We point out the rather curious fact that $p(\mathcal{O}, n) = p(\mathcal{D}, n)$ for $n \leqslant 7$, although there is little apparent relationship between the various partitions listed (see Corollary 1.2).

In this chapter, we shall present two of the most elemental tools for treating partitions: (1) infinite product generating functions; (2) graphical representation of partitions.

1.2 Infinite Product Generating Functions of One Variable

DEFINITION 1.5. The generating function $f(q)$ for the sequence $a_0, a_1, a_2, a_3, \ldots$ is the power series $f(q) = \sum_{n\geqslant 0} a_n q^n$.

Remark. For many of the problems we shall encounter, it suffices to consider $f(q)$ as a "formal power series" in q. With such an approach many of the manipulations of series and products in what follows may be justified almost trivially. On the other hand, much asymptotic work (see Chapter 6) requires that the generating functions be analytic functions of the complex variable q. In actual fact, both approaches have their special merits (recently, E. Bender (1974) has discussed the circumstances in which we may pass from one to the other). Generally we shall state our theorems on generating functions with explicit convergence conditions. For the most part we shall be dealing with absolutely convergent infinite series and infinite products; consequently, various rearrangements of series and interchanges of summation will be justified analytically from this simple fact.

DEFINITION 1.6. Let H be a set of positive integers. We let "H" denote the set of all partitions whose parts lie in H. Consequently, $p(\text{“}H\text{”}, n)$ is the number of partitions of n that have all their parts in H.

Thus if H_0 is the set of all odd positive integers, then "H_0" $= \mathcal{O}$.

$$p(\text{“}H_0\text{”}, n) = p(\mathcal{O}, n).$$

DEFINITION 1.7. Let H be a set of positive integers. We let "H"$(\leqslant d)$ denote the set of all partitions in which no part appears more than d times and each part is in H.

Thus if N is the set of all positive integers, then $p(\text{“}N\text{”}(\leqslant 1), n) = p(\mathcal{D}, n)$.

THEOREM 1.1. *Let H be a set of positive integers, and let*

$$f(q) = \sum_{n\geqslant 0} p(\text{“}H\text{”}, n) q^n, \tag{1.2.1}$$

$$f_d(q) = \sum_{n\geqslant 0} p(\text{“}H\text{”}(\leqslant d), n) q^n. \tag{1.2.2}$$

Then for $|q| < 1$

$$f(q) = \prod_{n \in H} (1 - q^n)^{-1}, \tag{1.2.3}$$

$$\begin{aligned} f_d(q) &= \prod_{n \in H} (1 + q^n + \cdots + q^{dn}) \\ &= \prod_{n \in H} (1 - q^{(d+1)n})(1 - q^n)^{-1}. \end{aligned} \tag{1.2.4}$$

Remark. The equivalence of the two forms for $f_d(q)$ follows from the simple formula for the sum of a finite geometric series:

$$1 + x + x^2 + \cdots + x^r = \frac{1 - x^{r+1}}{1 - x}.$$

Proof. We shall proceed in a formal manner to prove (1.2.3) and (1.2.4); at the conclusion of our proof we shall sketch how to justify our steps analytically. Let us index the elements of H, so that $H = \{h_1, h_2, h_3, h_4, \ldots\}$. Then

$$\begin{aligned} \prod_{n \in H} (1 - q^n)^{-1} &= \prod_{n \in H} (1 + q^n + q^{2n} + q^{3n} + \cdots) \\ &= (1 + q^{h_1} + q^{2h_1} + q^{3h_1} + \cdots) \\ &\quad \times (1 + q^{h_2} + q^{2h_2} + q^{3h_2} + \cdots) \\ &\quad \times (1 + q^{h_3} + q^{2h_3} + q^{3h_3} + \cdots) \\ &\quad \cdots \\ &= \sum_{a_1 \geq 0} \sum_{a_2 \geq 0} \sum_{a_3 \geq 0} \cdots q^{a_1 h_1 + a_2 h_2 + a_3 h_3 + \cdots} \end{aligned}$$

and we observe that the exponent of q is just the partition $(h_1^{a_1} h_2^{a_2} h_3^{a_3} \cdots)$. Hence q^N will occur in the foregoing summation once for each partition of n into parts taken from H. Therefore

$$\prod_{n \in H} (1 - q^n)^{-1} = \sum_{n \geq 0} p(\text{“}H\text{”}, n) q^n.$$

