obvious, but it is quite remarkable how much simpler and more transparent many of the derivations become when seen from the point of view of the black belt generatingfunctionologist. The 'Snake Oil' method that we present in section 4.3, below, explores some of these vistas. The method of rational functions, in section 4.4, is new, and does more and harder problems of this kind.

(g) **Other.** Is there something else you would like to know about your sequence? A generating function may offer hope. One example might be the discovery of congruence relations. Another possibility is that your generating function may bear a striking resemblance to some other known generating function, and that may lead you to the discovery that your problem is closely related to another one, which you never suspected before. It is noteworthy that in this way you may find out that the answer to your problem is simply related to the answer to another problem, without knowing formulas for the answers to either one of the problems!

In the rest of this chapter we are going to give a number of examples of problems that can be profitably thought about from the point of view of generating functions. We hope that after studying these examples the reader will be at least partly convinced of the power of the method, as well as of the beauty of the unified approach.

1.1 An easy two term recurrence

A certain sequence of numbers $a_0, a_1, \ldots$ satisfies the conditions

$$a_{n+1} = 2a_n + 1 \qquad (n \geq 0; a_0 = 0). \tag{1.1.1}$$

Find the sequence.

First try computing a few members of the sequence to see what they look like. It begins with 0, 1, 3, 7, 15, 31, ... These numbers look suspiciously like 1 less than the powers of 2. So we could conjecture that $a_n = 2^n - 1 \quad (n \geq 0)$, and prove it quickly, by induction based on the recurrence (1.1.1).

But this is a book about generating functions, so let's forget all of that, pretend we didn't spot the formula, and use the generating function method. Hence, instead of finding the sequence $\{a_n\}$, let's find the generating function $A(x) = \sum_{n\geq 0} a_n x^n$. Once we know what that function is, we will be able to read off the explicit formula for the a_n's by expanding $A(x)$ in a series.

To find $A(x)$, multiply both sides of the recurrence relation (1.1.1) by x^n and sum over the values of n for which the recurrence is valid, namely, over $n \geq 0$. Then try to relate these sums to the unknown generating function $A(x)$.

If we do this first to the left side of (1.1.1), there results $\sum_{n\geq 0} a_{n+1}x^n$. How can we relate this to $A(x)$? It is *almost* the same as $A(x)$. But the subscript of the 'a' in each term is 1 unit larger than the power of x. But, clearly,

$$\begin{aligned}\sum_{n\geq 0} a_{n+1}x^n &= a_1 + a_2x + a_3x^2 + a_4x^3 + \cdots \\ &= \{(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) - a_0\}/x \\ &= A(x)/x\end{aligned}$$

since $a_0 = 0$ in this problem. Hence the result of so operating on the left side of (1.1.1) is $A(x)/x$.

Next do the right side of (1.1.1). Multiply it by x^n and sum over all $n \geq 0$. The result is

$$\begin{aligned}\sum_{n\geq 0}(2a_n + 1)x^n &= 2A(x) + \sum_{n\geq 0} x^n \\ &= 2A(x) + \frac{1}{1-x},\end{aligned}$$

wherein we have used the familiar geometric series evaluation $\sum_{n\geq 0} x^n = 1/(1-x)$, which is valid for $|x| < 1$.

If we equate the results of operating on the two sides of (1.1.1), we find that

$$\frac{A(x)}{x} = 2A(x) + \frac{1}{1-x},$$

which is trivial to solve for the unknown generating function $A(x)$, in the form

$$A(x) = \frac{x}{(1-x)(1-2x)}.$$

This is the generating function for the problem. The unknown numbers a_n are arranged neatly on this clothesline: a_n is the coefficient of x^n in the series expansion of the above $A(x)$.

Suppose we want to find an explicit formula for the a_n's. Then we would have to expand $A(x)$ in a series. That isn't hard in this example, since the partial fraction expansion is

$$\begin{aligned}\frac{x}{(1-x)(1-2x)} &= x\left\{\frac{2}{1-2x} - \frac{1}{1-x}\right\} \\ &= \{2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \cdots\} \\ &\qquad - \{x + x^2 + x^3 + x^4 + \cdots\} \\ &= (2-1)x + (2^2-1)x^2 + (2^3-1)x^3 + (2^4-1)x^4 + \cdots\end{aligned}$$

It is now clear that the coefficient of x^n, i.e. a_n, is equal to $2^n - 1$, for each $n \geq 0$.

In this example, the heavy machinery wasn't needed because we knew the answer almost immediately, by inspection. The impressive thing about generatingfunctionology is that even though the problems can get a lot harder than this one, the method stays very much the same as it was here, so the same heavy machinery may produce answers in cases where answers are not a bit obvious.

