Chapter 2
Series

This chapter is devoted to a study of the different kinds of series that are widely used as generating functions.

2.1 Formal power series

To discuss the *formal* theory of power series, as opposed to their *analytic* theory, is to discuss these series as purely algebraic objects, in their roles as clotheslines, without using any of the function-theoretic properties of the function that may be represented by the series or, indeed, without knowing whether such a function exists.

We study formal series because it often happens in the theory of generating functions that we are trying to solve a recurrence relation, so we introduce a generating function, and then we go through the various manipulations that follow, but with a guilty conscience because we aren't sure whether the various series that we're working with will converge. Also, we might find ourselves working with the derivatives of a generating function, still without having any idea if the series converges to a function at all.

The point of this section is that there's no need for the guilt, because the various manipulations can be carried out in the ring of formal power series, where questions of convergence are nonexistent. We may execute the whole method and end up with the generating series, and only then discover whether it converges and thereby represents a real honest function or not. If not, we may still get lots of information from the formal series, but maybe we won't be able to get *analytic* information, such as asymptotic formulas for the sizes of the coefficients. Exact formulas for the sequences in question, however, might very well still result, even though the method rests, in those cases, on a purely algebraic, formal foundation.

The series

$$f = 1 + x + 2x^2 + 6x^3 + 24x^4 + 120x^5 + \cdots + n!x^n + \cdots, \qquad (2.1.1)$$

for instance, has a perfectly fine existence as a formal power series, despite the fact that it converges for no value of x other than $x = 0$, and therefore offers no possibilities for investigation by analytic methods. Not only that, but this series plays an important role in some natural counting problems.

A *formal power series* is an expression of the form

$$a_0 + a_1x + a_2x^2 + \cdots$$

where the sequence $\{a_n\}_0^\infty$ is called the *sequence of coefficients*. To say that two series are *equal* is to say that their coefficient sequences are the same.

We can do certain kinds of operations with formal power series. We can *add* or *subtract* them, for example. This is done according to the rules

$$\sum_n a_n x^n \pm \sum_n b_n x^n = \sum_n (a_n \pm b_n) x^n.$$

Power series can be multiplied by the usual Cauchy product rule,

$$\sum_n a_n x^n \sum_n b_n x^n = \sum_n c_n x^n \qquad (c_n = \sum_k a_k b_{n-k}). \tag{2.1.2}$$

It is certainly this product rule that accounts for the wide applicability of series methods in combinatorial problems. This is because frequently we can construct all a_n of the objects of type n in some family by choosing an object of type k and an object of type $n-k$ and stitching them together to make the object of type n. The number of ways of doing that will be $a_k a_{n-k}$, and if we sum on k we find that the Cauchy product of two formal series is directly relevant to the problem that we are studying.

If we follow the multiplication rule we obtain, for instance,

$$(1-x)(1+x+x^2+x^3+\cdots) = 1.$$

Thus we can say that the series $(1-x)$ has a reciprocal, and that reciprocal is $1+x+x^2+\cdots$ (and the other way around, too).

Proposition. *A formal power series $f = \sum_{n\ge 0} a_n x^n$ has a reciprocal if and only if $a_0 \ne 0$. In that case the reciprocal is unique.*

Proof. Let f have a reciprocal, namely $1/f = \sum_{n\ge 0} b_n x^n$. Then $f\cdot(1/f) = 1$ and according to (2.1.2), $c_0 = 1 = a_0 b_0$, so $a_0 \ne 0$. Further, in this case (2.1.2) tells us that for $n \ge 1$, $c_n = 0 = \sum_k a_k b_{n-k}$, from which we find

$$b_n = (-1/a_0) \sum_{k\ge 1} a_k b_{n-k} \qquad (n \ge 1). \tag{2.1.3}$$

This determines $b_1, b_2, \ldots$ uniquely, as claimed.

Conversely, suppose $a_0 \ne 0$. Then we can determine $b_0, b_1, \ldots$ from (2.1.3), and the resulting series $\sum_n b_n x^n$ is the reciprocal of f. ∎

The collection of formal power series under the rules of arithmetic that we have just described forms a *ring*, in which the invertible elements are the series with nonvanishing constant term.

The above idea of a *reciprocal* of a formal power series is not to be confused with the subtler notion of the *inverse* of such a series. The inverse of a series f, if it exists, is a series g such that $f(g(x)) = g(f(x)) = x$. When can such an inverse exist? First we need to be able to *define* the symbol $f(g(x))$, then we can worry about whether or not it is equal to x.

