to equal $\chi(d')f_\chi$, i.e., $f_\chi \in M_k(N, \chi)$. Finally, we sum the f_χ over all characters χ modulo N, reverse the order of summation over d and χ, and obtain:

$$\sum f_\chi = \sum_{d\in(\mathbb{Z}/N\mathbb{Z})^*} \frac{1}{\phi(N)} \sum_\chi \bar{\chi}(d) \; f|[\gamma_d]_k,$$

which is equal to f, because the inner sum is 1 if $d = 1$ and 0 otherwise. Thus, f can be written as a sum of functions in $M_k(N, \chi)$, as claimed. □

Notice that $M_k(N, \chi) = 0$ if χ has a different parity from k, i.e., if $\chi(-1) \neq (-1)^k$. This follows by taking $\gamma = -I$ in the definition (3.29) and recalling that $f|[-I]_k = (-1)^k f$.

For example, as an immediate corollary of Proposition 28 and the preceding remark, we have

Proposition 29.
$$M_k(\Gamma_1(4)) = \begin{cases} M_k(4, 1), & k \text{ even;} \\ M_k(4, \chi), & k \text{ odd,} \end{cases}$$

where 1 *denotes the trivial character and* χ *the unique nontrivial character modulo* 4.

Notice that the relationship in (3.29) is multiplicative in γ; that is, if it holds for γ_1 and γ_2, then it holds for their product. Thus, as in the case of modular forms without character, to show that $f(z)$ is in $M_k(N, \chi)$ it suffices to check the transformation rule on a set of elements that generate $\Gamma_0(N)$.

As another example, we look at $\Theta^2(z) = (\Sigma_{n\in\mathbb{Z}} q^{n^2})^2$, whose n-th q-expansion coefficient is the number of ways n can be written as a sum of two squares.

Proposition 30. $\Theta^2 \in M_1(\Gamma_1(4)) = M_1(4, \chi)$, *where* $\chi(d) = (-1)^{(d-1)/2}$.

Proof. It suffices to verify the transformation rule for $-I$, T, and $ST^4S = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}$, which generate $\Gamma_0(4)$ (see Problem 13 of §III.1). This is immediate for T, since Θ^2 has period 1. Next, the relation $f|[-I]_1 = -f = \chi(-1)f$ holds for any f, by definition. So it remains to treat the case ST^4S. Let

$$\alpha_N \underset{\text{def}}{=} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, \quad \text{so that} \quad \alpha_N^{-1} = -\frac{1}{N}\alpha_N,$$
$$\text{and} \quad \alpha_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha_N^{-1} = \begin{pmatrix} d & -c/N \\ -Nb & a \end{pmatrix}. \tag{3.30}$$

We write $ST^4S = -\alpha_4 T \alpha_4^{-1} = \frac{1}{4}\alpha_4 T \alpha_4$, and use the relationship $\Theta^2|[\alpha_4]_1 = -i\Theta^2$ (see (3.5)) to obtain

$$\Theta^2|[ST^4S]_1 = \Theta^2|[\alpha_4 T \alpha_4]_1 = -i\Theta^2|[T\alpha_4]_1 = -i\Theta^2|[\alpha_4]_1 = -\Theta^2.$$

(Recall that the scalar matrix $\frac{1}{4}I$ acts trivially on all functions, i.e., $[1/4]_1 =$ identity.)

To finish the proof of the proposition, we must show the cusp condition, i.e., that $\Theta^2|[\gamma_0]_1$ is finite at infinity for all $\gamma_0 \in \Gamma$. But the square of $\Theta^2|[\gamma_0]_1$ is $\Theta^4|[\gamma_0]_2$, and it will be shown in Problem 11 below that $\Theta^4 \in M_2(\Gamma_0(4))$; in particular, this means that $\Theta^4|[\gamma_0]_2$, and hence also $\Theta^2|[\gamma_0]_1$, is finite at infinity. This completes the proof. □

The spaces $M_k(N, \chi)$ include many of the most important examples of modular forms, and will be our basic object of study in several of the sections that follow. We also introduce the notation $S_k(N, \chi)$ to denote the subspace of cusp forms: $S_k(N, \chi) \underset{\text{def}}{=} M_k(N, \chi) \cap S_k(\Gamma_1(N))$.