The proof of (1.2.4) is identical with that of (1.2.3) except that the infinite geometric series is replaced by the finite geometric series:

$$\begin{aligned} \prod_{n \in H} &(1 + q^n + q^{2n} + \cdots + q^{dn}) \\ &= \sum_{d \geq a_1 \geq 0} \sum_{d \geq a_2 \geq 0} \sum_{d \geq a_3 \geq 0} \cdots q^{a_1 h_1 + a_2 h_2 + a_3 h_3 + \cdots} \\ &= \sum_{n \geq 0} p(\text{“}H\text{”}(\leqslant d), n) q^n. \end{aligned}$$

If we are to view the foregoing procedures as operations with convergent infinite products, then the multiplication of infinitely many series together requires some justification. The simplest procedure is to truncate the infinite product to $\prod_{i=1}^{n}(1-q^{h_i})^{-1}$. This truncated product will generate those partitions whose parts are among $h_1, h_2, \ldots, h_n$. The multiplication is now perfectly valid since only a finite number of absolutely convergent series are involved. Now assume q is real and $0 < q < 1$; then if $M = h_n$,

$$\sum_{j=0}^{M} p(\text{“}H\text{”}, j)q^j \leqslant \prod_{i=1}^{n}(1-q^{h_i})^{-1} \leqslant \prod_{i=1}^{\infty}(1-q^{h_i})^{-1} < \infty.$$

Thus the sequence of partial sums $\sum_{j=0}^{M} p(\text{“}H\text{”}, j)q^j$ is a bounded increasing sequence and must therefore converge. On the other hand

$$\sum_{j=0}^{\infty} p(\text{“}H\text{”}, j)q^j \geqslant \prod_{i=1}^{n}(1-q^{h_i})^{-1} \to \prod_{i=1}^{\infty}(1-q^{h_i})^{-1} \qquad \text{as} \quad n \to \infty.$$

Therefore

$$\sum_{j=0}^{\infty} p(\text{“}H\text{”}, j)q^j = \prod_{i=1}^{\infty}(1-q^{h_i})^{-1} = \prod_{n\in H}(1-q^n)^{-1}.$$

Similar justification can be given for the proof of (1.2.4). ■

COROLLARY 1.2 (Euler). *$p(\mathcal{O}, n) = p(\mathcal{D}, n)$ for all n.*

Proof. By Theorem 1.1,

$$\sum_{n\geqslant 0} p(\mathcal{O}, n)q^n = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}$$

and

$$\sum_{n\geqslant 0} p(\mathcal{D}, n)q^n = \prod_{n=1}^{\infty}(1+q^n).$$

Now

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^n)} = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}. \tag{1.2.5}$$

Hence

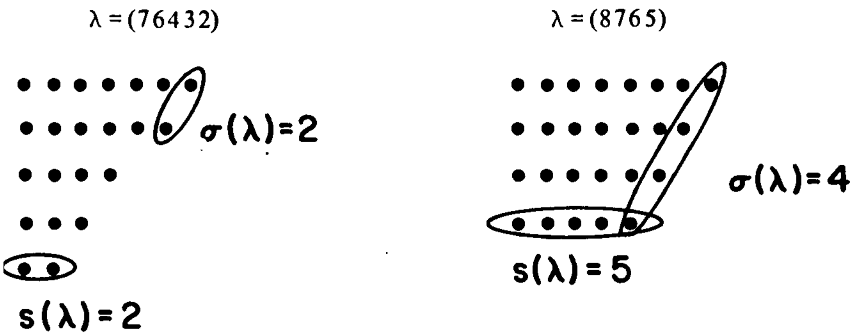
$$\sum_{n\geqslant 0} p(\mathcal{O}, n)q^n = \sum_{n\geqslant 0} p(\mathcal{D}, n)q^n,$$

THEOREM 1.6. *Let $p_e(\mathscr{D}, n)$ (resp. $p_o(\mathscr{D}, n)$) denote the number of partitions of n into an even (resp. odd) number of distinct parts. Then*

$$p_e(\mathscr{D}, n) - p_o(\mathscr{D}, n) = \begin{cases} (-1)^m & \text{if} \quad n = \frac{1}{2}m(3m \pm 1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We shall attempt to establish a one-to-one correspondence between the partitions enumerated by $p_e(\mathscr{D}, n)$ and those enumerated by $p_o(\mathscr{D}, n)$. For most integers n our attempt will be successful; however, whenever n is one of the pentagonal numbers $\frac{1}{2}m(3m \pm 1)$, a single exceptional case will arise.

