Other parameters besides the number of parts $\#(\lambda)$ of a partition λ will interest us from time to time; so we shall have occasion to consider other types of partition generating functions of several variables.

The preceding comments suggest the interest of considering infinite series and products in two (or more) variables. In the following section, we shall develop an elementary technique for proving many series and product identities. We shall obtain several classical theorems of great importance, such as Jacobi's triple product identity. As will become clear in Section 2.3, the results of Section 2.2 are quite useful in treating partition identities. It is possible, however, to skip Section 2.2 and read Section 2.3, referring back only for the statements of theorems. For the reader who needs series transformations to attack a partition problem, the first six examples at the end of this chapter form a good test of the techniques used in Section 2.2.

2.2 Elementary Series-Product Identities

We begin with a theorem due to Cauchy; as we shall see, this result provides the tool for doing everything else in this section.

THEOREM 2.1. *If* $|q| < 1$, $|t| < 1$, *then*

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-atq^n)}{(1-tq^n)}. \tag{2.2.1}$$

Remark. We shall try always to state our theorems with as little notational disguise as possible. However, for the proofs, it seems only sensible to use the following standard abbreviations

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a;q)_n,$$

$$(a)_0 = 1.$$

We may define $(a)_n$ for all real numbers n by

$$(a)_n = (a)_\infty/(aq^n)_\infty.$$

The series in (2.2.1) is an example of a basic hypergeometric series. The study of basic series (or q-series, or Eulerian series) is an extensive branch of analysis and we shall only touch upon it in this book. Most of the theorems of this section may be viewed as elementary results in the theory of basic hypergeometric series. Theorem 2.1 has become known as the "q-analog of the binon series," for if we write $a = q^\alpha$ where α is a nonnegative integer, then (2.2.1) formally tends to

$$1 + \sum_{n=1}^{\infty} \binom{\alpha + n - 1}{n} t^n = (1 - t)^{-\alpha}, \quad \text{as} \quad q \to 1^-.$$

Proof. Let us consider

$$F(t) = \prod_{n=0}^{\infty} \frac{(1 - atq^n)}{(1 - tq^n)} = \sum_{n=0}^{\infty} A_n t^n \tag{2.2.2}$$

where $A_n = A_n(a, q)$. We note that the A_n exist since the infinite product is uniformly convergent for fixed a and q inside $|t| \leqslant 1 - \varepsilon$, and therefore it defines a function of t analytic inside $|t| < 1$.

Now

$$\begin{aligned}(1 - t)F(t) &= (1 - at) \prod_{n=1}^{\infty} \frac{(1 - atq^n)}{(1 - tq^n)} \\ &= (1 - at) \prod_{n=0}^{\infty} \frac{(1 - atq^{n+1})}{(1 - tq^{n+1})} = (1 - at)F(tq). \end{aligned} \tag{2.2.3}$$

Clearly $A_0 = F(0) = 1$, and by comparing coefficients of t^n in the extremes of (2.2.3) we see that

$$A_n - A_{n-1} = q^n A_n - aq^{n-1} A_{n-1},$$

or

$$A_n = \frac{(1 - aq^{n-1})}{(1 - q^n)} A_{n-1}. \tag{2.2.4}$$

Iterating (2.2.4) we see that

$$\begin{aligned} A_n &= \frac{(1 - aq^{n-1})(1 - aq^{n-2})\cdots(1 - a)A_0}{(1 - q^n)(1 - q^{n-1})\cdots(1 - q)} \\ &= \frac{(a)_n}{(q)_n}. \end{aligned}$$

Substituting this value for A_n into (2.2.2), we obtain the theorem. ■

Euler found the two following special cases of Theorem 2.1. Each of these identities is directly related to partitions in Example 17 at the end of this chapter.

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}=\prod_{m=1}^{\infty}(1-q^m)^{-1}. \qquad (2.2.9)$$

Remark. Equation (2.2.8) is due to Cauchy, and Eq. (2.2.9) is due to Euler.

Proof. First we note that (2.2.9) is obtained from (2.2.8) by setting $z = q$. In Corollary 2.4, set $a = \alpha^{-1}$, $b = \beta^{-1}$, $c = z$. Hence

$$1+\sum_{n=1}^{\infty}\frac{(\alpha-1)(\alpha-q)\cdots(\alpha-q^{n-1})(\beta-1)(\beta-q)\cdots(\beta-q^{n-1})z^n}{(q)_n(z)_n}$$

$$=\frac{(z\alpha)_\infty(z\beta)_\infty}{(z)_\infty(z\alpha\beta)_\infty},$$

and if we set $\alpha = \beta = 0$ in this identity, we obtain (2.2.8). ∎

COROLLARY 2.7. *If* $|q| < 1$,

$$1+\sum_{n=1}^{\infty}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{m=1}^{\infty}(1-aq^{2m-1})(1+q^m).$$

Proof. Set $b = \beta^{-1}$ in Corollary 2.5. Hence

$$\sum_{n=0}^{\infty}\frac{(a)_n(\beta-1)(\beta-q)\cdots(\beta-q^{n-1})(-q)^n}{(q)_n(aq\beta)_n}=\frac{(aq;q^2)_\infty(-q)_\infty(aq^2\beta^2;q^2)_\infty}{(aq\beta)_\infty(-q\beta)_\infty}.$$

Now set $\beta = 0$ in this identity and we obtain the desired result. ∎

The next result, Jacobi's triple product identity, may be viewed as a corollary of Corollary 2.2; however, it is so important that we label it a theorem.