The Mellin transform of a modular form. Suppose that $f(z) = \Sigma\, a_n q_N^n$ (where $q_N = e^{2\pi i z/N}$) is a modular form of weight k for a congruence subgroup Γ' of level N. Further suppose that $|a_n| = O(n^c)$ for some constant $c \in \mathbb{R}$, i.e., that a_n/n^c is bounded as $n \to \infty$. It is not hard to see that the q_N-expansion coefficients for the Eisenstein series $G_k^{\underline{a} \bmod N}$ have this property with $c = k - 1 + \varepsilon$ for any $\varepsilon > 0$. For example, in the case $\Gamma' = \Gamma$, the coefficients are a constant multiple of $\sigma_{k-1}(n)$, and it is not hard to show that $\sigma_{k-1}(n)/n^{k-1+\varepsilon} \to 0$ as $n \to \infty$. We shall later show that, if f is a cusp form, we can take $c = k/2 + \varepsilon$. It has been shown (as a consequence of Deligne's proof of the Weil conjectures) that one can actually do better, and take $c = (k-1)/2 + \varepsilon$.

In Chapter II we saw that the Mellin transform of $\theta(t) = \Sigma\, e^{-\pi t n^2}$ and certain generalizations are useful in investigating some important Dirichlet series, such as the Riemann zeta-function, Dirichlet L-functions, and the Hasse–Weil L-function of the elliptic curves E_n: $y^2 = x^3 - n^2 x$. We now look at the Mellin transform for modular forms.

Because we use a variable z in the upper half-plane rather than t (e.g., $t = -2iz$), we define the Mellin transform by integrating along the positive imaginary axis rather than the positive real axis.

The most important case is $\Gamma' = \Gamma_1(N)$. For now we shall also assume that $f(\infty) = 0$. Thus, let $f(z) = \Sigma_{n=1}^{\infty} a_n q^n \in M_k(\Gamma_1(N))$. (Recall that since $T \in \Gamma_1(N)$, we have an expansion in powers of $q = e^{2\pi i z}$ rather than q_N.) We set

$$g(s) \underset{\text{def}}{=} \int_0^{i\infty} f(z) z^{s-1} dz. \tag{3.31}$$

We now show that if $f(z) = \Sigma_{n=1}^{\infty} a_n q^n$ with $|a_n| = O(n^c)$, then the integral $g(s)$ defined in (3.31) converges for $\operatorname{Re} s > c + 1$:

$$\begin{aligned}\int_0^{i\infty} f(z) z^{s-1} dz &= \sum_{n=1}^{\infty} a_n \int_0^{i\infty} z^s e^{2\pi i n z} \frac{dz}{z} \\ &= \sum_{n=1}^{\infty} a_n \left(-\frac{1}{2\pi i n}\right)^s \int_0^{\infty} t^s e^{-t} \frac{dt}{t} \qquad \text{(where } t = -2\pi i n z\text{)} \\ &= (-2\pi i)^{-s} \Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s} \qquad \text{(see (4.6) of Ch. II)}\end{aligned} \tag{3.32}$$

(where the use of Γ in the gamma-function $\Gamma(s)$ has no relation to its use in the notation for congruence subgroups; but in practice the use of the same letter Γ should not cause any confusion). Since $|a_n n^{-s}| = O(n^{c-\operatorname{Re} s})$, this last sum is absolutely convergent (and the interchanging of the order of integration and summation was justified).

If $f(z) = \Sigma_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_1(N))$ has $a_0 \neq 0$, we replace $f(z)$ by $f(z) - a_0$ in (3.31). In either case, we then obtain $g(s) = (-2\pi i)^{-s}\Gamma(s)L_f(s)$, where

$$L_f(s) \underset{\text{def}}{=} \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{for} \quad \operatorname{Re} s > c+1, \quad \text{if} \quad f(z) = \sum_{n=0}^{\infty} a_n q^n \quad \text{with} \quad |a_n| = O(n^c). \tag{3.33}$$

In addition to their invariance under $[\gamma]_k$ for $\gamma \in \Gamma_1(N)$, many modular forms also transform nicely under $[\alpha_N]_k$, where $\alpha_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ as in (3.30). It will be shown in the exercises that, for example, any function in $M_k(N, \chi)$ for χ a real character (i.e., its values are ± 1) can be written as a sum of two functions satisfying

$$f|[\alpha_N]_k = Ci^{-k}f, \qquad C = 1 \text{ or } -1, \tag{3.34}$$

where one of the functions satisfies (3.34) with $C = 1$ and the other with $C = -1$. An example we already know of a function satisfying (3.34) is Θ^2: the relation (3.5) is a special case of (3.34) with $k = 1$, $C = 1$, $N = 4$.

