N^{-2}-\frac{i}{k(N+k)}$, and J_3 the integral from $-N^{-2}-\frac{i}{k(N+k)}$ to $N^{-2}-\frac{i}{k(N+k)}$. Then we have

$$Q_0(n) = \sum_{k=1}^{N} A_k(n) \frac{1}{i} L_k(N) - \frac{1}{i} \sum_{k=1}^{N} A_k(n) \left(J_1 + J_2 + J_3\right). \tag{5.8}$$

It is easy to estimate J_1 and J_3. On the paths of integration $w = u \pm \frac{i}{k(N+k)}$, $-N^{-2} \leq u \leq N^{-2}$, and

$$\mathrm{Re}(\frac{1}{w}) = \frac{u}{u^2 + \frac{1}{k^2(N+k)^2}} < N^{-2} k^2 (N+k)^2 \leq 4k^2.$$

Thus the integrand is less than $e^{8\pi + 2\pi n N^{-2}}$, so

$$|J_1| \text{ and } |J_3| \leq 2N^{-2} e^{8\pi + 2\pi n N^{-2}}. \tag{5.9}$$

For J_2, the path of integration is $w = -N^{-2} + iv$ with $|v| \leq \frac{1}{k(N+k)}$. The real part of w is always $-N^{-2} < 0$ while $\mathrm{Re}(\frac{1}{w}) = \frac{-N^{-2}}{N^{-4}+v^2} < 0$. Thus the integrand is $O(1)$ (note this doesn't work on the right side, which is good). The path has length $\frac{2}{k(N+k)}$, so

$$|J_2| < \frac{2}{k(N+k)} < 2k^{-1}N^{-1}. \tag{5.10}$$

Combining (5.9) with (5.10) and the bounds on the Kloosterman sum $A_k(n)$ from (4.4) we get

$$\sum_{k=1}^{N} A_k(n) \left(J_1 + J_2 + J_3\right) = O\left(e^{2\pi n N^{-2}} \sum_{k=1}^{N} k^{\frac{2}{3}+\epsilon} (n,k)^{\frac{1}{3}} k^{-1} N^{-1}\right).$$

As long as $n \geq 1$ which we are assuming, $(n,k) \leq n$. Furthermore,

$$N^{-1} \sum_{k=1}^{N} k^{-\frac{1}{3}+\epsilon} = O(N^{-\frac{1}{3}+\epsilon}).$$

Thus we can estimate

$$\sum_{k=1}^{N} A_k(n) \left(J_1 + J_2 + J_3\right) = O(e^{2\pi n N^{-2}} n^{\frac{1}{3}} N^{-\frac{1}{3}+\epsilon}). \tag{5.11}$$

The last step is to deal with $L_k(n)$. Now if R is the rectangle (it had a positive orientation) then using the power series for e^x we get

$$\begin{aligned}\frac{1}{2\pi}L_k(n) &= \frac{1}{2\pi i}\int_R e^{\frac{2\pi}{k^2 w}+2\pi n w}dw \\ &= \frac{1}{2\pi i}\int_R \sum_{\mu=0}^{\infty}\frac{\left(\frac{2\pi}{k^2 w}\right)^{\mu}}{\mu!}\sum_{\nu=0}^{\infty}\frac{(2\pi n w)^{\nu}}{\nu!}dw.\end{aligned}$$

By the residue theorem, the integral

$$\frac{1}{2\pi i}\int_R \frac{\left(\frac{2\pi}{k^2 w}\right)^{\mu}}{\mu!}\cdot\frac{(2\pi n w)^{\nu}}{\nu!}$$

is zero unless there is a simple pole at 0, which requires $\nu-\mu=-1$. Thus we have

$$\begin{aligned}\frac{1}{2\pi}L_k(n) &= \frac{1}{k\sqrt{n}}\sum_{\nu=0}^{\infty}\frac{\left(\frac{2\pi\sqrt{n}}{k}\right)^{2\nu+1}}{\nu!(\nu+1)!} \\ &= \frac{1}{k\sqrt{n}}I_1(\frac{4\pi\sqrt{n}}{k})\end{aligned}$$

where $I_1(z)$ is the Bessel function defined in (4.1). Putting this together with (5.8) and (5.11) we obtain the desired

$$Q_0(n) = \frac{2\pi}{\sqrt{n}}\sum_{k=1}^{N}\frac{A_k(n)}{k}I_1(\frac{4\pi\sqrt{n}}{k}) + O\left(e^{2\pi n N^{-2}}n^{\frac{1}{3}}N^{-\frac{1}{3}+\epsilon}\right)$$

5.4. **Proof of Lemma 31.** The next step is to bound $Q_1(n)$ and $Q_2(n)$. I will do the case of $Q_1(n)$: the argument for $Q_2(n)$ is nearly identical and is the one Rademacher chooses to do. The definition of $Q_1(n)$ is

$$Q_1(n) := \sum_{k=1}^{N}\sideset{}{'}\sum_{h \bmod k} e^{-\frac{2\pi i}{k}(nh+h')}\int_{-\frac{1}{k(b+k)}}^{-\frac{1}{k(N+k)}} e^{\frac{2\pi}{k^2 w}+2\pi n w}d\phi.$$

Splitting the interval $[-\frac{1}{k(b+k)}, -\frac{1}{k(N+k)}]$ up into the intervals $[-\frac{1}{kl}, -\frac{1}{k(l+1)}]$ for l from $b+k$ to $N+k-1$, it follows that

$$Q_1(n) = \sum_{k=1}^{N}\sum_{l=N+1}^{N+k-1}\int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}} e^{\frac{2\pi}{k^2 w}+2\pi n w}d\phi \sideset{}{'}\sum_{\substack{h \bmod k \\ N<b+k\le l}} e^{-\frac{2\pi i}{k}(nh+h')} \tag{5.12}$$

where the extra condition on the last sum allows the extension of the second sum from $N+1$ to $N+k-1$. Because $b \equiv -h' \mod k$ (which follows from the uni-modular relation on the Farey sequence), the restriction $N < b+k \le l$ does in fact constrain the choice of h. This makes the last sum an incomplete Kloosterman sum, which using (4.4) tells me that

$$\sideset{}{'}\sum_{\substack{h \mod k \\ N-k<b\le l-k}} e^{-\frac{2\pi i}{k}(nh+h')} = O(k^{\frac{2}{3}}(n,k)^{\frac{1}{3}}) = O(k^{\frac{2}{3}}n^{\frac{1}{3}})$$

But now in the integral of (5.12), on the intervals the real part of $\frac{2\pi}{k^2 w} + 2\pi n w$ is

$$\begin{aligned}
\mathrm{Re}(\frac{2\pi}{k^2 w} + 2\pi n w) &= \mathrm{Re}\left(\frac{2\pi}{k^2(N^{-2} - i\phi)} + 2\pi n(N^{-2} - i\phi)\right) \\
&= 2\pi\left(\frac{N^{-2}}{k^2(N^{-4} + \phi^2)} + nN^{-2}\right) \\
&\leq 2\pi\left(\frac{N^{-2}}{k^2 N^{-4} + \frac{1}{(k+N)^2}}\right) + nN^{-2} \\
&\leq 2\pi\left((\frac{k+N}{N})^2 + nN^{-2}\right) \\
&\leq 8\pi + 2\pi n N^{-2}.
\end{aligned}$$

Combining this with the rest of the integral gives

$$\begin{aligned}
Q_1(n) &= O\left(e^{2\pi n N^{-2}} n^{\frac{1}{3}} \sum_{k=1}^{N} \sum_{l=N+1}^{N+k-1} \left(\frac{1}{kl} - \frac{1}{k(l+1)}\right) k^{\frac{2}{3}+\epsilon}\right) \\
&= O\left(e^{2\pi n N^{-2}} n^{\frac{1}{3}} \sum_{k=1}^{N} \frac{1}{k^{\frac{1}{3}-\epsilon} N}\right) \\
&= O\left(e^{2\pi n N^{-2}} n^{\frac{1}{3}} N^{-\frac{1}{3}+\epsilon}\right).
\end{aligned}$$

This and the analogous result for $Q_2(n)$ establish Lemma 31.