If $f = \sum_n a_n x^n$, then $f(g(x))$ means

$$f(g(x)) = \sum_n a_n g(x)^n. \tag{2.1.4}$$

If the series $g(x)$ has a nonzero constant term, g_0, then every term of the series (2.1.4) may contribute to the coefficient of each power of x. On the other hand, if $g_0 = 0$, then we will be able to compute the coefficient of, say, x^{57} in (2.1.4) from just the first 58 terms of the series shown. Indeed, notice that every single term

$$\begin{aligned} a_n g(x)^n &= a_n (g_1 x + g_2 x^2 + \dots)^n \\ &= a_n x^n (g_1 + g_2 x + \dots)^n \end{aligned}$$

with $n > 57$ will contain only powers of x higher than the 57th, and therefore we won't need to look at those terms to find the coefficient of x^{57}.

Thus if $g_0 = 0$ then the computation of each one of the coefficients of the series $f(g(x))$ is a *finite* process, and therefore all of those coefficients are well defined, and so is the series. If $g_0 \neq 0$, though, the computation of each coefficient of $f(g(x))$ is an *infinite* process unless f is a polynomial, and therefore it will make sense only if the series 'converge.' In a formal, algebraic theory, however, ideas of convergence have no place. Thus *the composition $f(g(x))$ of two formal power series is defined if and only if $g_0 = 0$ or f is a polynomial.*

For instance, the series $e^{e^x - 1}$ is a well defined *formal* series, whereas the series e^{e^x} is not defined, at least from the general definition of composition of functions.

To return to the question of finding a series inverse of a given series f, we see that if such an inverse series g exists, then

$$f(g(x)) = g(f(x)) = x \tag{2.1.5}$$

must both make sense and be true. We claim that if $f(0) = 0$ *the inverse series exists if and only if the coefficient of x is nonzero* in the series f.

Proposition. *Let the formal power series f, g satisfy (2.1.5) and $f(0) = 0$. Then $f = f_1 x + f_2 x^2 + \cdots \quad (f_1 \neq 0)$, and $g = g_1 x + g_2 x^2 + \cdots \quad (g_1 \neq 0)$.*

Proof. Suppose that $f = f_r x^r + \cdots$ and $g = g_s x^s + \cdots$, where $r, s \geq 0$ and $f_r g_s \neq 0$. Then $f(g(x)) = x = f_r g_s^r x^{rs} + \cdots$, whence $rs = 1$, and $r = s = 1$, as claimed. ■

In the ring of formal power series there are other operations defined, which mirror the corresponding operations of function calculus, but which make no use of limiting operations.

The *derivative* of the formal power series $f = \sum_n a_n x^n$ is the series $f' = \sum_n n a_n x^{n-1}$. Differentiation follows the usual rules of calculus, such as the sum, product, and quotient rules. Many of these properties are even easier to prove for formal series than they are for the functions of calculus.

For example:

Proposition. *If $f' = 0$ then $f = a_0$ is constant.*

Proof. Take another look at the '=' sign in the hypothesis $f' = 0$. It means that the formal power series f' is identical to the formal power series 0, and that means that each and every coefficient of the formal series f' is 0. But the coefficients of f' are $a_1, 2a_2, 3a_3, \ldots$, so each of these is 0, and therefore $a_j = 0$ for all $j \geq 1$, which is to say that f is constant. ∎

Next, try this one:

Proposition. *If $f' = f$ then $f = ce^x$.*

Proof. Since $f' = f$, the coefficient of x^n must be the same in f as in f', for all $n \geq 0$. Hence $(n+1)a_{n+1} = a_n$ for all $n \geq 0$, whence $a_{n+1} = a_n/(n+1) \quad (n \geq 0)$. By induction on n, $a_n = a_0/n!$ for all $n \geq 0$, and so $f = a_0 e^x$. ∎

2.2 The calculus of formal ordinary power series generating functions

Operations on formal series involve corresponding operations on their coefficients. If the series actually converge and represent functions, then operations on those functions correspond to certain operations on the power series coefficients of the expansions of those functions. In this section we will explore some of these relationships. They are of great importance in helping to spot which kind of generating function is appropriate for which kind of recurrence relation or other combinatorial situation.