We now show that if (3.34) holds, then we have a functional equation for the corresponding Mellin transform $g(s)$ which relates $g(s)$ to $g(k-s)$. For simplicity, we shall again suppose that $f(z) = \Sigma a_n q^n$ with $a_0 = 0$. We can write (3.34) explicitly as follows, by the definition of $[\alpha_N]_k$:

$$f(-1/Nz) = CN^{-k/2}(-iNz)^k f(z). \tag{3.35}$$

In (3.31), we break up the integral into the part from 0 to $i/\sqrt{N}$ and the part from $i/\sqrt{N}$ to $i\infty$. We choose $i/\sqrt{N}$ because it is the fixed point in H of $\alpha_N: z \mapsto -1/Nz$. We have

$$\begin{aligned} g(s) &= \int_{i/\sqrt{N}}^{i\infty} f(z)z^s \frac{dz}{z} - \int_{i/\sqrt{N}}^{i\infty} f(-1/Nz)(-1/Nz)^s \frac{d(-1/Nz)}{-1/Nz} \\ &= \int_{i/\sqrt{N}}^{i\infty} (f(z)z^s + f(-1/Nz)(-1/Nz)^s)\frac{dz}{z} \\ &= \int_{i/\sqrt{N}}^{i\infty} (f(z)z^s + i^k CN^{-k/2} f(z)(-1/Nz)^{s-k})\frac{dz}{z}, \end{aligned}$$

because of (3.35).

In the first place, this integral converges to an entire function of s, because $f(z)$ decreases exponentially as $z \to i\infty$. That is, because the lower limit of integration has been moved away from zero, we no longer have to worry about the behavior of the integrand near 0. (Compare with the proof of Proposition 13 in Chapter II, where we used a similar technique to find a rapidly convergent series for the critical value of the Hasse–Weil L-function; see the remark following equation (6.7) in §II.6.)

Moreover, if we replace s by $k - s$ in the last integral and factor out $i^k C N^{-k/2}(-N)^s$, we obtain:

$$g(k-s) = i^k C N^{-k/2}(-N)^s \int_{i/\sqrt{N}}^{i\infty} (i^{-k} C N^{k/2}(-N)^{-s} f(z) z^{k-s} + f(z) z^s) \frac{dz}{z}$$

$$= i^k C N^{-k/2}(-N)^s \int_{i/\sqrt{N}}^{i\infty} (i^k C N^{-k/2} f(z)(-1/Nz)^{s-k} + f(z) z^s) \frac{dz}{z}$$

$$= i^k C N^{-k/2}(-N)^s g(s),$$

because the last integral is the same as our earlier integral for $g(s)$. This equality can be written in the form

$$(-i\sqrt{N})^s g(s) = C(-i\sqrt{N})^{k-s} g(k-s).$$

Thus, by (3.32)–(3.33), if we define $\Lambda(s)$ for $\operatorname{Re} s > c + 1$ by

$$\Lambda(s) = (-i\sqrt{N})^s g(s) = (\sqrt{N}/2\pi)^s \Gamma(s) L_f(s), \tag{3.36}$$

we have shown that $\Lambda(s)$ extends to an entire function of s, and satisfies the functional equation

$$\Lambda(s) = C\Lambda(k-s). \tag{3.37}$$

As an example of this result, we can take $f(z) = \Delta(z) \in S_{12}(\Gamma)$, which satisfies (3.35) with $N = 1$, $k = 12$, $C = 1$. Then $\Delta(z) = \Sigma_{n=1}^{\infty} \tau(n) q^n$, $L_\Delta(s) = \Sigma_{n=1}^{\infty} \tau(n) n^{-s}$, and $\Lambda(s) = (2\pi)^{-s}\Gamma(s) L_\Delta(s)$ satisfies the relation: $\Lambda(s) = \Lambda(12 - s)$.