1.2 A slightly harder two term recurrence

A certain sequence of numbers $a_0, a_1, \ldots$ satisfies the conditions

$$a_{n+1} = 2a_n + n \qquad (n \geq 0; a_0 = 1). \tag{1.2.1}$$

Find the sequence.

As before, we might calculate the first several members of the sequence, to get 1, 2, 5, 12, 27, 58, 121, ... A general formula does not seem to be immediately in evidence in this case, so we use the method of generating functions. That means that instead of looking for the *sequence* $a_0, a_1, \ldots$, we will look for the *function* $A(x) = \sum_{j \geq 0} a_j x^j$. Once we have found the function, the sequence will be identifiable as the sequence of power series coefficients of the function.*

As in example 1, the first step is to make sure that the recurrence relation that we are trying to solve comes equipped with a clear indication of the range of values of the subscript for which it is valid. In this case, the recurrence (1.2.1) is clearly labeled in the parenthetical comment as being valid for $n = 0, 1, 2, \ldots$ Don't settle for a recurrence that has an unqualified free variable.

The next step is to define the generating function that you will look for. In this case, since we are looking for a sequence $a_0, a_1, a_2, \ldots$ one natural choice would be the function $A(x) = \sum_{j \geq 0} a_j x^j$ that we mentioned above.

Next, take the recurrence relation (1.2.1), multiply both sides of it by x^n, and sum over *all the values of n for which the relation is valid*, which, in this case, means sum from $n = 0$ to ∞. Try to express the result of doing that in terms of the function $A(x)$ that you have just defined.

If we do that to the left side of (1.2.1), the result is

$$\begin{aligned} a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \cdots &= (A(x) - a_0)/x \\ &= (A(x) - 1)/x. \end{aligned}$$

So much for the left side. What happens if we multiply the right side of (1.2.1) by x^n and sum over nonnegative integers n? Evidently the result is $2A(x) + \sum_{n \geq 0} n x^n$. We need to identify the series

$$\sum_{n \geq 0} n x^n = x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

* If you are feeling rusty in the power series department, see chapter 2, which contains a review of that subject.

There are two ways to proceed: (a) look it up (b) work it out. To work it out we use the following stunt, which seems artificial if you haven't seen it before, but after using it 4993 times it will seem quite routine:

$$\sum_{n\geq 0} nx^n = \sum_{n\geq 0} x(\frac{d}{dx})x^n = x(\frac{d}{dx})\sum_{n\geq 0} x^n = x(\frac{d}{dx})\frac{1}{1-x} = \frac{x}{(1-x)^2}. \tag{1.2.2}$$

In other words, the series that we are interested in is essentially the derivative of the geometric series, so its sum is essentially the derivative of the sum of the geometric series. This raises some nettlesome questions, which we will mention here and deal with later. For what values of x is (1.2.2) valid? The geometric series converges only for $|x| < 1$, so the analytic manipulation of functions in (1.2.2) is legal only for those x. However, often the *analytic* nature of the generating function doesn't interest us; we love it only for its role as a clothesline on which our sequence is hanging out to dry. In such cases we can think of a generating function as only a *formal* power series, i.e., as an algebraic object rather than as an analytic one. Then (1.2.2) would be valid as an identity in the ring of formal power series, which we will discuss later, and the variable x wouldn't need to be qualified at all.

Anyway, the result of multiplying the right hand side of (1.2.1) by x^n and summing over $n \geq 0$ is $2A(x) + x/(1-x)^2$, and if we equate this with our earlier result from the left side of (1.2.1), we find that

$$\frac{(A(x)-1)}{x} = 2A(x) + \frac{x}{(1-x)^2}, \tag{1.2.3}$$

and we're ready for the easy part, which is to solve (1.2.3) for the unknown $A(x)$, getting

$$A(x) = \frac{1-2x+2x^2}{(1-x)^2(1-2x)}. \tag{1.2.4}$$

Exactly what have we learned? The original problem was to 'find' the numbers $\{a_n\}$ that are determined by the recurrence (1.2.1). We have, in a certain sense, 'found' them: the number a_n is the coefficient of x^n in the power series expansion of the function (1.2.4).

This is the end of the 'find-the-generating-function' part of the method. We have it. What we do with it depends on exactly why we wanted to know the solution of (1.2.1) in the first place.

Suppose, for example, that we want an exact, simple formula for the members a_n of the unknown sequence. Then the method of partial fractions will work here, just as it did in the first example, but its application is now a little bit trickier. Let's try it and see.