Definition. *The symbol $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$ means that the series f is the ordinary power series ('ops') generating function for the sequence $\{a_n\}_0^\infty$. That is, it means that $f = \sum_n a_n x^n$.*

Suppose $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$. Then what generates $\{a_{n+1}\}_0^\infty$? To answer that we do a little calculation:

$$\sum_{n\geq 0} a_{n+1}x^n = \frac{1}{x}\sum_{m\geq 1} a_m x^m = \frac{(f(x) - f(0))}{x}.$$

Therefore

$$f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty \Rightarrow ((f - a_0)/x) \overset{ops}{\longleftrightarrow} \{a_{n+1}\}_0^\infty. \tag{2.2.1}$$

Thus a shift of the subscript by 1 unit changes the series represented to the difference quotient $(f - a_0)/x$. If we shift by 2 units, of course, we just iterate the difference quotient operation, and find that

$$\begin{aligned}\{a_{n+2}\}_0^\infty &\overset{ops}{\longleftrightarrow} \frac{((f - a_0)/x) - a_1}{x} \\ &= \frac{f - a_0 - a_1 x}{x^2}.\end{aligned}$$

Note how this point of view allows us to see 'at a glance' that the Fibonacci recurrence relation $F_{n+2} = F_{n+1} + F_n \quad (n \geq 0; F_0 = 0; F_1 = 1)$ translates directly into the ordinary power series generating function relation

$$\frac{f - x}{x^2} = \frac{f}{x} + f.$$

Indeed, the purpose of this section is to develop this facility for passing from *sequence* relations to *series* relations quickly and conveniently.

Rule 1. *If* $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$, *then, for integer* $h > 0$,

$$\{a_{n+h}\}_0^\infty \overset{ops}{\longleftrightarrow} \frac{f - a_0 - \cdots - a_{h-1}x^{h-1}}{x^h}.$$

Next let's look into the effect of multiplying the sequence by powers of n. Again, suppose that $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$. Then what generates the sequence $\{na_n\}_0^\infty$? The question means this: can we express the series $\sum_n na_nx^n$ in some simple way in terms of the series $f = \sum_n a_nx^n$? The answer is easy, because the former series is exactly xf'. Therefore, *to multiply the nth member of a sequence by n causes its ops generating function to be 'multiplied' by* $x(d/dx)$, which we will write as xD. In symbols:

$$f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty \Rightarrow (xDf) \overset{ops}{\longleftrightarrow} \{na_n\}_0^\infty. \tag{2.2.2}$$

As an example, consider the recurrence

$$(n+1)a_{n+1} = 3a_n + 1 \qquad (n \geq 0; a_0 = 1).$$

If f is the opsgf of the sequence $\{a_n\}_0^\infty$, then from Rule 1 and (2.2.2),

$$f' = 3f + \frac{1}{1-x},$$

which is a first order differential equation in the unknown generating function, and it can be solved by standard methods.

Next suppose $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$. Then what generates the sequence

$$\{n^2a_n\}_0^\infty?$$

Obviously we re-apply the multiply-by-n operator xD, so the answer is $(xD)^2f$. In general,

$$(xD)^kf \overset{ops}{\longleftrightarrow} \{n^ka_n\}_{n\geq 0}.$$

OK, what generates $\{(3 - 7n^2)a_n\}_{n\geq 0}$? Again obviously, we do the same thing to xD that is done to n, i.e., $(3 - 7(xD)^2)f$ is the answer. The general prescription is:

Rule 2. *If* $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$, *and* P *is a polynomial, then*

$$P(xD)f \overset{ops}{\longleftrightarrow} \{P(n)a_n\}_{n\geq 0}.$$

Example 1.

Find a closed formula for the sum of the series $\sum_{n\geq 0}(n^2+4n+5)/n!$. According to the rule, the answer is the value at $x=1$ of the series

$$\begin{aligned}\{(xD)^2+4(xD)+5\}e^x &= \{x^2+x\}e^x+4xe^x+5e^x\\ &= (x^2+5x+5)e^x.\end{aligned}$$

Therefore the answer to the question is $11e$.

But we cheated. Did you catch the illegal move? We took our generating function and evaluated it at $x=1$, didn't we? Such an operation doesn't exist in the ring of formal series. There, series don't have 'values' at particular values of x. The letter x is purely a formal symbol whose powers mark the clothespins on the line.

What *can* be evaluated at a particular numerical value of x is a power series that converges at that x, which is an analytic idea rather than a formal one. The way we make peace with our consciences in such situations, which occur frequently, is this: if, after writing out the recurrence relation and solving it by means of a formal power series generating function, we find that the series so obtained converges to an analytic function inside a certain disk in the complex plane, then the whole derivation that we did formally is actually valid analytically for all complex x in that disk. Therefore we can shift gears and regard the series as a convergent analytic creature if it pleases us to do so. ∎

Example 2.