The derivation of (3.37) from (3.34) indicates a close connection between Dirichlet series with a functional equation and modular forms. We came across Dirichlet series with a functional equation in a very different context in Chapter II. Namely, the Hasse–Weil L-function of the elliptic curve E_n: $y^2 = x^3 - n^2 x$ satisfies (3.36)–(3.37) with $k = 2$, $N = 32n^2$ for n odd and $16n^2$ for n even, $C = (\frac{-2}{n})$ for n odd and $(\frac{-2}{n/2})$ for n even (see (5.10)–(5.12) in Ch. II). We also saw that the Hasse–Weil L-function of the elliptic curve $y^2 = x^3 + 16$ satisfies (3.36)–(3.37) with $k = 2$, $N = 27$, $C = 1$ (see Problem 8(d) of §II.5).

So the question naturally arises: Can one go the other way? Does every Dirichlet series with the right type of functional equation come from some modular form, i.e., is it of the form $L_f(s)$ for some modular form f? In particular, can the Hasse–Weil L-functions we studied in Chapter II be obtained by taking the Mellin transform of a suitable modular form of weight 2? That is, if we write $L(E_n, s)$ in the form $\Sigma_{m=1}^{\infty} b_m m^{-s}$ (see (5.3) in Ch. II), is $\Sigma_{m=1}^{\infty} b_m q^m$ the q-expansion of a weight two modular form?

Hecke [1936] and Weil [1967] showed that the answer to these questions is basically yes, but with some qualifications. We shall not give the details, which are available in [Ogg 1969], but shall only outline the situation and state Weil's fundamental theorem on the subject.

Suppose that $L(s) = \Sigma a_n n^{-s}$ satisfies (3.36)–(3.37) (and a suitable hypothesis about convergence). Using the "inverse Mellin transform", one can reverse the steps that led to (3.36)–(3.37), and find that $f(z) = \Sigma a_n q^n$ satisfies (3.34). For now, let us suppose that $N = \lambda^2$ is a perfect square, and that $C = i^k$ (k even). Then, if $f(z)$ satisfies (3.35), it follows that $f_1(z) \underset{\text{def}}{=} f(z/\lambda) = \Sigma a_n q_\lambda^n$ satisfies: $f_1(-1/z) = z^k f_1(z)$. Thus, f_1 is invariant under $[S]_k$ and $[T^\lambda]_k$, and hence is invariant under the group generated by S and T^λ. Hecke denoted that group $\mathfrak{G}(\lambda)$. We have encountered the group $\mathfrak{G}(2)$ before.

In this way one can show, for example, that $L(E_{2n_0}, s)$ corresponds to a modular form (actually, a cusp form) of weight 2 for $\mathfrak{G}(8n_0)$.

Unfortunately, however, Hecke's groups $\mathfrak{G}(\lambda)$ turn out not to be large enough to work with satisfactorily. In general, they are not congruence subgroups. ($\mathfrak{G}(2) \supset \Gamma(2)$ is an exception.)

But one can do much better. Weil showed, roughly speaking, that if one has functional equations analogous to (3.36)–(3.37) for enough "twists" $\Sigma \chi(n) a_n n^{-s}$ of the Dirichlet series $\Sigma a_n n^{-s}$, then the corresponding q-expansion is in $M_k(\Gamma_0(N))$. We now give a more precise statement of Weil's theorem.

Let χ_0 be a fixed Dirichlet character modulo N (χ_0 is allowed to be the trivial character). Let χ be a variable Dirichlet character of conductor m, where m is either an odd prime not dividing N, or else 4 (we allow $m = 4$ only if N is odd). By a "large" set of values of m we shall mean that the set contains at least one m in any given arithmetic progression $\{u + jv\}_{j \in \mathbb{Z}}$, where u and v are relatively prime. According to Dirichlet's theorem, any such arithmetic progression contains a prime; thus, a "large" set of primes is one which satisfies (this weak form of) Dirichlet's theorem. By a "large" set of characters χ we shall mean the set of all nontrivial χ modulo m for a "large" set of m.