5.5. **Proof of Lemma 32.** By the definition of $R(n)$ and D in Lemma 29,

$$R(n) = \sum_{k=1}^{N} {\sum_{h \bmod k}}' e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \sum_{m=0}^{\infty} c(m) e^{\frac{2\pi i h' m}{k} - \frac{2\pi m}{k^2 w}} e^{2\pi n w} d\phi$$

Note that for any m, on the interval $\left[-\frac{1}{k(N+k)}, \frac{1}{k(N+k)}\right]$

$$\mathrm{Re}\left(\frac{2\pi m}{k^2 w}\right) = \frac{2\pi m N^{-2}}{k^2(N^{-4} + \phi^2)} \geq \frac{2\pi m}{k^2 N^{-2} + N^2 k^2 \phi^2} \geq \frac{2\pi m}{2} = \pi m.$$

I decompose the interval $\left[-\vartheta'_{h,k}, \vartheta''_{h,k}\right]$ into $\left[-\frac{1}{k(k+b)}, -\frac{1}{k(N+k)}\right]$, $\left[-\frac{1}{k(N+k)}, \frac{1}{k(N+k)}\right]$, and $\left[\frac{1}{k(N+k)}, \frac{1}{k(b+k)}\right]$, and then further decompose the first and last intervals. The first decomposes into $\left[-\frac{1}{kl}, -\frac{1}{k(l+1)}\right]$ for l from $b+k$ to $N+k-1$, the last into $\left[\frac{1}{k(l+1)}, \frac{1}{kl}\right]$. Let S_1 be the integral from the middle part:

$$S_1 := \sum_{k=1}^{N} \sum_{m=0}^{\infty} c(m) \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} e^{-\frac{2\pi m}{k^2 w} + 2\pi n w} d\phi {\sum_{h \bmod k}}' e^{-\frac{2\pi i}{k}(nh - mh')}.$$

The last sum is a Kloosterman sum, so it is $O(k^{\frac{2}{3}+\epsilon}n^{\frac{1}{3}})$ uniformly in m. Thus

$$\begin{aligned}
S_1 &= O\left(\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}|c(m)|\frac{2}{k(N+k)}e^{-\pi m+2\pi nN^{-2}}k^{\frac{2}{3}+\epsilon}n^{\frac{1}{3}}\right)\\
&= O\left(e^{2\pi nN^{-2}}n^{\frac{1}{3}}N^{-1}\sum_{m=0}^{\infty}|c(m)|e^{-\pi m}\sum_{k=1}^{N}k^{-\frac{1}{3}+\epsilon}\right)\\
&= O\left(e^{2\pi nN^{-2}}n^{\frac{1}{3}}N^{-\frac{1}{3}+\epsilon}\right)
\end{aligned}$$

where the last step uses that $\sum_{m=0}^{\infty}|c(m)|e^{-\pi m}$ is a finite constant because $j(z)$ is absolutely convergent at $\frac{i}{2}$.

The other two integrals are similar, so I will only deal with the one over $\left[-\frac{1}{k(k+b)}, -\frac{1}{k(N+k)}\right]$. Write

$$S_2 = \sum_{k=1}^{N}\sum_{m=0}^{\infty}c(m)\sum_{l=N+1}^{N+k-1}\int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}}e^{-\frac{2\pi m}{k^2w}+2\pi nw}d\phi\sum_{\substack{h \bmod k\\ N<k+b\le l}}{}^{'}\, e^{-\frac{2\pi i}{k}(nh-mh')}.$$

Again, the last sum is a Kloosterman sum, which we can bound by $O(k^{\frac{2}{3}+\epsilon}n^{\frac{1}{3}})$. Using the estimate on the real part,

$$\begin{aligned}
S_2 &= O\left(\sum_{k=1}^{N}\sum_{m=0}^{\infty}|c(m)|\frac{1}{kN}e^{-\pi m+2\pi nN^{-2}}k^{\frac{2}{3}+\epsilon}n^{\frac{1}{3}}\right)\\
&= O\left(e^{2\pi nN^{-2}}n^{\frac{1}{3}}N^{-\frac{1}{3}+\epsilon}\right).
\end{aligned}$$

Combining this with a similar statement for S_3 and the bound for S_1, we get the statement of Lemma 32. This completes the proof of Theorem 6.

6. The Convergent Series for the Partition Function

6.1. **Outline of the Proof.** The same sort of argument works to prove the series representation for $p(n)$ in Theorem 8. The generating function for the partition function can be expressed as

$$F(q) = \sum_{n=1}^{\infty}p(n)q^n = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)}$$

by expanding $\frac{1}{1-q^n}$ as a geometric series. This differs from $\frac{1}{\eta(z)}$ by a factor of $q^{\frac{1}{24}}$. The first order of business is to convert the transformation law for the eta function into one for the function F.

Lemma 33. *Let* $F(t) = \dfrac{1}{\prod_{n=1}^{\infty}(1-t^m)}$, *and let*

$$x = e^{\frac{2\pi ih}{k}-\frac{2\pi z}{k^2}} \quad\text{and}\quad x' = e^{\frac{2\pi iH}{k}-\frac{2\pi}{z}}$$

where $\mathrm{Re}(z) > 0$, *h and k are relatively prime, and* $hH \equiv -1 \mod k$. *Then the transformation law becomes*

$$F(x) = e^{\pi is(h,k)}\left(\frac{z}{k}\right)^{1/2}e^{\frac{\pi}{12z}-\frac{\pi z}{12k^2}}F(x').$$

Proof. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$, the functional equation in Theorem 2 implies

$$\frac{1}{\eta(\tau)} = \frac{1}{\eta(\tau')}(-i(c\tau+d))^{\frac{1}{2}} e^{\pi i(\frac{a+d}{12c}+s(-d,c))}$$

where $\tau' = \frac{a\tau+b}{c\tau+d}$. Rewriting this in terms of $F(e^{2\pi i\tau}) = \frac{e^{\pi i\tau/12}}{\eta(\tau)}$ gives

$$F(e^{2\pi i\tau}) = F(e^{2\pi i\tau'})e^{\frac{\pi i(\tau-\tau')}{12}}\,(-i(c\tau+d))^{\frac{1}{2}}\,e^{\pi i(\frac{a+d}{12c}+s(-d,c))}.$$

Take $a = H$, $c = k$, $d = -h$, $b = -\frac{hH+1}{k}$, and $\tau = \frac{iz+h}{k}$. Then $\tau' = \dfrac{iz^{-1}+H}{k}$ and the equation becomes

$$F(e^{\frac{2\pi ih}{k}-\frac{2\pi z}{k}}) = F(e^{\frac{2\pi iH}{k}-\frac{2\pi}{kz}})z^{\frac{1}{2}}e^{\frac{\pi}{12kz}-\frac{\pi z}{12k}+\pi is(h,k)}.$$

Replacing z by z/k gives the desired formula. $\square$

Now fix n, and allow N to vary. As with the case of the $j-$function, we will extract coefficients using a Cauchy's theorem. This in turn can be written in terms of an integral along Rademacher's contour followed by another change of variables onto the circle K of radius $\frac{1}{2}$ centered at $z = \frac{1}{2}$.

Lemma 34. *With the points $z_1(h,k)$ and $z_2(h,k)$ as in Section 3.3,*

$$p(n) = \sum_{0\le h<k\le N}{}' \int_{z_1(h,k)}^{z_2(h,k)} F\left(e^{\frac{2\pi ih}{k}-\frac{2\pi z}{k^2}}\right)\frac{i}{k^2}e^{-2\pi inh/k}e^{2n\pi z/k^2}dz.$$

The next step is to use the transformation law for the eta function and split the integral up to deal with the elementary factor

$$\Psi_k(z) := z^{\frac{1}{2}}e^{\frac{\pi}{12z}-\frac{\pi z}{12k^2}} \tag{6.1}$$

separately. To do so, define

$$I_1(h,k) := \int_{z_1(h,k)}^{z_2(h,k)} \Psi_k(z)e^{2\pi nz/k^2}dz \tag{6.2}$$

$$I_2(h,k) := \int_{z_{h,k}}^{z_2(h,k)} \Psi_k(z)\left(F(e^{\frac{2\pi iH}{k}-\frac{2\pi}{z}})-1\right)e^{2\pi nz/k^2}dz$$

where H is defined as in Lemma 33. Denote $e^{\pi is(h,k)}$ by $\omega(h,k)$.

Lemma 35. *With the notation above,*

$$p(n) = \sum_{0\le h<k\le N}{}' ik^{-5/2}\omega(h,k)e^{-2\pi inh/k}(I_1(h,k)+I_2(h,k))$$

The next step is of course to analyze the two integrals. I_1 will contribute the main term.

Lemma 36. *We have that*

$$I_1(h,k) = \int_{K_-}\Psi_k(z)e^{2\pi nz/k^2}dz + O(k^{\frac{3}{2}}N^{-\frac{3}{2}})$$

where K_- is the circle with radius $\frac{1}{2}$ centered at $z = \frac{1}{2}$ with negative orientation.

I_2 becomes an error term.

Lemma 37. *We have that*
$$I_2(h,k) = O(k^{\frac{3}{2}}N^{-\frac{3}{2}}).$$

Next, we put these together. Note that $|\omega(h,k)| = 1$ as $s(h,k)$ is a real number. Thus we can conclude that

$$\left|\sum_{0\le h<k\le N}{}^{'} ik^{-\frac{5}{2}}\omega(h,k)e^{-2\pi inh/k}Ck^{\frac{3}{2}}N^{-\frac{3}{2}}\right| \le \sum_{n=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}} Ck^{-1}N^{-3/2}$$

$$\le CN^{-\frac{3}{2}}\sum_{k=1}^{N}1 = O(N^{-\frac{1}{2}}). \tag{6.3}$$

Combining this with Lemmas 36 and 37 gives

$$p(n) = \sum_{0\le h<k\le N}{}^{'} ik^{-5/2}\omega(h,k)e^{-2\pi inh/k}\int_{K_-}\Psi_k(z)e^{2\pi nz/k^2}dz + O(N^{-\frac{1}{2}}).$$

Letting N tend to infinity gives us that

$$p(n) = i\sum_{k=1}^{\infty}B_k(n)k^{-\frac{5}{2}}\int_{K_-} z^{\frac{1}{2}}e^{\frac{\pi}{12z}+\frac{2\pi z}{k^2}(n-\frac{1}{24})}dz \tag{6.4}$$

where $B_k(n)$ is the exponential sum $B_k(n) := \sum_{0\le h<k}{}^{'} e^{\pi is(h,k)-2\pi inh/k}$.

