§4. Doubly periodic functions

Let L be a lattice in the complex plane, by which we mean the set of all integral linear combinations of two given complex numbers ω_1 and ω_2, where ω_1 and ω_2 do not lie on the same line through the origin. For example, if $\omega_1 = i$ and $\omega_2 = 1$, we get the lattice of Gaussian integers $\{mi + n | m, n \in \mathbb{Z}\}$. It will turn out that the example of the lattice of Gaussian integers is intimately related to the elliptic curves $y^2 = x^3 - n^2x$ that come from the congruent number problem.

The fundamental parallelogram for ω_1, ω_2 is defined as

$$\Pi = \{a\omega_1 + b\omega_2 | 0 \le a \le 1, 0 \le b \le 1\}.$$

Since ω_1, ω_2 form a basis for $\mathbb{C}$ over $\mathbb{R}$, any number $x \in \mathbb{C}$ can be written in the form $x = a\omega_1 + b\omega_2$ for some a, $b \in \mathbb{R}$. Then x can be written as the sum of an element in the lattice $L = \{m\omega_1 + n\omega_2\}$ and an element in Π, and in only one way unless a or b happens to be an integer, i.e., the element of Π happens to lie on the boundary $\partial\Pi$.

We shall always take ω_1, ω_2 in clockwise order; that is, we shall assume that ω_1/ω_2 has positive imaginary part.

Notice that the choice of ω_1, ω_2 giving the lattice L is not unique. For example, $\omega_1' = \omega_1 + \omega_2$ and ω_2 give the same lattice. More generally, we can obtain new bases ω_1', ω_2' for the lattice L by applying a matrix with integer entries and determinant 1 (see Problem 1 below).

For a given lattice L, a meromorphic function on $\mathbb{C}$ is said to be an *elliptic function* relative to L if $f(z + l) = f(z)$ for all $l \in L$. Notice that it suffices to check this property for $l = \omega_1$ and $l = \omega_2$. In other words, an elliptic function is periodic with two periods ω_1 and ω_2. Such a function is determined by its values on the fundamental parallelogram Π; and its values on opposite points of the boundary of Π are the same, i.e., $f(a\omega_1 + \omega_2) = f(a\omega_1)$, $f(\omega_1 + b\omega_2) = f(b\omega_2)$. Thus, we can think of an elliptic function $f(z)$ as a function on the set Π with opposite sides glued together. This set (more precisely, "complex manifold") is known as a "torus". It looks like a donut.

Doubly periodic functions on the complex numbers are directly analogous to singly periodic functions on the real numbers. A function $f(x)$ on $\mathbb{R}$ which satisfies $f(x + n\omega) = f(x)$ is determined by its values on the interval $[0, \omega]$. Its values at 0 and ω are the same, so it can be thought of as a function on the interval $[0, \omega]$ with the endpoints glued together. The "real manifold" obtained by gluing the endpoints is simply a circle (see Fig. I.7).

Returning now to elliptic functions for a lattice L, we let $\mathscr{E}_L$ denote the set of such functions. We immediately see that $\mathscr{E}_L$ is a subfield of the field of all meromorphic functions, i.e., the sum, difference, product, or quotient of two elliptic functions is elliptic. In addition, the subfield $\mathscr{E}_L$ is closed under differentiation. We now prove a sequence of propositions giving some very special properties which any elliptic function must have. The condition that a meromorphic function be doubly periodic turns out to be much more