THEOREM 2.8. *For* $z \neq 0$, $|q| < 1$,

$$\sum_{n=-\infty}^{\infty}z^nq^{n^2}=\prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1}). \qquad (2.2.10)$$

Proof. For $|z| > |q|$, $|q| < 1$,

$$\prod_{n=0}^{\infty}(1+zq^{2n+1})=\sum_{m=0}^{\infty}\frac{z^mq^{m^2}}{(q^2;q^2)_m} \qquad \text{(by (2.2.6))}$$

$$=\frac{1}{(q^2;q^2)_\infty}\sum_{m=0}^{\infty}z^mq^{m^2}(q^{2m+2};q^2)_\infty$$

$$=\frac{1}{(q^2;q^2)_\infty}\sum_{m=-\infty}^{\infty}z^mq^{m^2}(q^{2m+2};q^2)_\infty$$

(since $(q^{2m} \quad q^2)_\infty$ vanishes for m negative)

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m^2} \sum_{r=0}^{\infty} \frac{(-1)^r q^{r^2+2mr+r}}{(q^2;q^2)_r}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r z^{-r} q^r}{(q^2;q^2)_r} \sum_{m=-\infty}^{\infty} q^{(m+r)^2} z^{m+r}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-q/z)^r}{(q^2;q^2)_r} \sum_{m=-\infty}^{\infty} q^{m^2} z^m$$

$$= \frac{1}{(q^2;q^2)_\infty(-q/z;q^2)_\infty} \sum_{m=-\infty}^{\infty} q^{m^2} z^m.$$

This is the desired result. Note that absolute convergence pertains everywhere only so long as $|z| > |q|$, $|q| < 1$. However, the full result of the theorem follows either by invoking analytic continuation, or by observing that the entire argument may be carried out again with z^{-1} replacing z. ■

COROLLARY 2.9. *For* $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2 - in}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2 - in}(1 - q^{(2n+1)i})$$

$$= \prod_{n=0}^{\infty} (1 - q^{(2k+1)(n+1)})(1 - q^{(2k+1)n+i})(1 - q^{(2k+1)(n+1)-i}). \tag{2.2.11}$$

Proof. Replace q by $q^{k+\frac{1}{2}}$ and then set $z = -q^{k+\frac{1}{2}-i}$ in (2.2.10). This substitution immediately yields the equality of the extremes in (2.2.11). Now

$$\sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2 - in}(1 - q^{(2n+1)i})$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2 - in} + \sum_{n=1}^{\infty} (-1)^n q^{(2k+1)n(n-1)/2 + in}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2 - in} + \sum_{n=-1}^{-\infty} (-1)^n q^{(2k+1)n(n+1)/2 - in}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2 - in}.$$ ■

We remark that Corollary 2.9 reduces to Corollary 1.7 when $k = i = 1$ once we observe that

$$\prod_{n=0}^{\infty}(1 - q^{3n+3})(1 - q^{3n+1})(1 - q^{3n+2}) = \prod_{n=1}^{\infty}(1 - q^n).$$

COROLLARY 2.10 (Gauss)

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \prod_{m=1}^{\infty}\frac{(1 - q^m)}{(1 + q^m)}, \tag{2.2.12}$$

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{m=1}^{\infty}\frac{(1 - q^{2m})}{(1 - q^{2m-1})}. \tag{2.2.13}$$

Proof. By (2.2.10) with $z = -1$,

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = (q^2; q^2)_\infty(q; q^2)_\infty(q; q^2)_\infty$$

$$= (q)_\infty(q; q^2)_\infty = (q)_\infty/(-q)_\infty$$

where the final equation follows from (1.2.5). Next

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \tfrac{1}{2}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2}$$

$$= \tfrac{1}{2}(q)_\infty(-q)_\infty(-1)_\infty$$

$$= (q)_\infty(-q)_\infty(-q)_\infty = (q^2; q^2)_\infty(-q)_\infty = (q^2; q^2)_\infty/(q; q^2)_\infty$$

where again the final equation follows from (1.2.5). ∎

So far this section seems filled with much mathematics and little commentary. It has been the hope that the power of Theorem 2.1 and simple series manipulation would be fully appreciated if numerous significant results followed in rapid-fire order. The reader will have a chance to practice the techniques involved in the many examples at the end of this chapter.

2.3 Applications to Partitions

We shall prove four theorems on partitions utilizing either the actual results or the methods of Section 2.2. We conclude with an examination of "Durfee squares," which allows us to obtain (2.2.9) from purely combinatorial considerations. We begin with an interpretation of Corollary 2.9.

THEOREM 2.11. *Let $\mathscr{D}(k, i)$ denote all those partitions with distinct parts in which each part is congruent to* 0, $\pm i$ (modulo $2k + 1$). *Let* $p_e(\mathscr{D}(k, i), n)$ (resp. $p_o(\mathscr{D}(k, i), n)$) *denote the number of partitions of n taken from $\mathscr{D}(k, i)$ with an even* (resp. *odd*) *number of parts. Then*