The final step is to evaluate the integral in terms of Bessel functions and the hyperbolic sine function. Make the change of variable $w = \frac{1}{z}$, $dz = \frac{-1}{w^2}dw$. This sends the circle K_- to the the line with real part 1. Then (6.4) becomes

$$p(n) = \frac{1}{i}\sum_{k=1}^{\infty}B_k(n)k^{-5/2}\int_{1-\infty i}^{1+\infty i} w^{-5/2}e^{\frac{\pi w}{12}+\frac{2\pi}{k^2}(n-\frac{1}{24})\frac{1}{w}}dw$$

Now substitute $t = \frac{\pi}{12}w$. This gives

$$p(n) = 2\pi\left(\frac{\pi}{12}\right)^{\frac{3}{2}}\sum_{k=1}^{\infty}B_k(n)k^{-\frac{5}{2}}\frac{1}{2\pi i}\int_{\frac{\pi}{12}-\infty i}^{\frac{\pi}{12}+\infty i} t^{-\frac{5}{2}}e^{t+\frac{\pi^2}{6k^2}(n-\frac{1}{24})\frac{1}{t}}dt$$

which looks like (4.2) with $v = \frac{3}{2}$ and

$$\frac{z}{2} = \left(\frac{\pi^2}{6k^2}(n-\frac{1}{24})\right)^{\frac{1}{2}}.$$

Rewriting in terms of $I_{\frac{3}{2}}(\frac{\pi}{k}\sqrt{\frac{2}{3}(n-\frac{1}{24})})$ gives

$$p(n) = \frac{2\pi}{(n-\frac{1}{24})^{\frac{3}{4}}(24)^{\frac{3}{4}}}\sum_{k=1}^{\infty}B_k(n)k^{-1}I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(n-\frac{1}{24})}\right).$$

Using the special value of Bessel functions of half odd-order from (4.3) gives

$$p(n) = \frac{1}{\sqrt{2}\pi}\sum_{k=1}^{\infty}k^{\frac{1}{2}}B_k(n)\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(n-\frac{1}{24})}\right)}{\sqrt{n-\frac{1}{24}}}\right)$$

This finishes the proof of Theorem 8. $\square$

6.2. **Proof of Lemma 34.** The starting point is Cauchy's integral formula, which combines with the Power series for $F(x)$ to imply

$$p(n) = \frac{1}{2\pi i}\int_C \frac{F(x)}{x^{n+1}}dx$$

where C is any positively oriented closed contour surrounding 0 lying inside the unit circle where F is holomorphic. Making the change of variable $x = e^{2\pi i\tau}$, a circle of radius $e^{-2\pi}$ centered at 0 in the x-plane is sent to the line joining i and $1+i$ in the τ-plane. Thus

$$p(n) = \int_i^{i+1} F\left(e^{2\pi i\tau}\right) e^{-2\pi in\tau}d\tau$$

as $dx = e^{2\pi i\tau}d\tau$. Replace this line with the Rademacher contour $P(N)$ from Section 3.3, and let $\gamma(h,k)$ denote the arc on the circle $C(h,k)$ connecting $\alpha(h,k)$ and $\alpha'(h,k)$. Then

$$p(n) = \sum_{0\leq h<k\leq N}{}' \int_{\gamma_{h,k}} F(e^{2\pi i\tau})e^{-2\pi in\tau}d\tau.$$

Make the change of variable $z = -ik^2(\tau - \frac{h}{k})$ which sends $C(h,k)$ onto a circle of radius $\frac{1}{2}$ with $z = \frac{1}{2}$ as its center. The arc $\gamma(h,k)$ maps onto an arc joining the points $z_1(h,k)$ and $z_2(h,k)$. Furthermore, we have $dz = -ik^2(d\tau)$ and $\tau = \frac{iz}{k^2} + \frac{h}{k}$. Thus the integral becomes the desired

$$p(n) = \sum_{h,k}{}' \int_{z_1(h,k)}^{z_2(h,k)} F\left(e^{\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}}\right) \frac{i}{k^2}e^{-2\pi inh/k}e^{2n\pi z/k^2}dz.$$

6.3. **Proof of Lemma 35.** Lemma 33 tells me that

$$F(e^{\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}}) = \omega(h,k)\left(\frac{z}{k}\right)^{\frac{1}{2}} e^{\frac{\pi}{12z} - \frac{\pi z}{12k^2}}F(e^{\frac{2\pi iH}{k} - \frac{2\pi}{z}})$$

where H is chosen so $hH = -1 \mod k$ and $\omega(h,k)$ was defined to be $e^{\pi is(h,k)}$. Substituting this into Lemma 34 which gives a formula for $p(n)$ in terms of a contour integral gives

$$p(n) = \sum_{0\leq h<k\leq N}{}' ik^{-2}e^{-2\pi inh/k}\omega(h,k)\int_{z_1(h,k)}^{z_2(h,k)} \left(\frac{z}{k}\right)^{\frac{1}{2}} e^{\frac{\pi}{12z} - \frac{\pi z}{12k^2} + 2\pi nz/k^2}F(e^{\frac{2\pi iH}{k} - \frac{2\pi}{z}})dz$$

Rearranging terms to match the definitions in (6.1) and (6.2) gives

$$p(n) = \sum_{0\leq h<k\leq N}{}' ik^{-5/2}\omega(h,k)e^{-2\pi inh/k}(I_1(h,k) + I_2(h,k)).$$

6.4. **Proof of Lemma 36.** Instead of integrating around the circle from $z_1(h,k)$ to $z_2(h,k)$, we will integrate around the entire circle and analyze the error. Note that the circle has a a negative orientation. Let K_- denote the negatively oriented circle, J_1 the arc from 0 to $z_1(h,k)$ and J_2 the arc from $z_2(h,k)$ to 0. Then $I_1(h,k)$ can be written as the integral over K_- minus the integrals over J_1 and J_2.

To estimate J_1, note that the length of the arc joining the point 0 and $z_1(h,k)$ is less than $\pi|z_1(h,k)| < \sqrt{2}\pi\frac{k}{N}$. On the circle $\mathrm{Re}(1/z) = 1$ while $0 \leq \mathrm{Re}(z) \leq 1$. Thus

$$\begin{aligned}|\Psi_k(z)e^{2\pi nz/k^2}| &= e^{2\pi n\,\mathrm{Re}(z)/k^2}|z|^{\frac{1}{2}}e^{\frac{\pi}{12}\mathrm{Re}(\frac{1}{z})-\frac{\pi}{12k^2}\mathrm{Re}(z)} \\ &\leq \frac{e^{2\pi n}2^{\frac{1}{4}}k^{\frac{1}{2}}e^{\pi/12}}{N^{\frac{1}{2}}}\end{aligned}$$

since $|z|$ is maximized at $z_1(h,k)$. Thus the integral over J_1 is $O(k^{\frac{3}{2}}N^{-\frac{3}{2}})$. The same argument works for J_2. This shows that

$$I_1(h,k) = \int_{K_-}\Psi_k(z)e^{2\pi nz/k^2}dz + O(k^{\frac{3}{2}}N^{-\frac{3}{2}}).$$

6.5. **Proof of Lemma 37.** Instead of integrating around the circle from $z_1(h,k)$ to $z_2(h,k)$, we will integrate along the chord joining them. Note that $0 < \mathrm{Re}(z) \leq 1$ and $\mathrm{Re}(\frac{1}{z}) \geq 1$ anywhere in the circle.