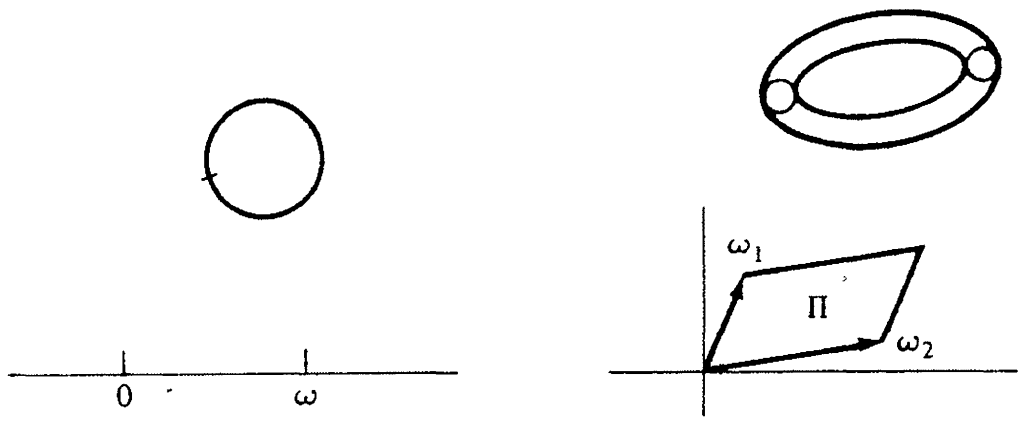

Figure I.7

restrictive than the analogous condition in the real case. The set of real-analytic periodic functions with given period is much "larger" than the set $\mathscr{E}_L$ of elliptic functions for a given period lattice L.

Proposition 3. *A function $f(z) \in \mathscr{E}_L$, $L = \{m\omega_1 + n\omega_2\}$, which has no pole in the fundamental parallelogram Π must be a constant.*

PROOF. Since Π is compact, any such function must be bounded on Π, say by a constant M. But then $|f(z)| < M$ for all z, since the values of $f(z)$ are determined by the values on Π. By Liouville's theorem, a meromorphic function which is bounded on all of $\mathbb{C}$ must be a constant. □

Proposition 4. *With the same notation as above, let $\alpha + \Pi$ denote the translate of Π by the complex number α, i.e., $\{\alpha + z | z \in \Pi\}$. Suppose that $f(z) \in \mathscr{E}_L$ has no poles on the boundary C of $\alpha + \Pi$. Then the sum of the residues of $f(z)$ in $\alpha + \Pi$ is zero.*

PROOF. By the residue theorem, this sum is equal to

$$\frac{1}{2\pi i}\int_C f(z)dz.$$

But the integral over opposite sides cancel, since the values of $f(z)$ at corresponding points are the same, while dz has opposite signs, because the path of integration is in opposite directions on opposite sides (see Fig. I.8). Thus, the integral is zero, and so the sum of residues is zero. □

Notice that, since a meromorphic function can only have finitely many poles in a bounded region, it is always possible to choose an α such that the boundary of $\alpha + \Pi$ misses the poles of $f(z)$. Also note that Proposition 4 immediately implies that a nonconstant $f(z) \in \mathscr{E}_L$ must have at least two poles (or a multiple pole), since if it had a single simple pole, then the sum of residues would not be zero.

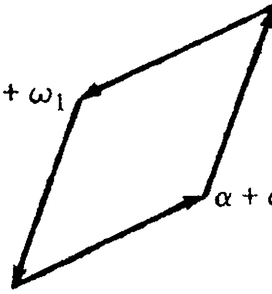

Figure I.8

Proposition 5. *Under the conditions of Proposition 4, suppose that $f(z)$ has no zeros or poles on the boundary of $\alpha + \Pi$. Let $\{m_i\}$ be the orders of the various zeros in $\alpha + \Pi$, and let $\{n_j\}$ be the orders of the various poles. Then $\Sigma m_i = \Sigma n_j$.*

PROOF. Apply Proposition 4 to the elliptic function $f'(z)/f(z)$. Recall that the logarithmic derivative $f'(z)/f(z)$ has a pole precisely where $f(z)$ has a zero or pole, such a pole is simple, and the residue there is equal to the order of zero or pole of the original $f(z)$ (negative if a pole). (Recall the argument: If $f(z) = c_m(z-a)^m + \cdots$, then $f'(z) = c_m m(z-a)^{m-1} + \cdots$, and so $f'(z)/f(z) = m(z-a)^{-1} + \cdots$.) Thus, the sum of the residues of $f'(z)/f(z)$ is $\Sigma m_i - \Sigma n_j = 0$. □