Find a closed formula for the sum of the squares of the first N *positive integers.*

To do that, begin with the fact that

$$\sum_{n=0}^{N} x^n = \frac{x^{N+1}-1}{x-1},$$

and notice that if we apply $(xD)^2$ to both sides of this relation and then set $x=1$, the left side will be the sum of squares that we seek, and the right side will be the answer! Hence

$$\sum_{n=1}^{N} n^2 = (xD)^2\left\{\frac{x^{N+1}-1}{x-1}\right\}\Bigg|_{x=1}.$$

After doing the two differentiations and lots of algebra, the answer emerges as

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6} \qquad (N = 1, 2, \ldots),$$

which you no doubt knew already. Do notice, however, that the generating function machine is capable of doing, quite mechanically, many formidable-looking problems involving sums. ∎

Our third rule will be a restatement of the way that two opsgf's are multiplied.

Rule 3. *If* $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$ *and* $g \overset{ops}{\longleftrightarrow} \{b_n\}_0^\infty$*, then*

$$fg \overset{ops}{\longleftrightarrow} \left\{\sum_{r=0}^{n} a_r b_{n-r}\right\}_{n=0}^{\infty}. \tag{2.2.3}$$

Now consider the product of more than two series. For instance, in the case of three series, if f, g, h are the series, and if they generate sequences $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, then a brief computation shows that fgh generates the sequence

$$\left\{\sum_{r+s+t=n} a_r b_s c_t\right\}_{n=0}^{\infty}. \tag{2.2.4}$$

A comparison with Rule 3 above will suggest the general formulas that apply to products of any number of power series. One case of this is worth writing down, namely the expressions for the kth power of a series.

Rule 4. *Let* $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$*, and let* k *be a positive integer. Then*

$$f^k \overset{ops}{\longleftrightarrow} \left\{\sum_{n_1+n_2+\cdots+n_k=n} a_{n_1} a_{n_2} \cdots a_{n_k}\right\}_{n=0}^{\infty}. \tag{2.2.5}$$

Example 3.

Let $f(n,k)$ denote the number of ways that the nonnegative integer n can be written as an ordered sum of k nonnegative integers. Find $f(n,k)$. For instance, $f(4,2) = 5$ because 4=4+0=3+1=2+2=1+3=0+4.

To find f, consider the power series $1/(1-x)^k$. Since $1/(1-x) \overset{ops}{\longleftrightarrow} \{1\}$, by (2.2.5) we have

$$1/(1-x)^k \overset{ops}{\longleftrightarrow} \{f(n,k)\}_{n=0}^{\infty}.$$

By (1.5.5), $f(n,k) = \binom{n+k-1}{n}$, and we are finished. ∎

Next consider the effect of multiplying a power series by $1/(1-x)$. Suppose $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$. Then what sequence does $f(x)/(1-x)$ generate?

To find out, we have

$$\frac{f(x)}{(1-x)} = (a_0 + a_1x + a_2x^2 + \cdots)(1 + x + x^2 + \cdots)$$
$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \cdots$$

which clearly leads us to:

Rule 5. *If* $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$ *then*

$$\frac{f}{(1-x)} \overset{ops}{\longleftrightarrow} \left\{\sum_{j=0}^{n} a_j\right\}_{n\geq 0}.$$

That is, the effect of dividing an opsgf by $(1-x)$ *is to replace the sequence that is generated by the sequence of its partial sums.*

Example 4.

Here is another derivation of the formula for the sum of the squares of the first n whole numbers. Since $1/(1-x) \overset{ops}{\longleftrightarrow} \{1\}_{n\geq 0}$, we have by Rule 2, $(xD)^2(1/(1-x)) \overset{ops}{\longleftrightarrow} \{n^2\}_{n\geq 0}$, and by Rule 5,

$$\frac{1}{1-x}(xD)^2\frac{1}{1-x} \overset{ops}{\longleftrightarrow} \left\{\sum_{j=0}^{n} j^2\right\}_{n\geq 0}.$$