Then we can estimate the integrand as follows:

$$\begin{aligned}&\left|\Psi_k(z)\left(F(e^{\frac{2\pi iH}{k}-\frac{2\pi}{z}})-1\right)e^{2\pi nz/k^2}\right| \\ &= |z|^{\frac{1}{2}}e^{\frac{\pi}{12}\mathrm{Re}(\frac{1}{z})-\frac{\pi}{12k^2}\mathrm{Re}(z)}e^{2\pi n\,\mathrm{Re}(z)/k^2}\left|\sum_{m=1}^{\infty}p(m)e^{2\pi iHm/k}e^{-2\pi m/z}\right| \\ &\leq |z|^{\frac{1}{2}}e^{\frac{\pi}{12}\mathrm{Re}(\frac{1}{z})}e^{2\pi n/k^2}\sum_{m=1}^{\infty}p(m)e^{-2\pi m\,\mathrm{Re}(\frac{1}{z})} \\ &< |z|^{\frac{1}{2}}e^{2\pi n}\sum_{m=1}^{\infty}p(m)e^{-2\pi(m-\frac{1}{24})\mathrm{Re}(\frac{1}{z})} \\ &\leq |z|^{\frac{1}{2}}e^{2\pi n}\sum_{m=1}^{\infty}p(m)e^{-2\pi(m-\frac{1}{24})} \\ &\leq |z|^{\frac{1}{2}}e^{2\pi n}\sum_{m=1}^{\infty}p(24m-1)e^{-2\pi(24m-1)/24} \\ &= |z|^{\frac{1}{2}}e^{2\pi n}\sum_{m=1}^{\infty}p(24m-1)y^{24m-1} \\ &= c|z|^{\frac{1}{2}}\end{aligned}$$

where the second to last step uses $p(m) < p(24m-1)$ for $m \geq 1$ and $y := e^{-2\pi/24}$. The constant c equals

$$e^{2\pi n}\sum_{m=1}^{\infty}p(24m-1)y^{24m-1}$$

which bounded because it is a sub-sequence of the $q-$series for F evaluated at y which is inside the unit circle. Note it is independent of z or N (it depends on n, but n is fixed).

Now the length of the chord is less than $\frac{2\sqrt{2}k}{N}$ by Proposition 23. Furthermore, $|z| \leq \sqrt{2}\frac{k}{N}$ on the chord. This implies that the integral is

$$I_2(h,k) = O(k^{\frac{3}{2}}N^{-\frac{3}{2}}). \tag{6.5}$$

Remark 38. *One of the reasons Rademacher's improved contour of integration does not immediately apply to the j function is that, in contrast to Lemma 33, there is no extra factors in the transformation law in the j function because it is "better" behaved that the eta function. In particular, the $\sqrt{z}$ is necessary for this lemma to work. Attempting to directly copy this argument naively tells me that $f(q) = j(e^{2\pi iz})$ is bounded on the chord and gives me a bound of $O(kN^{-1})$ for the integral. After summing over (h,k) with $0 \leq h < k \leq N$ and $(h,k) = 1$ as in (6.3), this is not obviously bounded by anything better than $O(1)$. A possible salvage would be to divide the segment up into smaller pieces and bound each separately. This looks like the situation in Lemma 32, where the integral is divided up carefully into pieces to get a good enough bound: it is not clear whether using Rademacher's contour instead of Farey arcs would make this division any easier.*

References

1. Tom Apostol, *Modular functions and dirichlet series in number theory*, Springer-Verlag, 1989.
2. G. H. Hardy and S. Ramanujan, *Asymptotic formula and combinatory analysis*, Proc. of the London Math. Society (1918), 75–115.
3. Hans Rademacher, *The fourier coefficients of the modular invariant $j(\tau)$*, American Journal of Mathematics (1938), 501–512.
4. ______, *On the partition function $p(n)$*, Proc. of the London Math. Society (1938), 241–254.
5. ______, *On the expansion of the partition function in a series*, The Annals of Mathematics (1943), 416–422.
6. George Watson, *A treatise on the theory of bessel functions*, second edition ed., Cambridge University Press, 1995.

denominator lying strictly between h/k and h_1/k_1 is $(h + h_1)/(k + k_1)$ and this number is called the mediant of h/k and h_1/k_1. Thus using the mediants as the end points for our intervals seems a natural dissection of C. Now if h_0/k_0, h/k, h_1/k_1 are three successive terms in F_N, let us write

$$\theta'_{0,1} = \frac{1}{N+1};$$

$$\theta'_{h,k} = \frac{h}{k} - \frac{h_0 + h}{k_0 + h} \quad \text{for} \quad h > 0,$$

$$\theta''_{h,k} = \frac{h_1 + h}{k_1 + h} - \frac{h}{k}. \tag{5.2.9}$$

Then clearly

$$p(n) = \frac{1}{2\pi i}\int_C \frac{P(x)}{x^{n+1}}\,dx$$

$$= \rho^{-n}\int_0^1 P[\rho\exp(2\pi i\phi)]\exp(-2\pi i n\phi)\,d\phi$$

$$= \rho^{-n}\sum_{\substack{k=1\\(h,k)=1\\0\leqslant h<k}}^{N}\int_{-\theta'_{h,k}}^{\theta''_{h,k}} P\left[\rho\exp\left(\frac{2\pi ih}{k} + 2\pi i\phi\right)\right]\exp\left(-\frac{2\pi inh}{k} - 2\pi in\phi\right)d\phi. \tag{5.2.10}$$

All that awaits now before the application of (5.2.2) is an appropriate choice of ρ; we could leave the matter indefinite and choose ρ as well as possible when compelled to do so. However, for simplicity we shall make the "right" choice immediately:

$$\rho = \exp\left(-\frac{2\pi}{N^2}\right), \tag{5.2.11}$$

and shall content ourselves with subsequent observations on the necessity of this choice. Hence

$$p(n) = \exp\left(\frac{2\pi n}{N^2}\right)\sum_{\substack{k=1\\(h,k)=1\\0\leqslant h<k}}^{N}\exp\left(-\frac{2\pi ihn}{k}\right)$$

$$\times\int_{\theta'_{h,k}}^{\theta''_{h,k}} P\left\{\exp\left[\frac{2\pi ih}{k} - \frac{2\pi}{k}\left(\frac{k}{N^2} - ik\phi\right)\right]\right\}\exp(-2\pi in\phi)\,d\phi; \tag{5.2.12}$$

thus to apply (5.2.2) we must define

$$z = k(N^{-2} - i\phi). \tag{5.2.13}$$

Consequently, applying (5.2.2) to the integrand in (5.2.12), we find that

$$p(n) = \exp\left(\frac{2\pi n}{N^2}\right) \sum_{\substack{k=1 \\ (h,k)=1 \\ 0 \leqslant h < k}}^{N} \exp\left(-\frac{2\pi i h n}{k}\right) \omega_{h,k}$$

$$\times \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} \exp\left[\pi\frac{(z^{-1} - z)}{12k}\right] P\left\{\exp\left[2\pi i\frac{(h' + iz^{-1})}{k}\right]\right\} \exp(-2\pi i n\phi)\, d\phi. \tag{5.2.14}$$

Now as $z \to 0$ with $\operatorname{Re} z > 0$, we see that $\exp[2\pi i(h' + iz^{-1})/k] \to 0$ rapidly. Therefore the obvious way to evaluate (5.2.14) is to replace in the integrand $P(x)$ by $1 + (P(x) - 1)$. Hence

$$p(n) = \exp\left(\frac{2\pi n}{N^2}\right) \sum_{\substack{k=1 \\ (h,k)=1 \\ 0 \leqslant h < k}}^{N} \exp\left(-\frac{2\pi i h n}{k}\right) \omega_{h,k} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} \exp\left[\pi\frac{(z^{-1} - z)}{12k} - 2\pi i n\phi\right] d\phi$$

$$+ \exp\left(\frac{2\pi n}{N^2}\right) \sum_{\substack{k=1 \\ (h,k)=1 \\ 0 \leqslant h < k}}^{N} \exp\left(-\frac{2\pi i h n}{k}\right) \omega_{hk} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} \exp\left[\pi\frac{(z^{-1} - z)}{12k}\right]$$

$$\times \left(P\left\{\exp\left[2\pi i\frac{(h' + iz^{-1})}{k}\right]\right\} - 1\right) \exp(-2\pi i n\phi)\, d\phi = \Sigma_1 + \Sigma_2. \tag{5.2.15}$$

It is our natural expectation that Σ_1 will contribute our principal estimate for $p(n)$ and that the contribution of Σ_2 will be negligible. We undertake our long analysis of (5.2.15) by proving that Σ_2 is indeed negligible: First (recalling that $z = kN^{-2} - ik\phi$),

$$\left| z^{\frac{1}{2}} \exp\left[\pi\frac{(z^{-1} - z)}{12k}\right] \left(P\left\{\exp\left[2\pi i\frac{(h' + iz^{-1})}{k}\right]\right\} - 1\right)\right|$$

$$\leqslant |z|^{\frac{1}{2}} \exp\left(-\frac{\pi}{12N^2}\right) \sum_{m=1}^{\infty} p(m) \exp\left[-2\pi \operatorname{Re}(z^{-1})\frac{(m - 1/24)}{k}\right]. \tag{5.2.16}$$

Now

$$\frac{1}{z} = \frac{1}{kN^{-2} - ik\phi} = \frac{N^{-2} + i\phi}{k(N^{-4} + \phi^2)}.$$

From (5.2.9) it is immediate that each of $\theta'_{h,k}$ and $\theta''_{h,k}$ satisfies $1/2kN \leqslant \theta_{h,k} < 1/kN$, and since $-\theta'_{h,k} \leqslant \phi \leqslant \theta''_{h,k}$, we see that

$$\frac{1}{k}\,\mathrm{Re}(z^{-1}) = \frac{N^{-2}}{k^2(N^{-4} + \phi^2)} > \frac{N^{-2}}{k^2N^{-4} + N^{-2}} = \frac{1}{1 + k^2N^{-2}} \geqslant \tfrac{1}{2}. \quad (5.2.17)$$