Let $C = \pm 1$, and for any χ of conductor m set

$$C_\chi = C\chi_0(m)\chi(-N)g(\chi)/g(\bar{\chi}), \tag{3.38}$$

where $g(\chi) = \Sigma_{j=1}^{m} \chi(j) e^{2\pi i j/N}$ is the Gauss sum. Given a q-expansion $f(z) = \Sigma_{n=0}^{\infty} a_n q^n$, $q = e^{2\pi i z}$, for which $|a_n| = O(n^c)$, we define $L_f(s)$ by (3.33) and $\Lambda(s)$ by (3.36), and we further define

$$L_f(\chi, s) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}; \qquad \Lambda(\chi, s) = (m\sqrt{N}/2\pi)^s \Gamma(s) L_f(\chi, s). \tag{3.39}$$

Weil's Theorem. *Suppose that* $f(z) = \Sigma_{n=0}^{\infty} a_n q^n$, $q = e^{2\pi i z}$, *has the property that* $|a_n| = O(n^c)$, $c \in \mathbb{R}$. *Suppose that for* $C = 1$ *or* -1 *the function* $\Lambda(s)$ *defined by* (3.36) *has the property that* $\Lambda(s) + a_0(1/s + C/(k-s))$ *extends* *to an entire function which is bounded in any vertical strip of the complex plane, and satisfies the functional equation* $\Lambda(s) = C\Lambda(k - s)$. *Further suppose that for a "large" set of characters* χ *of conductor* m *(in the sense explained above), the function* $\Lambda(\chi, s)$ *defined by* (3.39) *extends to an entire function which is bounded in any vertical strip, and satisfies the functional equation* $\Lambda(\chi, s) = C_\chi \Lambda(\bar{\chi}, k - s)$, *with* C_χ *defined in* (3.38).

Then $f \in M_k(N, \chi_0)$, *and* f *satisfies* (3.34). *If, in addition,* $L_f(s)$ *converges absolutely for* $\mathrm{Re}\, s > k - \varepsilon$ *for some* $\varepsilon > 0$, *then* f *is a cusp form.*

One can show that the Hasse–Weil L-functions of the elliptic curves in Chapter II satisfy the hypotheses of Weil's theorem (with $\chi_0 = 1$). The same techniques as in the proof of the theorem in §II.5 can be used to show this. However, one must consider the Hecke L-series obtained in (5.6) of Ch. II by replacing $\tilde{\chi}_n(I)$ by the character $\tilde{\chi}_n(I)\chi(\mathbb{N}I)$ with χ any Dirichlet character modulo m as in Weil's theorem. For example, if we do this for $L(E_1, s)$, where E_1 is the elliptic curve $y^2 = x^3 - x$, we can conclude by Weil's theorem that

$$f_{E_1}(z) = q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} + \sum_{m \geq 25} b_m q^m \qquad (3.40)$$

(see (5.4) of Ch. II) is a cusp form of weight two for $\Gamma_0(32)$.

If we form the q-expansion corresponding to the L-series of E_n: $y^2 = x^3 - n^2x$, namely, $f_{E_n}(z) = \Sigma\, \chi_n(m) b_m q^m$, it turns out that $f_{E_n} \in M_2(\Gamma_0(32n^2))$ for n odd and $f_{E_n} \in M_2(\Gamma_0(16n^2))$ for n even. Note that when $n \equiv 1 \bmod 4$, so that χ_n is a character of conductor n, this is an immediate consequence of the fact that $f_{E_1} \in M_2(\Gamma_0(32))$, by Proposition 17(b).

More generally, it can be shown that the Hasse–Weil L-function for any elliptic curve with complex multiplication satisfies the hypotheses of Weil's theorem with $k = 2$, and so corresponds to a weight two modular form (actually, a cusp form) for $\Gamma_0(N)$. (N is the so-called "conductor" of the elliptic curve.)

Many elliptic curves without complex multiplication are also known to have this property. In fact, it was conjectured (by Taniyama and Weil) that every elliptic curve defined over the rational numbers has L-function which satisfies Weil's theorem for some N. Geometrically, the cusp forms of weight two can be regarded as holomorphic differential forms on the Riemann surface $\Gamma_0(N)\backslash H$ (i.e., the fundamental domain with $\Gamma_0(N)$-equivalent boundary sides identified and the cusps included). The Taniyama–Weil conjecture then can be shown to take the form: every elliptic curve over $\mathbb{Q}$ can be obtained as a quotient of the Jacobian of some such Riemann surface.

For more information about the correspondence between modular forms and Dirichlet series, see [Hecke 1981], [Weil 1967], [Ogg 1969], and [Shimura 1971].

