Theorem 4.4.2. *(The Pfaff-Saalschütz identity)*

$$\sum_k \frac{(a+k)!(b+k)!(c-a-b+n-1-k)!}{(k+1)!(n-k)!(c+k)!} = \frac{(a-1)!(b-1)!(c-a-b-1)!(c-a+n)!(c-b+n)!}{(c-a-1)!(c-b-1)!(n+1)!(c+n)!}.$$

Proof: Take

$$R(n,k) = -\frac{(b+k)(a+k)}{(c-b+n+1)(c-a+n+1)}.$$

■

Theorem 4.4.3. *(Dixon's identity)*

$$\sum_k (-1)^k \binom{n+b}{n+k}\binom{n+c}{c+k}\binom{b+c}{b+k} = \frac{(n+b+c)!}{n!b!c!}.$$

Proof: Take $R(n,k) = (c+1-k)(b+1-k)/(2(n+k)(n+b+c+1))$. ■

4.5 Generating functions and unimodality, convexity, etc.

The binomial coefficients are the prototype of *unimodal* sequences. A sequence is unimodal if its entries rise to a maximum and then decrease. The binomial coefficients $\{\binom{n}{k}\}_{k=0}^n$ do just that. The maximum ('mode') of the binomial coefficient sequence occurs at $k = n/2$ if n is even, and at $k = (n \pm 1)/2$ if n is odd.

In general, a sequence $c_0, c_1, \ldots, c_n$ is unimodal if there exist indices r, s such that

$$c_0 \le c_1 \le c_2 \le \cdots \le c_r = c_{r+1} = \cdots = c_{r+s} \ge c_{r+s+1} \ge \cdots \ge c_n. \quad (4.5.1)$$

Many of the sequences that occur in combinatorics are unimodal. Sometimes it is easy and sometimes it can be very hard to prove that a given sequence is unimodal. Generating functions can help with this kind of a problem, though they are far from a panacæa.

A stronger property than unimodality is *logarithmic concavity*. First recall that a function f on the real line is concave if whenever $x < y$ we have $f((x+y)/2) \ge (f(x) + f(y))/2$. This means that the graph of the function bulges up over every one of its chords.

Similarly, a sequence $c_0, c_1, \ldots, c_n$ of positive numbers is log concave if $\log c_\mu$ is a concave function of μ, which is to say that

$$(\log c_{\mu-1} + \log c_{\mu+1})/2 \le \log c_\mu.$$

If we exponentiate both sides of the above, to eliminate all of the logarithms, we find that the sequence is log concave if

$$c_{\mu-1}c_{\mu+1} \le c_\mu^2 \qquad (\mu = 1, 2, \ldots, n-1). \quad (4.5.2)$$

If, in (4.5.2) we can replace the '$\le$' by '$<$', then we will say that the sequence is *strictly* log concave.

Proposition. *Let $\{c_r\}_0^n$ be a log concave sequence of positive numbers. Then the sequence is unimodal.*

Proof. If the sequence is not unimodal then it has three consecutive members that satisfy $c_{r-1} > c_r < c_{r+1}$ which contradicts the assumed log concavity. ∎

In many cases of interest, generating functions can help to prove log concavity of a sequence, and therefore unimodality too. The source of such results is usually some variant of the following:

Theorem 4.5.2. *Let $p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$ be a polynomial all of whose zeros are real and negative. Then the coefficient sequence $\{c_r\}_0^n$ is strictly log concave.*

To prove the theorem we need to recall Rolle's theorem of elementary calculus. It holds that if $f(x)$ is continuously differentiable in (a, b), and if $f(a) = f(b)$, then somewhere between a and b the derivative f' must vanish.

If f is a polynomial this can be considerably strengthened. Let u and v be two consecutive distinct zeros of f. Then by Rolle's theorem there is a zero of f' in (u, v). Suppose f is of degree n, has only real zeros, and has exactly r *distinct* real zeros. Then Rolle's theorem accounts for $r - 1$ of the zeros of f', because we find one between each pair of consecutive distinct zeros of f. The remaining $n - r$ zeros of f are copies of the distinct zeros. But if x_0 is a root of f of multiplicity $m > 1$, then $(x - x_0)^m$ is a factor of f, and so $(x - x_0)^{m-1}$ is a factor of f'. Thus x_0 is a zero of multiplicity $m - 1$ of f'. This accounts for the other $n - 1 - (r - 1) = n - r$ zeros of f'. In particular, *the zeros of f' are all real if the zeros of f are.* For maximum utility in our present discussion, we summarize this discussion in the following way:

Lemma 4.5.1. *Let*

$$f(x, y) = c_0x^n + c_1x^{n-1}y + \cdots + c_ny^n \tag{4.5.3}$$

be a polynomial all of whose roots x/y are real. Let $g(x, y)$ be the result of differentiating f some number of times with respect to x and y. If g is not identically zero, then all of its zeros are real.