Also

$$|z|^{\frac{1}{2}} = (k^2N^{-4} + k^2\phi^2)^{\frac{1}{4}} < (k^2N^{-4} + N^{-2})^{\frac{1}{4}} \leqslant 2^{\frac{1}{4}}N^{-\frac{1}{2}}. \quad (5.2.18)$$

Hence by (5.2.16), (5.2.17), and (5.2.18) we have the following estimate for Σ_2:

$$\begin{aligned}
|\Sigma_2| &\leqslant \exp\left(\frac{2\pi n}{N^2}\right) \sum_{\substack{k=1\\(h,k)=1\\0\leqslant h<k}}^{N} 2^{\frac{1}{4}}N^{-\frac{1}{2}} \exp\left(-\frac{\pi}{12N^2}\right) \\
&\quad \times \sum_{m=1}^{\infty} p(m) \exp\left[-\pi\left(m - \frac{1}{24}\right)\right] \int_{-\theta'_{h,k}}^{\theta''_{h,k}} d\phi \\
&\leqslant \exp\left(\frac{2\pi n}{N^2} - \frac{\pi}{12N^2}\right) 2^{\frac{1}{4}}N^{-\frac{1}{2}} \sum_{m=1}^{\infty} p(m) \exp\left[-\pi\left(m - \frac{1}{24}\right)\right] \sum_{\substack{k=1\\(h,k)=1\\0\leqslant h<k}}^{N} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} d\phi \\
&= \exp\left(\frac{2\pi n}{N^2} - \frac{\pi}{12N^2}\right) 2^{\frac{1}{4}}N^{-\frac{1}{2}} \sum_{m=1}^{\infty} p(m) \exp\left[-\pi\left(m - \frac{1}{24}\right)\right] \\
&\leqslant CN^{-\frac{1}{2}} \exp\left(\frac{2\pi n}{N^2}\right). \qquad (5.2.19)
\end{aligned}$$

This estimate suffices for our purposes, since $N^{-\frac{1}{2}} \exp(2\pi n N^{-2}) \to 0$ as $N \to \infty$ for fixed n.

We now handle Σ_1, the main term in Eq. (5.2.15). Here we shall show that the integral is the most significant portion of a Hankel-type loop integral. Once we establish this fact (Eq. (5.2.26)), it will then be a simple matter to complete our evaluation of $p(n)$.

In the integral for Σ_1 (see (5.2.15)) we set $\omega = N^{-2} - i\phi$, and thus the integral becomes

$$I_{h,k} \equiv \int_{N^{-2}+i\theta'_{h,k}}^{N^{-2}-i\theta''_{h,k}} (k\omega)^{\frac{1}{2}} \exp\left[\frac{\pi}{12k}\left(\frac{1}{k\omega} - k\omega\right) + 2\pi n(\omega - N^{-2})\right] i\,d\omega$$

$$= \exp(-2\pi n N^{-2})\, k^{\frac{1}{2}} i^{-1} \int_{N^{-2}-i\theta''_{h,k}}^{N^{-2}+i\theta'_{h,k}} \omega^{\frac{1}{2}} \exp\left[\frac{\pi}{12k^2\omega}\right] \exp\left[2\pi\left(n - \frac{1}{24}\right)\omega\right] d\omega$$

$$= \exp(-2\pi n N^{2})\, k^{\frac{1}{2}} i^{-1} \int_{N^{-2}+i\theta'_{h,k}}^{N^{-2}-i\theta''_{h,k}} g(\omega)\, d\omega, \tag{5.2.20}$$

where

$$g(\omega) = \omega^{\frac{1}{2}} \exp\left[2\pi\left(n - \frac{1}{24}\right)\omega + \frac{\pi}{12k^2\omega}\right].$$

The integrand of this last integral is single valued and analytic in the complex ω-plane cut along the entire negative real axis. Hence we may write (invoking Cauchy's theorem):

$$\exp(2\pi n N^{-2}) I_{h,k} = \frac{k^{\frac{1}{2}}}{i}\left(\int_{-\infty}^{(0+)} - \int_{-\infty}^{-\varepsilon} - \int_{-\varepsilon}^{-\varepsilon - i\theta''_{h,k}} - \int_{-\varepsilon - i\theta''_{h,k}}^{N^{-2}-i\theta''_{h,k}} \right.$$

$$\left. - \int_{N^{-2}+i\theta'_{h,k}}^{-\varepsilon + i\theta'_{h,k}} - \int_{-\varepsilon + i\theta'_{h,k}}^{-\varepsilon} - \int_{-\varepsilon}^{-\infty}\right) g(\omega)\, d\omega \tag{5.2.21}$$

where the integral $\int_{-\infty}^{(0+)}$ is the loop integral along the contour $\mathscr{L}$ in Fig. 5.1. We assume $0 < \varepsilon < N^{-2}$ and we shall mostly be interested in what happens as $\varepsilon \to 0$. For brevity, we rewrite (5.2.21) as

$$\exp(2\pi n N^{-2}) I_{h,k} = k^{\frac{1}{2}} i^{-1}\{L_k - I_1 - I_2 - I_3 - I_4 - I_5 - I_6\} \tag{5.2.22}$$

and our next task is to show that each of the four integrals I_2, I_3, I_4, and I_5 is negligible:

$$|I_2| \leqslant \int_0^{-\theta''_{h,k}} (\varepsilon^2 + v^2)^{\frac{1}{4}} \exp\left[\frac{\pi}{12k^2} \operatorname{Re}\left(\frac{1}{-\varepsilon + iv}\right)\right] \exp\left[-2\pi\left(n - \frac{1}{24}\right)\varepsilon\right] |dv|.$$

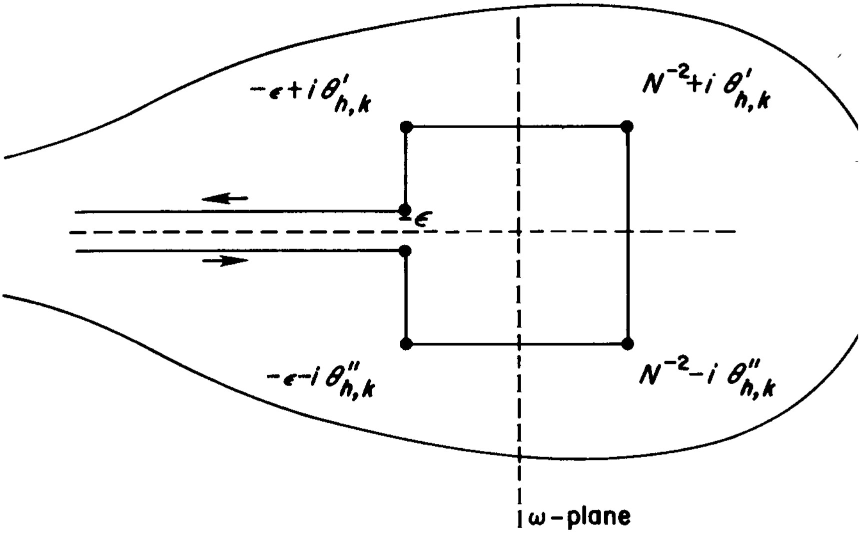

Figure 5.1

But $\mathrm{Re}[(-\varepsilon + iv)^{-1}] = -\varepsilon/(\varepsilon^2 + v^2) < 0$, and so

$$|I_2| < (\varepsilon^2 + \theta_{hk}''^2)^{\frac{1}{2}}\theta''_{h,k} < \left(\varepsilon^2 + \frac{1}{k^2N^2}\right)^{\frac{1}{2}} \frac{1}{kN} \to k^{-3/2}N^{-3/2} \quad \text{as} \quad \varepsilon \to 0. \tag{5.2.23}$$

Integral I_5 is treated in exactly the same way and (5.2.23) is also valid with I_2 replaced by I_5. As for I_3,

$$|I_3| \leqslant \int_{-\varepsilon}^{N^{-2}} (u^2 + \theta_{h,k}''^2)^{\frac{1}{2}} \exp\left[\frac{\pi}{12k^2}\mathrm{Re}\left(\frac{1}{u - i\theta''_{hk}}\right)\right] \exp\left[2\pi\left(n - \frac{1}{24}\right)u\right] du.$$

Now

$$\frac{1}{k^2}\mathrm{Re}\,[(u - i\theta''_{h,k})^{-1}] = \frac{u}{k^2(u^2 + \theta_{hk}''^2)} \leqslant \frac{N^{-2}}{k^2\theta_{h,k}''^2} \leqslant 4.$$

Hence

$$|I_3| < (N^{-4} + \theta_{h,k}''^2)^{\frac{1}{2}} \exp\left(\frac{\pi}{3}\right) \exp\left(\frac{2\pi n}{N^2}\right)(\varepsilon + N^{-2})$$

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$$< (N^{-4} + k^{-2}N^{-2})^{\frac{1}{2}} \exp\left(\frac{\pi}{3} + \frac{2\pi n}{N^2}\right)(\varepsilon + N^{-2})$$

$$\leqslant (\varepsilon + N^{-2})2^{\frac{1}{2}}k^{-\frac{1}{2}}N^{-\frac{1}{2}} \exp\left(\frac{\pi}{3} + \frac{2\pi n}{N^2}\right)$$

$$\to 2^{\frac{1}{2}}k^{-\frac{1}{2}}N^{-5/2} \exp\left(\frac{\pi}{3} + \frac{2\pi n}{N^2}\right) \quad \text{as} \quad \varepsilon \to 0 \tag{5.2.24}$$

and the final inequality in (5.2.24) is also easily seen to hold with I_3 replaced by I_5.