We now define what will turn out to be a key example of an elliptic function relative to the lattice $L = \{m\omega_1 + n\omega_2\}$. This function is called the Weierstrass $\wp$-function. It is denoted $\wp(z; L)$ or $\wp(z; \omega_1, \omega_2)$, or simply $\wp(z)$ if the lattice is fixed throughout the discussion. We set

$$\wp(z) = \wp(z; L) \underset{\text{def}}{=} \frac{1}{z^2} + \sum_{\substack{l \in L\\ l \neq 0}} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2}\right). \tag{4.1}$$

Proposition 6. *The sum in* (4.1) *converges absolutely and uniformly for z in any compact subset of $\mathbb{C} - L$.*

PROOF. The sum in question is taken over a two-dimensional lattice. The proof of convergence will be rather routine if we keep in mind a one-dimensional analog. If instead of L we take the integers $\mathbb{Z}$, and instead of reciprocal squares we take reciprocals, we obtain a real function $f(x) = \frac{1}{x} + \Sigma \frac{1}{x-l} + \frac{1}{l}$, where the sum is over nonzero $l \in \mathbb{Z}$. To prove absolute and uniform convergence in any compact subset of $\mathbb{R} - \mathbb{Z}$, first write the summand as $x/(l(x-l))$, and then use a comparison test, showing that the series in question basically has the same behavior as l^{-2}. More precisely, use the following lemma: if Σb_l is a convergent sum of positive terms (all our sums being over nonzero $l \in \mathbb{Z}$), and if $\Sigma f_l(x)$ has the property that $|f_l(x)/b_l|$ approaches a finite limit as $l \to \pm\infty$, uniformly for x in some set, then the sum $\Sigma f_l(x)$ converges absolutely and uniformly for x in that set. The details are easy to fill in. (By the way, our particular example of $f(x)$ can be shown to be the function $\pi \cot \pi x$; just take the logarithmic derivative of both sides of the infinite product for the sine function: $\sin \pi x = \pi x \prod_{n=1}^{\infty}(1 - x^2/n^2)$.)

The proof of Proposition 6 proceeds in the same way. First write the summand over a common denominator:

$$\frac{1}{(z-l)^2} - \frac{1}{l^2} = \frac{2z - z^2/l}{(z-l)^2 l}.$$

Then show absolute and uniform convergence by comparison with $|l|^{-3}$, where the sum is taken over all nonzero $l \in L$. More precisely, Proposition 6 will follow from the following two lemmas.

Lemma 1. *If $\Sigma\, b_l$ is a convergent sum of positive terms, where the sum is taken over all nonzero elements in the lattice L, and if $\Sigma f_l(z)$ has the property that $|f_l(z)/b_l|$ approaches a finite limit as $|l| \to \infty$, uniformly for z in some subset of $\mathbb{C}$, then the sum $\Sigma f_l(z)$ converges absolutely and uniformly for z in that set.*

Lemma 2. $\Sigma\, |l|^{-s}$ *converges if $s > 2$.*

The proof of Lemma 1 is routine, and will be omitted. We give a sketch of the proof of Lemma 2. We split the sum into sums over l satisfying $n - 1 < |l| \leq n$, as $n = 1, 2, \ldots$. It is not hard to show that the number of l in that annulus has order of magnitude n. Thus, the sum in the lemma is bounded by a constant times $\Sigma_{n=1}^{\infty} n \cdot n^{-s} = \Sigma\, n^{1-s}$, and the latter sum converges for $s - 1 > 1$.

This concludes the proof of Proposition 6. □

Proposition 7. *$\wp(z) \in \mathscr{E}_L$, and its only pole is a double pole at each lattice point.*

PROOF. The same argument as in the proof of Proposition 6 shows that for any fixed $l \in L$, the function $\wp(z) - (z - l)^{-2}$ is continuous at $z = l$. Thus, $\wp(z)$ is a meromorphic function with a double pole at all lattice points and no other poles. Next, note that $\wp(z) = \wp(-z)$, because the right side of (4.1) remains unchanged if z is replaced by $-z$ and l is replaced by $-l$; but summing over $l \in L$ is the same as summing over $-l \in L$.