Proof of theorem 4.5.2: Since the zeros of f are all negative, we have

$$f(x) = c_0 + c_1x + \cdots + c_nx^n = \prod_{j=1}^{n}(x + x_j), \tag{4.5.4}$$

where the x_j's are positive real numbers. Hence none of the c_i's can vanish. Now apply the differential operator $D_x^m D_y^{n-m-2}$ to the polynomial $f(x, y)$ of (4.5.3). Then only three terms survive, viz.:

$$\frac{c_{n-m-2}(m+2)}{n-m-1}x^2 + 2c_{n-m-1}xy + \frac{(n-m)c_{n-m}}{m+1}y^2. \tag{4.5.5}$$

We can put this in a cleaner form by writing $c_j = \binom{n}{j} p_j$, in which case the result (4.5.5) becomes

$$\binom{n}{m+1}(p_{n-m-2}x^2 + 2p_{n-m-1}xy + p_{n-m}y^2).$$

But this quadratic polynomial, according to lemma 4.5.1 above, must have two real roots, and so its discriminant must be nonnegative, i.e.,

$$p_{n-m-1}^2 \geq p_{n-m-2}p_{n-m},$$

and the sequence of p's is log concave. If we substitute back the c's, we find that

$$\begin{aligned} c_{n-m-1}^2 &\geq \frac{(m+2)(n-m)}{(m+1)(n-m-1)} c_{n-m-2}c_{n-m} \\ &> c_{n-m-2}c_{n-m}, \end{aligned}$$

and the strict log concavity is established. ■

Corollary 4.5.1. *The binomial coefficient sequence* $\left\{\binom{n}{k}\right\}_{k=0}^{n}$ *is log concave, and therefore unimodal.*

Proof. The zeros of the generating polynomial $(1+x)^n$ are evidently real and negative. ■

Corollary 4.5.2. *The sequence of Stirling numbers of the first kind*

$$\{\begin{bmatrix} n \\ k \end{bmatrix}\}_{k=1}^{n}$$

is log concave, and therefore unimodal.

Proof. According to (3.5.2), the opsgf of these Stirling numbers is the polynomial

$$\sum_j \begin{bmatrix} n \\ j \end{bmatrix} x^{j-1} = (x+1)(x+2)\cdots(x+n-1),$$

whose zeros are clearly real and negative. ■

Corollary 4.5.3. *The sequence of Stirling numbers of the second kind* $\{\left\{ {n \atop k} \right\}\}_{k=1}^{n}$ *is log concave, and therefore unimodal.*

Proof. We'll have to work just a little harder for this one, because the zeros of the polynomial

$$A_n(x) = \sum_j \left\{ {n \atop j} \right\} x^j$$

are not easy to find. They are, however, real and negative, and here is one way to see that: by (1.6.8) we have the recurrence formula

$$A_n(y) = \{y(1 + D_y)\}A_{n-1}(y) \qquad (n > 0; A_0 = 1),$$

which can be rewritten in the form

$$e^y A_n(y) = y\,(e^y A_{n-1}(y))' \qquad (n > 0; A_0 = 1). \tag{4.5.6}$$

We claim that for each $n = 0, 1, 2, \ldots$, the function $e^y A_n(y)$ has exactly n zeros, which are real, distinct, and negative except for the one at $y = 0$. This is true for $n = 0$, and if it is true for $0, 1, \ldots, n-1$, then (4.5.6) and Rolle's theorem guarantee that $(e^y A_{n-1}(y))'$ has $n-2$ negative, distinct zeros, one between each pair of zeros of $A_{n-1}(y)$. After multiplying by y, as in (4.5.6), we have $n-1$ negative, distinct zeros for $e^y A_n(y)$, but we need to find still one more. But $e^y A_{n-1}(y)$ obviously approaches zero as $y \to -\infty$. Hence its derivative must have one more zero to the left of the leftmost zero of $A_{n-1}(y)$, and we are finished. ■

The theorem is very strong, but one must not be left with the impression that unimodality or log concavity has something essential to do with reality of the zeros of the generating polynomials. Many sequences are known that are unimodal, and have generating polynomials whose zeros all lie on the unit circle, and are quite uniformly distributed, in angle, around the circle. In such cases our theorem will be of no help.

For example, an *inversion* of a permutation σ of n letters is a pair (i, j) for which $1 \le i < j \le n$, but $\sigma(i) > \sigma(j)$. A permutation may have between 0 and $\binom{n}{2}$ inversions. It is well known that if $b(n, k)$ is the number of permutations of n letters that have exactly k inversions, then

$$\{b(n, k)\}_{k \ge 0} \overset{ops}{\longleftrightarrow} (1 + x)(1 + x + x^2) \cdots (1 + x + x^2 + \cdots + x^{n-1}). \tag{4.5.7}$$

The zeros of the generating polynomial are very uniformly sprinkled around the unit circle, so the hypotheses of theorem 4.5.2 are extravagantly violated. Nonetheless, the sequence is unimodal; it rises steadily for $k \le \binom{n}{2}/2$, and falls steadily thereafter.