The integrals I_1 and I_6 are not negligible; however,

$$I_1 + I_6 = \int_{-\infty}^{-\varepsilon} \sqrt{|u|} \exp\left(-\frac{\pi i}{2}\right) \exp\left[\frac{\pi}{12k^2 u} + 2\pi\left(n - \frac{1}{24}\right)u\right] du$$

$$+ \int_{-\varepsilon}^{-\infty} \sqrt{|u|} \exp\left(\frac{\pi i}{2}\right) \exp\left[\frac{\pi}{12k^2 u} + 2\pi\left(n - \frac{1}{24}\right)u\right] du$$

$$= -2i \int_{\varepsilon}^{\infty} t^{\frac{1}{2}} \exp\left[-2\pi\left(n - \frac{1}{24}\right)t - \frac{\pi}{12k^2 t}\right] dt$$

$$= -2iH_k. \tag{5.2.25}$$

Hence, letting $\varepsilon \to 0$ in (5.2.22), we may simplify that equation to

$$\exp(2\pi n N^{-2}) I_{hk} = \frac{k^{\frac{1}{2}}}{i} L_k + 2k^{\frac{1}{2}} H_k + O(k^{-1}N^{-3/2}) + O\left[\exp\left(\frac{2\pi n}{N^2}\right) N^{-5/2}\right], \tag{5.2.26}$$

where the constants implied in the O terms are absolute. Consequently, writing

$$\psi_k(n) = \frac{k^{\frac{1}{2}}}{i} L_k + 2k^{\frac{1}{2}} H_k, \tag{5.2.27}$$

we see from (5.2.15), (5.2.19), and (5.2.26) that

$$p(n) = \sum_{\substack{k=1 \\ (h,k)=1 \\ 0 \leqslant h < k}}^{N} \left\{\omega_{h,k} \exp\left(-\frac{2\pi i h n}{k}\right) \psi_k(n)\right\} + O\left[\exp\left(\frac{2\pi n}{N^2}\right) N^{-\frac{1}{2}}\right]$$

$$+ O\left(\sum_{\substack{k=1 \\ (h,k)=1 \\ 0 \leqslant h < k}}^{N} k^{-1}N^{-3/2}\right) + O\left[\sum_{\substack{k=1 \\ (h,k)=1 \\ 0 \leqslant h < k}}^{N} \exp\left(\frac{2\pi n}{N^2}\right) N^{-5/2}\right]$$

$$= \sum_{k=1}^{N} A_k(n)\psi_k(n) + O\left[N^{-\frac{1}{2}} \exp\left(\frac{2\pi n}{N^2}\right)\right] + O(N^{-\frac{1}{2}}). \tag{5.2.28}$$

Now $A_k(n)$ is precisely as it appears in (5.1.1) and the error terms here are all $\to 0$ as $N \to \infty$. Hence the final problem for us is to show that

$$\psi_k(n) = \frac{k^{\frac{1}{2}}}{\pi\sqrt{2}}\left[\frac{d}{dx}\frac{\sinh(\pi/k(\frac{2}{3}(x - 1/24))^{\frac{1}{2}})}{(x - 1/24)^{\frac{1}{2}}}\right]_{x=n}. \tag{5.2.29}$$

We may do this fairly easily, assuming certain classical results about Hankel's integral. First

$$\frac{1}{i} L_k = \frac{1}{i}\int_{-\infty}^{(0+)} \omega^{\frac{1}{2}} \exp\left[\frac{\pi}{12k^2\omega} + 2\pi\left(n - \frac{1}{24}\right)\omega\right] d\omega$$

$$= \frac{1}{i}\int_{-\infty}^{(0+)} \omega^{\frac{1}{2}} \exp\left[2\pi\left(n - \frac{1}{24}\right)\omega\right] \sum_{s=0}^{\infty} \frac{(\pi/12k^2\omega)^s}{s!}\, d\omega$$

$$= \frac{1}{i}\sum_{s=0}^{\infty} \frac{(\pi/12k^2)^s}{s!}\int_{-\infty}^{(0+)} \omega^{\frac{1}{2}-s} \exp\left[2\pi\left(n - \frac{1}{24}\right)\omega\right] d\omega$$

$$= 2\pi\sum_{s=0}^{\infty} \frac{(\pi/12k^2)^s}{s!}\left[2\pi\left(n - \frac{1}{24}\right)\right]^{s-3/2} \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^z z^{-s+\frac{1}{2}}\, dz$$

$$= (2\pi)^{-\frac{1}{2}}\left(n - \frac{1}{24}\right)^{-3/2} \sum_{s=0}^{\infty} \frac{[\pi^2(n - 1/24)/6k^2]^s}{s!\,\Gamma(s - 1/2)}$$

where the final equation follows from Hankel's loop integral formula for the reciprocal of the gamma function:

$$\frac{1}{\Gamma(s - \frac{1}{2})} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^z z^{-s+\frac{1}{2}}\, dz.$$

Now

$$\Gamma(s - \tfrac{1}{2}) = (s - 3/2)(s - 5/2)\cdots\tfrac{1}{2}\Gamma(\tfrac{1}{2})$$

$$= 2^{-s+1}\pi^{\frac{1}{2}}(2s - 3)(2s - 5)\cdots 3\cdot 1.$$

Hence

$$\sum_{s=0}^{\infty}\frac{(\frac{1}{4}Y^2)^s}{s!\Gamma(s-\frac{1}{2})}=\frac{1}{2\sqrt{\pi}}\left[-1+Y^2\left(\frac{1}{2!}+\frac{3Y^2}{4!}+\frac{5Y^4}{6!}+\cdots\right)\right]$$

$$=\frac{1}{2\sqrt{\pi}}\left[-1+Y^2\frac{d}{dY}\sum_{n=0}^{\infty}\frac{Y^{2n+1}}{(2n+2)!}\right]$$

$$=\frac{1}{2\sqrt{\pi}}\left[-1+Y^2\frac{d}{dY}\left(\frac{\cosh Y-1}{Y}\right)\right];$$

and so with $Y=(\pi/k)(2/3(x-1/24))^{\frac{1}{2}}$, we see that

$$\frac{1}{i}L_k=(2\pi)^{-\frac{1}{2}}\left(n-\frac{1}{24}\right)^{-3/2}\frac{1}{2\sqrt{\pi}}\left[-1+Y^2\frac{d}{dY}\left(\frac{\cosh Y-1}{Y}\right)\right]_{x=n}$$

$$=2^{-3/2}\pi^{-1}\left(n-\frac{1}{24}\right)^{-3/2}\left[Y^2\frac{d}{dY}\left(\frac{\cosh Y}{Y}\right)\right]_{x=n}$$

$$=2^{-3/2}\pi^{-1}\left(n-\frac{1}{24}\right)^{-3/2}\left\{Y^2\left[\frac{d}{dx}\left(\frac{\cosh Y}{Y}\right)\Big/\frac{dY}{dx}\right]\right\}_{x=n}$$

$$=2^{-3/2}\pi^{-1}\left(n-\frac{1}{24}\right)^{-3/2}\left[\frac{3k^2Y^3}{\pi^2}\frac{d}{dx}\left(\frac{\cosh Y}{Y}\right)\right]_{x=n}$$

$$=3^{-\frac{1}{2}}k^{-1}\left\{\frac{d}{dx}\frac{\cosh((\pi/k)[\frac{2}{3}(x-1/24)]^{\frac{1}{2}})}{(\pi/k)[\frac{2}{3}(x-1/24)]^{\frac{1}{2}}}\right\}_{x=n}$$

$$=\frac{1}{\pi\sqrt{2}}\left\{\frac{d}{dx}\frac{\cosh((\pi/k)[\frac{2}{3}(x-1/24)]^{\frac{1}{2}})}{(x-1/24)^{\frac{1}{2}}}\right\}_{x=n}. \tag{5.2.30}$$

Finally we treat the H_k in (5.2.27) in a similar manner. We begin with a classical evaluation of a definite integral:

$$\int_0^{\infty}\exp(-c^2t-a^2t^{-1})t^{-\frac{1}{2}}\,dt=2\int_0^{\infty}\exp(-c^2u^2-a^2u^{-2})\,du=\frac{\sqrt{\pi}}{c}\exp(-2ac).$$