That is, the sum of the squares of the first n positive integers is the coefficient of x^n in the series

$$\frac{1}{1-x}(xD)^2\frac{1}{1-x} = \frac{x(1+x)}{(1-x)^4}.$$

However, by (1.5.5) with $k=3$,

$$[x^n]\left(\frac{1}{(1-x)^4}\right) = \binom{n+3}{3}.$$

Hence, by (1.2.7),

$$[x^n]\frac{x(1+x)}{(1-x)^4} = \binom{n+2}{3} + \binom{n+1}{3}$$
$$= \frac{n(n+1)(2n+1)}{6},$$

so this must be the sum of the squares of the first n positive integers. ∎

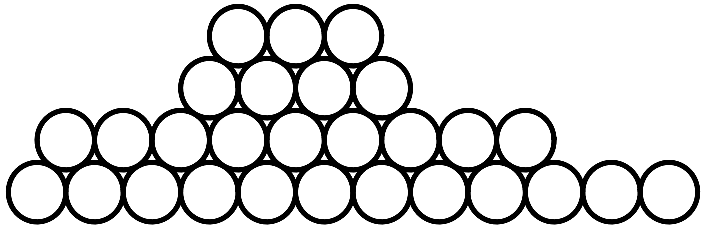

Fig. 2.1: A $(28, 12)$ fountain

Example 5.

The *harmonic numbers* $\{H_n\}_1^\infty$ are defined by

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \qquad (n \geq 1).$$

How can we find their ops generating function? By Rule 5, that function is $1/(1-x)$ times the opsgf of the sequence $\{1/n\}_1^\infty$ of reciprocals of the positive integers. So what is $f = \sum_{n\geq 1} x^n/n$? Well, its derivative is $1/(1-x)$, so it must be $-\log(1-x)$. That means that the opsgf of the harmonic numbers is

$$\sum_{n=1}^{\infty} H_n x^n = \frac{1}{1-x} \log\left(\frac{1}{1-x}\right).$$

■

Example 6.

Prove that the Fibonacci numbers satisfy

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1 \qquad (n \geq 0).$$

By Rule 5, the opsgf of the sequence on the left side is $F/(1-x)$, where F is the opsgf of the Fibonacci numbers, which we found in section 1.3 to be $x/(1-x-x^2)$. By Rule 1, the opsgf of the sequence on the right hand side is

$$\frac{F-x}{x^2} - \frac{1}{1-x},$$

and it is the work of just a moment to check that these are equal. ■

Example 7.

By a *fountain* of coins we mean an arrangement of n coins in rows such that the coins in the first row form a single contiguous block, and that in all higher rows each coin touches exactly two coins from the row beneath it. If the first row contains k coins, we will speak of an (n, k)-fountain. In Fig. 2.1 we show a $(28, 12)$ fountain.

Among all possible fountains we distinguish a special type: those in which *every* row consists of just a single contiguous block of coins. Let's call these *block fountains*.

The question here is this: how many block fountains have a first row that consists of exactly k coins?

Let $f(k)$ be that number, for $k = 0, 1, 2, \ldots$ If we strip off the first row from such a block fountain, then we are looking at another block fountain that has k fewer coins in it. Conversely, if we wish to form all possible block fountains whose first row has k coins, then begin by laying down that row. Then choose a number j, $0 \le j \le k-1$. Above the row of k coins we will place a block fountain whose first row has j coins. If $j = 0$ there is just one way to do that. Otherwise there are $k - j$ ways to do it, depending on how far in we indent the row of j over the row of k coins. It follows that $f(0) = 1$ and

$$f(k) = \sum_{j=1}^{k} (k-j) f(j) + 1 \qquad (k = 1, 2, \ldots). \tag{2.2.6}$$

Define the opsgf $F(x) = \sum_{j \ge 0} f(j) x^j$. The appearance, under the summation sign in (2.2.6), of a function of $k - j$ times a function of j should trigger a reflex reaction that Rule 3, above, applies, and that the product of two ordinary power series generating functions is involved. The two series in question are the opsgf's of the integers $\{j\}_1^\infty$ and of the unknowns $\{f(j)\}_1^\infty$, respectively.

However the former opsgf is $x/(1-x)^2$, and the latter is $F(x) - 1$. Hence, after multiplying equation (2.2.6) by x^k and summing over $k \ge 1$ we obtain

$$F(x) - 1 = \frac{x}{(1-x)^2}(F(x) - 1) + \frac{x}{1-x},$$

and therefore

$$F(x) = \frac{1-2x}{1-3x+x^2}. \tag{2.2.7}$$

The sequence $\{f(k)\}_0^\infty$ begins with 1, 1, 2, 5, 13, 34, 89, ... If these numbers look suspiciously like Fibonacci numbers, then see exercise 19. ■

2.3 The calculus of formal exponential generating functions

In this section we will investigate the analogues of the rules in the preceding section, which applied to *ordinary* power series, in the case of *exponential* generating functions.

Definition. *The symbol* $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$ *means that the series* f *is the exponential generating function of the sequence* $\{a_n\}_0^\infty$*, i.e., that*

$$f = \sum_{n \ge 0} \frac{a_n}{n!} x^n.$$