To begin with, we note that each partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n has a smallest part $s(\lambda) = \lambda_r$; also, we observe that the largest part λ_1 of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is the first of a sequence of, say, $\sigma(\lambda)$ consecutive integers that are parts of λ (formally $\sigma(\lambda)$ is the largest j such that $\lambda_j = \lambda_1 - j + 1$). Graphically the parameters $s(\lambda)$ and $\sigma(\lambda)$ are easily described:

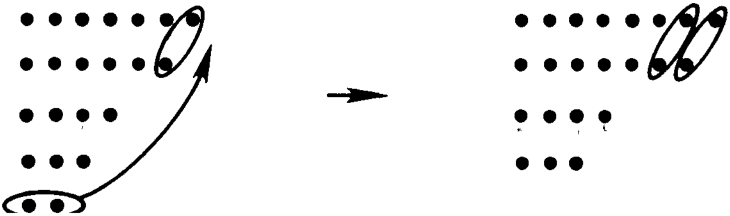


We transform partitions as follows.

Case 1. $s(\lambda) \leqslant \sigma(\lambda)$. In this event, we add one to each of the $s(\lambda)$ largest parts of λ and we delete the smallest part. Thus

$$\lambda = (76432) \rightarrow \lambda' = (8743);$$

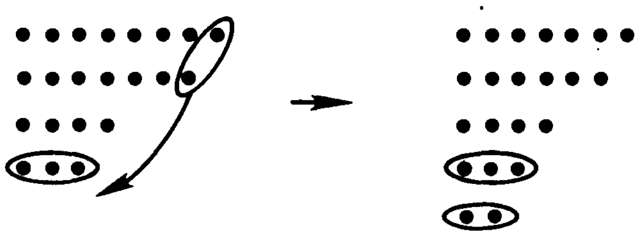
that is



Case 2. $s(\lambda) > \sigma(\lambda)$. In this event, we subtract one from each of the $\sigma(\lambda)$ largest parts of λ and insert a new smallest part of size $\sigma(\lambda)$. Thus

$$\lambda = (8743) \to (76432);$$

that is



The foregoing procedure in either case changes the parity of the number of parts of the partition, and noting that exactly one case is applicable to any partition λ, we see directly that the mapping establishes a one-to-one correspondence. However, there are certain partitions for which the mapping will not work. The example $\lambda = (8765)$ is a case in point. Case 2 should be applicable to it; however, the image partition is *no longer* one with distinct parts. Indeed, Case 2 breaks down in precisely those cases when the partition has r parts, $\sigma(\lambda) = r$ and $s(\lambda) = r + 1$, in which case the number being partitioned is

$$(r + 1) + (r + 2) + \cdots + 2r = \tfrac{1}{2}r(3r + 1).$$

On the other hand, Case 1 breaks down in precisely those cases when the partition has r parts, $\sigma(\lambda) = r$ and $s(\lambda) = r$, in which case the number being partitioned is

$$r + (r + 1) + \cdots + (2r - 1) = \tfrac{1}{2}r(3r - 1).$$

Consequently, if n is not a pentagonal number, $p_e(\mathcal{D}, n) = p_o(\mathcal{D}, n)$; if $n = \tfrac{1}{2}r(3r \pm 1)$, $p_e(\mathcal{D}, n) = p_o(\mathcal{D}, n) + (-1)^r$. ■

COROLLARY 1.7 (Euler's pentagonal number theorem).

$$\prod_{n=1}^{\infty} (1 - q^n) = 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}(1 + q^m)$$

$$= \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}. \qquad (1.3.1)$$

Proof. Clearly

$$\sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)} = 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)} + \sum_{m=-1}^{-\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}$$

$$= 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)} + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m+1)}$$

$$= 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}(1 + q^m)$$

$$= 1 + \sum_{n=1}^{\infty} (p_e(\mathscr{D}, n) - p_o(\mathscr{D}, n))q^n,$$

by Theorem 1.6.