Hence

$$\int_0^{\infty}\exp(-c^2t-a^2t^{-1})t^{\frac{1}{2}}\,dt=-\frac{\sqrt{\pi}}{2c}\frac{d}{dc}\left[\frac{\exp(-2ac)}{c}\right], \tag{5.2.31}$$

and applying (5.2.31) to H_k we see that

$$H_k=-\frac{1}{4\sqrt{\pi}(n-1/24)}\left(\frac{d}{dc}\frac{\exp[-\pi(2/3)^{\frac{1}{2}}c/k]}{c}\right)_{c=(n-1/24)^{\frac{1}{2}}}$$

$$= -\frac{1}{2\pi\sqrt{2}}\left\{\frac{d}{dx}\frac{\exp[-\pi(2/3)^{\frac{1}{2}}(x-1/24)^{\frac{1}{2}}/k]}{(x-1/24)^{\frac{1}{2}}}\right\}_{x=n}. \tag{5.2.32}$$

Thus we obtain, from (5.2.27), (5.2.30), and (5.2.32), that

$$\begin{aligned}\psi_k(n) &= \frac{k^{\frac{1}{2}}}{\pi\sqrt{2}}\left\{\frac{d}{dx}\frac{\cosh((\pi/k)[\frac{2}{3}(x-1/24)]^{\frac{1}{2}})}{(x-1/24)^{\frac{1}{2}}}\right\}_{x=n}\\ &\quad - \frac{k^{\frac{1}{2}}}{\pi\sqrt{2}}\left(\frac{d}{dx}\frac{\exp\{-(\pi/k)[\frac{2}{3}(x-1/24)]^{\frac{1}{2}}\}}{(x-1/24)^{\frac{1}{2}}}\right)_{x=n}\\ &= \frac{k^{\frac{1}{2}}}{\pi\sqrt{2}}\left\{\frac{d}{dx}\frac{\sinh((\pi/k)[\frac{2}{3}(x-1/24)]^{\frac{1}{2}})}{(x-1/24)^{\frac{1}{2}}}\right\}_{x=n},\end{aligned} \tag{5.2.33}$$

and since (5.2.33) is just (5.2.29), we see that Theorem 5.1 is established. ■

Examples

1. The number of partitions of n into at most two parts ($= p(\text{"}\{1, 2\}\text{"}, n)$ by Theorem 1.4) may be shown to be $[n/2] + 1$ by examination of the decomposition of the generating function

$$\frac{1}{(1-q)(1-q^2)} = \frac{1/2}{(1-q)^2} + \frac{1/2}{(1-q^2)}.$$

2. The number of partitions of n into at most three parts ($= p(\text{"}\{1, 2, 3\}\text{"}, n)$ by Theorem 1.4) may be shown to be the nearest integer to $(1/12)(n+3)^2$ by examination of the following decomposition of the generating function:

$$\frac{1}{(1-q)(1-q^2)(1-q^3)} = \frac{1/6}{(1-q)^3} + \frac{1/4}{(1-q)^2} + \frac{1/4}{(1-q^2)} + \frac{1/3}{(1-q^3)}.$$

We remark that Examples 1 and 2 are special cases of Cayley's general decomposition:

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^i)} = \sum \frac{1}{1^{p_1}2^{p_2}3^{p_3}\cdots i^{p_i}p_1!p_2!\cdots p_i!} \times \frac{1}{(1-q)^{p_1}(1-q^2)^{p_2}\cdots(1-q^i)^{p_i}}$$

where the summation runs over all partitions $(1^{p_1}2^{p_2}2^{p_3}\cdots i^{p_i})$ of i. We shall give this formula in Example 1 of Chapter 12 as a special case of results on Bell polynomials.

3*. There are many modular functions besides $\eta(\tau)$ that have transformation formulas like (5.2.2). Indeed it has been shown by Hagis and Subramanyasastri that if H is the set of integers $\equiv \pm a_1, \pm a_2, \ldots, \pm a_r \pmod{k}$, then there exists an asymptotic series expansion for $p(\text{"}H\text{"}, n)$ of the same form as that

(ii) *There exist* $c \in \mathbb{C}$, $d \in \mathbb{R}$, *and* $N > 0$ *such that*

$$f(\tau) \sim c(-i\tau)^{-d} e^{\frac{2\pi i N}{\tau}} \qquad (\tau \to 0).$$

Then

$$a(n) \sim \frac{c}{\sqrt{2} N^{\frac{1}{2}\left(d-\frac{1}{2}\right)}} n^{\frac{1}{2}\left(d-\frac{3}{2}\right)} e^{4\pi\sqrt{Nn}} \qquad (n \to \infty).$$

From this we immediately conclude the growth behavior of the partition function.

Theorem 3.16. *We have*

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{\frac{2n}{3}}} \qquad (n \to \infty). \tag{3.1}$$

We note that the partition function also has the following q-hypergeometric series representation

$$P(q) = \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n^2}, \tag{3.2}$$

where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. Showing modularity by just using this representation is still an open problem [3].

Rademacher [36], building on work of Hardy and Ramanujan [22], used the Circle Method to obtain an exact formula for $p(n)$. To state his result, we let

$$I_s(x) := i^{-s} J_s(ix) = \sum_{m \geq 0} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{3.3}$$

be the *I-Bessel function of order s*. Moreover, with $\chi_{12}(x) := \left(\frac{12}{x}\right)$, we define the *Kloosterman sum*

$$A_k(n) := \frac{\sqrt{k}}{4\sqrt{3}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv 1-24n \pmod{24k}}} \chi_{12}(x) e^{\frac{2\pi i x}{12k}}.$$

Note that for $k > 0$, $(h,k) = 1$, and $\mathrm{Re}(z) > 0$ we may rewrite the transformation law of the partition generating function as

$$P\left(\exp\left(\frac{2\pi i}{k}(h+iz)\right)\right) = \omega_{h,k}\sqrt{z} e^{\frac{\pi}{12k}\left(z^{-1}-z\right)} P\left(\exp\left(\frac{2\pi i}{k}\left(h' + \frac{i}{z}\right)\right)\right), \tag{3.4}$$

where $hh' \equiv -1 \pmod{k}$. Here

$$\omega_{h,k} := \exp\left(\pi i s\left(h,k\right)\right) \tag{3.5}$$

with

$$s(h,k) := \sum_{\mu \pmod k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

Here

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Sometimes it is also useful to rewrite $\omega_{h,k}$ as [33]

$$\omega_{h,k} = \begin{cases} \left(\frac{-k}{h}\right) e^{-\pi i\left(\frac{1}{4}(2-hk-h)+\frac{1}{12}\left(k-k^{-1}\right)\left(2h-h'+h^2h'\right)\right)} & \text{if } h \text{ is odd}, \\ \left(\frac{-h}{k}\right) e^{-\pi i\left(\frac{1}{4}(k-1)+\frac{1}{12}\left(k-k^{-1}\right)\left(2h-h'+h^2h'\right)\right)} & \text{if } k \text{ is odd}. \end{cases}$$

Here $\left(\frac{a}{b}\right)$ denotes the Jacobi symbol.

Note that we may also write

$$A_k(n) = \sum_{h \pmod k^\star} \omega_{h,k} e^{-\frac{2\pi i n h}{k}}. \tag{3.6}$$

Theorem 3.17. *For $n \geq 1$, we have*

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k\geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

We note that this is a very astonishing identity expressing the integer $p(n)$ as an infinite sum of transcendentral numbers. Recently Bruinier and Ono [16] found a formula for $p(n)$ as a finite sum of algebraic numbers.

Proof. Here we only give some details of the proof, for more see [4]. By Cauchy's Theorem, we obtain

$$p(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{P(q)}{q^{n+1}} dq,$$

where $\mathcal{C}$ is any path inside the unit circle surrounding 0 counterclockwise. We may choose the circle around 0 with radius $\rho = \exp(-\frac{2\pi}{N^2})$, with $N > 0$ fixed (later we let $N \to \infty$). Then

$$p(n) = \rho^{-n} \int_0^1 P\left(\rho \exp\left(2\pi i t\right)\right) \exp\left(-2\pi i n t\right) dt.$$

Define

$$\vartheta'_{h,k} := \frac{1}{k\left(k_1 + k\right)}, \qquad \vartheta''_{h,k} := \frac{1}{k\left(k_2 + k\right)},$$

where $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order N (for example, see p. 72 of [4]). From the theory of Farey fractions it is known that ($j = 1, 2$)

$$\frac{1}{k + k_j} \leq \frac{1}{N+1}.$$

We decompose the path of integration in paths along the Farey arcs $-\vartheta'_{h,k} \leq \phi \leq \vartheta''_{h,k}$, where $\phi = t - \frac{h}{k}$. Setting $z = k(N^{-2} - i\phi)$ then yields

$$p(n) = \exp\left(\frac{2\pi n}{N^2}\right) \sum_{\substack{1\leq k\leq N\\ 0\leq h<k\\ (h,k)=1}} \exp\left(-\frac{2\pi i n h}{k}\right)$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} P\left(\exp\left(\frac{2\pi i}{k}(h + iz)\right)\right) \exp\left(-2\pi i n\phi\right) d\phi.$$