Problems

1. Let $\alpha \in GL_2^+(\mathbb{Q})$, and let $g(z) = f(\alpha z)$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Notice that $f(\alpha z)|[\gamma]_k$ was defined to be $(c\alpha z + d)^{-k} f(\gamma\alpha z)$, which is *not* the same as $g(z)|[\gamma]_k = (cz + d)^{-k}$ $f(\alpha\gamma z)$. Show that if $\alpha = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$, i.e., if $\alpha z = nz$, then $g(z)|[\gamma]_k = f(\alpha z)|[\alpha\gamma\alpha^{-1}]_k = f(nz)|[\left(\begin{smallmatrix} a & nb \\ c/n & d \end{smallmatrix}\right)]_k$.

2. Let Γ' be a congruence subgroup of Γ of level N, and denote $\Gamma'_s = \{\gamma \in \Gamma' | \gamma s = s\}$ for $s \in \mathbb{Q} \cup \{\infty\}$. Let $s = \alpha^{-1}\infty$, $\alpha \in \Gamma$.
   (a) Prove that $\alpha\Gamma'_s\alpha^{-1} = (\alpha\Gamma'\alpha^{-1})_\infty$.
   (b) Show that there exists a unique positive integer h (called the "ramification index" of Γ' at s) such that

      (i) in the case $-I \in \Gamma'$
      $$\Gamma'_s = \pm\alpha^{-1}\{T^{hm}\}_{n\in\mathbb{Z}}\alpha;$$

      (ii) in the case $-I \notin \Gamma'$ either
      $$\Gamma'_s = \alpha^{-1}\{T^{hn}\}_{n\in\mathbb{Z}}\alpha; \text{ or} \qquad \text{(IIa)}$$
      $$\Gamma'_s = \alpha^{-1}\{(-T^h)^n\}_{n\in\mathbb{Z}}\alpha. \qquad \text{(IIb)}$$

   Show that h is a divisor of N.
   (c) Show that the integer h and the type (I, IIa, or IIb) of s does not depend on the choice of $\alpha \in \Gamma$ with $s = \alpha^{-1}\infty$; and they only depend on the Γ'-equivalence class of s.
   (d) Show that if $\alpha^{-1}\infty$ is of type I or IIa and $f \in M_k(\Gamma')$, then $f|[\alpha^{-1}]_k$ has a Fourier expansion in powers of q_h. A cusp of Γ' is called "regular" if it is of type I or IIa; it is called "irregular" if it is of type IIb.
   (e) Show that if $\alpha^{-1}\infty$ is an irregular cusp, and $f \in M_k(\Gamma')$, then $f|[\alpha^{-1}]_k$ has a Fourier expansion in powers of q_{2h} in which only odd powers appear if k is odd and only even powers appear if k is even. If k is odd, note that this means that to show that $f \in M_k(\Gamma')$ is a cusp form one need only check the q-expansions at the regular cusps.

3. Let h be any positive integer, and suppose $2h|N$, $N \geq 4$. Let Γ' be the following level N congruence subgroup: $\Gamma' = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} -1 & -h \\ 0 & -1 \end{smallmatrix}\right)^j \bmod N \text{ for some } j\}$. Show that ∞ is a cusp of type IIb.

4. (a) Show that $\Gamma_1(N)$ has the same cusps as $\Gamma_0(N)$ for $N = 3, 4$.
   (b) Note that $-I \notin \Gamma_1(N)$ for $N > 2$. Which of the cusps of $\Gamma_1(3)$ and $\Gamma_1(4)$, if any, are irregular?

5. Find the ramification indices of Γ' at all of its cusps when:
   (a) $\Gamma' = \Gamma_0(p)$ (p a prime);
   (b) $\Gamma' = \Gamma_0(p^2)$;
   (c) $\Gamma' = \Gamma(2)$.

6. Prove that if $\Gamma' \subset \Gamma$ is a normal subgroup, then all cusps have the same ramification index, namely $[\Gamma_\infty : \pm\Gamma'_\infty]$.

7. (a) Show that any weight zero modular function for $\Gamma' \subset \Gamma$ satisfies a polynomial of degree $[\Gamma : \Gamma']$ over the field $\mathbb{C}(j)$ of weight zero modular functions for Γ.