To complete the proof we must show that

$$1 + \sum_{n=1}^{\infty} (p_e(\mathscr{D}, n) - p_o(\mathscr{D}, n))q^n = \prod_{n=1}^{\infty} (1 - q^n).$$

Now

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \cdots (-1)^{a_1+a_2+a_3+\cdots} q^{a_1\cdot 1 + a_2\cdot 2 + a_3\cdot 3 \ldots},$$

as in the proof of (1.2.4) in Theorem 1.1. Note now that each partition with distinct parts is counted with a weight $(-1)^{a_1+a_2+a_3+\cdots}$, which is $+1$ if the partition has an even number of parts and -1 if the partition has an odd number of parts. Consequently

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \cdots (-1)^{a_1+a_2+a_3+\cdots} q^{a_1\cdot 1 + a_2\cdot 2 + a_3\cdot 3 + \cdots}$$

$$= 1 + \sum_{n=1} (p_e(\mathscr{D}, n) - p_o(\mathscr{D}, n))q^n,$$

and so we have the desired result. ∎

Corollary 1.8 (Euler). *If* $n > 0$, *then*

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7)$$
$$+ \cdots + (-1)^m p(n - \tfrac{1}{2}m(3m-1))$$
$$+ (-1)^m p(n - \tfrac{1}{2}m(3m+1)) + \cdots = 0, \qquad (1.3.2)$$

where we recall that $p(M) = 0$ *for all negative* M.

Proof. Let a_n denote the left-hand side of (1.3.2). Then clearly

$$\sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} p(n)q^n \cdot \left[1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}(1 + q^m)\right]$$

$$= \prod_{n=1}^{\infty} (1 - q^n)^{-1} \cdot \prod_{n=1}^{\infty} (1 - q^n)$$

$$= 1$$

where the penultimate equation follows immediately by (1.2.3) and Corollary 1.7. Hence, $a_n = 0$ for $n > 0$. ∎

Corollary 1.8 provides an extremely efficient algorithm for computing $p(n)$ that we shall discuss further in Chapter 14.

Examples

1. (Subbarao) The number of partitions of n in which each part appears two, three, or five times equals the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12.

2. The number of partitions of n in which only odd parts may be repeated equals the number of partitions of n in which no part appears more than three times.

3. The number of partitions of n in which only parts $\not\equiv 0 \pmod{2^m}$ may be repeated equals the number of partitions of n in which no part appears more than $2^{m+1} - 1$ times.

4. (Ramanujan) The number of partitions of n with unique smallest part and largest part at most twice the smallest part equals the number of partitions of n in which the largest part is odd and the smallest part is larger than half the largest part.

5. Let $P_1(r; n)$ denote the number of partitions of n into parts that are either even and not congruent to $4r - 2 \pmod{4r}$ or odd and congruent to $2r - 1$ or $4r - 1 \pmod{4r}$. Let $P_2(r; n)$ denote the number of partitions of n in which only even parts may be repeated and all odd parts are congruent to $2r - 1$ modulo $2r$. Then $P_1(r; n) = P_2(r; n)$.

Comment on Examples 6–7. P. A. MacMahon introduced what he termed "modular" partitions. Given the positive integers k and n, there exist (by the Euclidean algorithm) $h \geqslant 0$ and $0 < j \leqslant k$ such that

$$n = kh + j.$$

The "modular" partitions are a modification of the Ferrers graph so that n is represented by a row of h k's and one j. Thus the representation of $8 + 8 + 7 + 7 + 6 + 5 + 2$ to the modulus 2 is

$$\begin{matrix} 2\,2\,2\,2 \\ 2\,2\,2\,2 \\ 2\,2\,2\,1 \\ 2\,2\,2\,1 \\ 2\,2\,2 \\ 2\,2\,1 \\ 2 \end{matrix}$$

Note that the ordinary Ferrers graph is just the modular representation with modulus 1.

6. Let $W_1(r, m, n)$ denote the number of partitions of n into m parts, each larger than 1, with exactly r odd parts, each distinct. Let $W_2(r, m, n)$ denote the number of partitions of n with $2m$ as largest part and exactly r