The first step is to expand the right side of (1.2.4) in partial fractions. Such a fraction is guaranteed to be expandable in partial fractions in the form

$$\frac{1-2x+2x^2}{(1-x)^2(1-2x)} = \frac{A}{(1-x)^2} + \frac{B}{1-x} + \frac{C}{1-2x}, \tag{1.2.5}$$

and the only problem is how to find the constants A, B, C.

Here's the quick way. First multiply both sides of (1.2.5) by $(1-x)^2$, and then let $x = 1$. The instant result is that $A = -1$ (don't take my word for it, try it for yourself!). Next multiply (1.2.5) through by $1 - 2x$ and let $x = 1/2$. The instant result is that $C = 2$. The hard one to find is B, so let's do that one by cheating. Since we know that (1.2.5) is an identity, i.e., is true for all values of x, let's choose an easy value of x, say $x = 0$, and substitute that value of x into (1.2.5). Since we now know A and C, we find at once that $B = 0$.

We return now to (1.2.5) and insert the values of A, B, C that we just found. The result is the relation

$$A(x) = \frac{1-2x+2x^2}{(1-x)^2(1-2x)} = \frac{(-1)}{(1-x)^2} + \frac{2}{1-2x}. \tag{1.2.6}$$

What we are trying to do is to find an explicit formula for the coefficient of x^n in the left side of (1.2.6). We are trading that in for two easier problems, namely finding the coefficient of x^n in each of the summands on the right side of (1.2.6). Why are they easier? The term $2/(1-2x)$, for instance, expands as a geometric series. The coefficient of x^n there is just $2 \cdot 2^n = 2^{n+1}$. The series $(-1)/(1-x)^2$ was handled in (1.2.2) above, and its coefficient of x^n is $-(n+1)$. If we combine these results we see that our unknown sequence is

$$a_n = 2^{n+1} - n - 1 \qquad (n = 0, 1, 2, \ldots).$$

Having done all of that work, it's time to confess that there are better ways to deal with recurrences of the type (1.2.1), without using generating functions.* However, the problem remains a good example of how generating functions can be used, and it underlines the fact that a single unified method can replace a lot of individual special techniques in problems about sequences. Anyway, it won't be long before we're into some problems that essentially cannot be handled without generating functions. ∎

It's time to introduce some notation that will save a lot of words in the sequel.

Definition. *Let $f(x)$ be a series in powers of x. Then by the symbol $[x^n]f(x)$ we will mean the coefficient of x^n in the series $f(x)$.*

Here are some examples of the use of this notation.

$$[x^n]e^x = 1/n!; \quad [t^r]\{1/(1-3t)\} = 3^r; \quad [u^m](1+u)^s = \binom{s}{m}.$$

---

* See, for instance, chapter 1 of my book [Wi2].

A perfectly obvious property of this symbol, that we will use repeatedly, is

$$[x^n]\{x^a f(x)\} = [x^{n-a}]f(x). \tag{1.2.7}$$

Another property of this symbol is the convention that if β is any real number, then

$$[\beta x^n]f(x) = (1/\beta)[x^n]f(x), \tag{1.2.8}$$

so, for instance, $[x^n/n!]e^x = 1$ for all $n \geq 0$.

Before we move on to the next example, here is a summary of the method of generating functions as we have used it so far.

THE METHOD

Given: a recurrence formula that is to be solved by the method of generating functions.

1. Make sure that the set of values of the free variable (say n) for which the given recurrence relation is true, is clearly delineated.
2. Give a name to the generating function that you will look for, and write out that function in terms of the unknown sequence (e.g., call it $A(x)$, and define it to be $\sum_{n\geq 0} a_n x^n$).
3. Multiply both sides of the recurrence by x^n, and sum over all values of n for which the recurrence holds.
4. Express both sides of the resulting equation explicitly in terms of your generating function $A(x)$.
5. Solve the resulting equation for the unknown generating function $A(x)$.
6. If you want an exact formula for the sequence that is defined by the given recurrence relation, then attempt to get such a formula by expanding $A(x)$ into a power series by any method you can think of. In particular, if $A(x)$ is a rational function (quotient of two polynomials), then success will result from expanding in partial fractions and then handling each of the resulting terms separately.

1.3 A three term recurrence

Now let's do the Fibonacci recurrence

$$F_{n+1} = F_n + F_{n-1}. \qquad (n \geq 1; F_0 = 0;\ F_1 = 1). \tag{1.3.1}$$

Following 'The Method,' we will solve for the generating function

$$F(x) = \sum_{n\geq 0} F_n x^n.$$