To prove double periodicity, we look at the derivative. Differentiating (4.1) term-by-term, we obtain:

$$\wp'(z) = -2 \sum_{l \in L} \frac{1}{(z-l)^3}.$$

Now $\wp'(z)$ is obviously doubly periodic, since replacing z by $z + l_0$ for some fixed $l_0 \in L$ merely rearranges the terms in the sum. Thus, $\wp'(z) \in \mathscr{E}_L$. To prove that $\wp(z) \in \mathscr{E}_L$, it suffices to show that $\wp(z + \omega_i) - \wp(z) = 0$ for $i = 1, 2$. We prove this for $i = 1$; the identical argument applies to $i = 2$.

Since the derivative of the function $\wp(z + \omega_1) - \wp(z)$ is $\wp'(z + \omega_1) - \wp'(z) = 0$, we must have $\wp(z + \omega_1) - \wp(z) = C$ for some constant C. But substituting $z = -\frac{1}{2}\omega_1$ and using the fact that $\wp(z)$ is an even function, we conclude that $C = \wp(\frac{1}{2}\omega_1) - \wp(-\frac{1}{2}\omega_1) = 0$. This concludes the proof. □

Notice that the double periodicity of $\wp(z)$ was not immediately obvious from the definition (4.1).

Since $\wp(z)$ has exactly one double pole in a fundamental domain of the form $\alpha + \Pi$, by Proposition 5 it has exactly two zeros there (or one double zero). The same is true of any elliptic function of the form $\wp(z) - u$, where u is a constant. It is not hard to show (see the problems below) that $\wp(z)$ takes every value $u \in \mathbb{C} \cup \{\infty\}$ exactly twice on the torus (i.e., a fundamental parallelogram with opposite sides glued together), counting multiplicity (which means the order of zero of $\wp(z) - u$); and that the values assumed with multiplicity two are ∞, $e_1 \underset{\text{def}}{=} \wp(\omega_1/2)$, $e_2 \underset{\text{def}}{=} \wp(\omega_2/2)$, and $e_3 \underset{\text{def}}{=} \wp((\omega_1 + \omega_2)/2)$. Namely, $\wp(z)$ has a double pole at 0, while the other three points are the zeros of $\wp'(z)$.

§5. The field of elliptic functions

Proposition 7 gives us a concrete example of an elliptic function. Just as $\sin x$ and $\cos x$ play a basic role in the theory of periodic functions on $\mathbb{R}$, because of Fourier expansion, similarly the functions $\wp(z)$ and $\wp'(z)$ play a fundamental role in the study of elliptic functions. But unlike in the real case, we do not even need infinite series to express an arbitrary elliptic function in terms of these two basic ones.

Proposition 8. $\mathscr{E}_L = \mathbb{C}(\wp, \wp')$, *i.e., any elliptic function for L is a rational expression in* $\wp(z; L)$ *and* $\wp'(z; L)$. *More precisely, given* $f(z) \in \mathscr{E}_L$, *there exist two rational functions* $g(X)$, $h(X)$ *such that* $f(z) = g(\wp(z)) + \wp'(z)h(\wp(z))$.

Proof. If $f(z)$ is an elliptic function for L, then so are the two even functions

$$\frac{f(z) + f(-z)}{2} \quad \text{and} \quad \frac{f(z) - f(-z)}{2\wp'(z)}.$$

Since $f(z)$ is equal to the first of these functions plus $\wp'(z)$ times the second, to prove Proposition 8 it suffices to prove

Proposition 9. *The subfield* $\mathscr{E}_L^+ \subset \mathscr{E}_L$ *of even elliptic functions for L is generated by* $\wp(z)$, *i.e.,* $\mathscr{E}_L^+ = \mathbb{C}(\wp)$.

Proof. The idea of the proof is to cook up a function which has the same zeros and poles as $f(z)$ using only functions of the form $\wp(z) - u$ with u a constant.