For $N \to \infty$ we have that $z \to 0$, i.e., $\exp(\frac{2\pi i}{k}(h+iz)) \to \exp\left(\frac{2\pi i h}{k}\right)$. We thus require the behavior of $P(q)$ as $q \to \exp\left(\frac{2\pi i h}{k}\right)$. For this, we apply (3.4) to obtain

$$p(n) = \exp\left(\frac{2\pi n}{N^2}\right) \sum_{\substack{1\leq k\leq N\\ 0\leq h<k\\ (h,k)=1}} \exp\left(-\frac{2\pi i n h}{k}\right) \omega_{h,k}$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{\frac{1}{2}} \exp\left(\frac{\pi}{12k}\left(z^{-1} - z\right)\right) P\left(\exp\left(\frac{2\pi i}{k}\left(h' + \frac{i}{z}\right)\right)\right) \exp\left(-2\pi i n\phi\right) d\phi.$$

Now $\exp(\frac{2\pi i}{k}(h' + \frac{i}{z})) \to 0$ for $z \to 0^+$. Thus all terms in $P(q) - 1$ are "small". We therefore write

$$p(n) = \sum\nolimits_1 + \sum\nolimits_2$$

with

$$\sum\nolimits_1 := \exp\left(\frac{2\pi n}{N^2}\right) \sum_{\substack{1\leq k\leq N\\ 0\leq h<k\\ (h,k)=1}} \exp\left(-\frac{2\pi i n h}{k}\right) \omega_{h,k}$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{\frac{1}{2}} \exp\left(\frac{\pi}{12k}\left(z^{-1} - z\right)\right) \exp\left(-2\pi i n\phi\right) d\phi,$$

$$\sum\nolimits_2 := \exp\left(\frac{2\pi n}{N^2}\right) \sum_{\substack{1\le k\le N\\ 0\le h<k\\ (h,k)=1}} \exp\left(-\frac{2\pi i n h}{k}\right)\omega_{h,k}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{\frac12}\exp\left(\frac{\pi\left(z^{-1}-z\right)}{12k}\right)$$
$$\times\left(P\left(\exp\left(\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)\right)\right)-1\right)\exp\left(-2\pi i n\phi\right)d\phi.$$

Here we only focus on the main term $\sum_1$ and only note that $\sum_2$ contributes to the error terms giving

$$\sum\nolimits_2 = O\left(N^{-\frac12}\exp\left(\frac{2\pi n}{N^2}\right)\right)\to 0$$

for $N\to\infty$. The proof is given in [4].

Turning to the main term, setting $w := N^{-2}-i\phi$ yields

$$\sum\nolimits_1 = \exp\left(\frac{2\pi n}{N^2}\right) \sum_{\substack{1\le k\le N\\ 0\le h<k\\ (h,k)=1}} \exp\left(-\frac{2\pi i n h}{k}\right)\omega_{h,k} I_{h,k}$$

with

$$I_{h,k} := -ik^{\frac12}\exp\left(-2\pi n N^{-2}\right)\int_{N^{-2}-i\vartheta''_{h,k}}^{N^{-2}+i\vartheta'_{h,k}} g(w)dw.$$

Here

$$g(w) := w^{\frac12}\exp\left(2\pi\left(n-\frac{1}{24}\right)w+\frac{\pi}{12k^2w}\right).$$

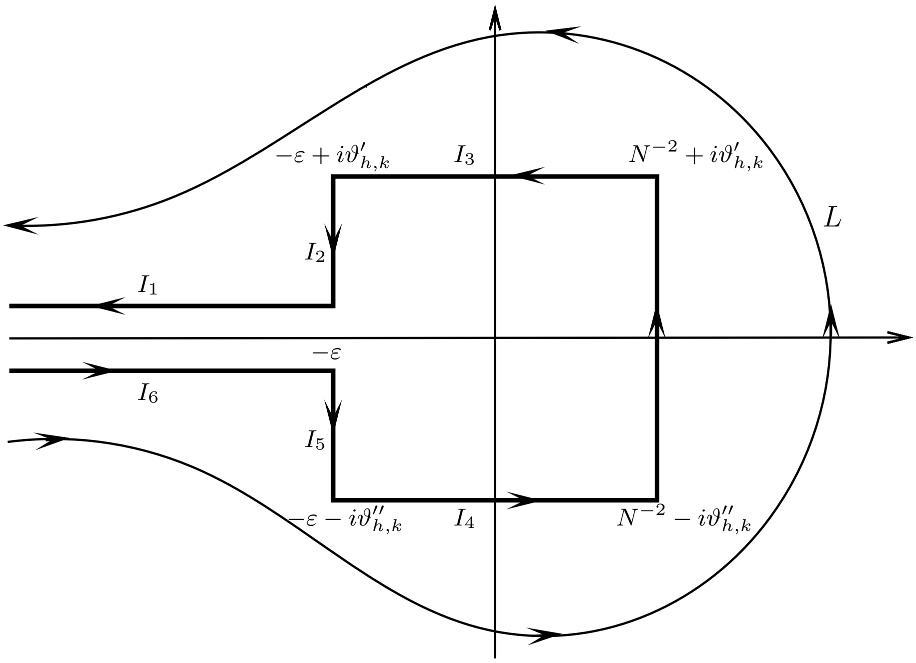
Using the Residue Theorem, we may write

$$\exp\left(\frac{2\pi n}{N^2}\right) I_{h,k} = -k^{\frac12} i\left(\mathcal{L}_k-\mathcal{I}_1-\mathcal{I}_2-\mathcal{I}_3-\mathcal{I}_4-\mathcal{I}_5-\mathcal{I}_6\right)$$

with

$$\mathcal{L}_k := \int_L z^{\frac12}\exp\left(2\pi\left(n-\frac{1}{24}\right)z+\frac{\pi}{12k^2z}\right)dz,$$
$$\mathcal{I}_j := \int_{I_j} z^{\frac12}\exp\left(2\pi\left(n-\frac{1}{24}\right)z+\frac{\pi}{12k^2z}\right)dz,$$

where L and I_j denote the paths of integration in the picture



We note without a proof that $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$, and $\mathcal{I}_5$ contribute to the error and vanish as $N \to \infty$. Moreover as $\varepsilon \to 0$

$$\mathcal{I}_1 + \mathcal{I}_6 = -2i \int_0^\infty t^{\frac{1}{2}} \exp\left(-2\pi \left(n - \frac{1}{24}\right) t - \frac{\pi}{12k^2 t}\right) dt =: -2i\mathcal{L}_k^\star$$

This gives that

$$p(n) = \sum_{k \geq 1} A_k(n) \psi_k(n),$$

where $A_k(n)$ is defined as in (3.6) and

$$\psi_k(n) := -i\sqrt{k}\mathcal{L}_k + 2\sqrt{k}\mathcal{L}_k^\star.$$

To finish the proof, we have to show that

$$\psi_k(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \frac{1}{k} I_{\frac{3}{2}} \left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

Inserting the power series expansion for the exponential function, interchanging summation and integration and then making a change of variables gives that

$$-i\mathcal{L}_k = 2\pi \sum_{s \geq 0} \frac{\left(\frac{\pi}{12k^2}\right)^s}{s!} \left(2\pi \left(n - \frac{1}{24}\right)\right)^{s-\frac{3}{2}} \frac{1}{2\pi i} \int_L e^z z^{-s+\frac{1}{2}} dz,$$

where the loop L is as in the above picture.

Using the Hankel's loop integral formula

$$\frac{1}{\Gamma\left(s-\frac{1}{2}\right)}=\frac{1}{2\pi i}\int_L e^z z^{-s+\frac{1}{2}}dz$$

then yields that

$$-i\mathcal{L}_k=\frac{1}{\sqrt{2\pi}}\frac{1}{\left(n-\frac{1}{24}\right)^{\frac{3}{2}}}\sum_{s\geq 0}\frac{\left(\frac{\pi^2\left(n-\frac{1}{24}\right)}{6k^2}\right)^s}{s!\Gamma\left(s-\frac{1}{2}\right)}.$$

Treating $\mathcal{L}_k^\star$ similarly, the claim follows using the series representation of the Bessel function (3.3). □

Corollary 3.18. *The asymptotic estimate* (3.1) *is true.*

Proof. The claim follows immediately from Theorem 3.17 using that

$$I_\ell(x)\sim\frac{e^x}{\sqrt{2\pi x}}\qquad (x\to\infty).$$

□

4. Mock modular form

Next we aim to study coefficients of *mock modular forms*, which are holomorphic parts of *harmonic (weak) Maass forms*. These are a generalization of modular forms which satisfy a transformation law like (1.1) and (weak) growth conditions at the cusps, but, instead of being meromorphic, they are annihilated by the weight k *hyperbolic Laplacian* ($\tau=x+iy$)

$$\Delta_k:=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

We let H_k denote the space of harmonic Maass forms of weight k. A theory of harmonic Maass forms was built by Bruinier and Funke [15]. Previously, Niebur [34, 35] and Hejhal [24] constructed certain non-holomorphic Poincaré series which turn out to be examples of harmonic weak Maass forms. Since functions that are holomorphic on $\mathbb{H}$ are annihilated by Δ_k, weakly holomorphic modular forms are harmonic Maass forms. The simplest (non-weakly