11 Dirichlet Series and Euler Products

11.1 Introduction

In 1737 Euler proved Euclid's theorem on the existence of infinitely many primes by showing that the series $\sum p^{-1}$, extended over all primes, diverges. He deduced this from the fact that the zeta function $\zeta(s)$, given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for real $s > 1$, tends to ∞ as $s \to 1$. In 1837 Dirichlet proved his celebrated theorem on primes in arithmetical progressions by studying the series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{2}$$

where χ is a Dirichlet character and $s > 1$.

The series in (1) and (2) are examples of series of the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} \tag{3}$$

where $f(n)$ is an arithmetical function. These are called Dirichlet series with coefficients $f(n)$. They constitute one of the most useful tools in analytic number theory.

This chapter studies general properties of Dirichlet series. The next chapter makes a more detailed study of the Riemann zeta function $\zeta(s)$ and the Dirichlet L-functions $L(s, \chi)$.

Notation Following Riemann, we let s be a complex variable and write

$$s = \sigma + it,$$

where σ and t are real. Then $n^s = e^{s \log n} = e^{(\sigma + it) \log n} = n^\sigma e^{it \log n}$. This shows that $|n^s| = n^\sigma$ since $|e^{i\theta}| = 1$ for real θ.

The set of points $s = \sigma + it$ such that $\sigma > a$ is called a *half-plane*. We will show that for each Dirichlet series there is a half-plane $\sigma > \sigma_c$ in which the series converges, and another half-plane $\sigma > \sigma_a$ in which it converges absolutely. We will also show that in the half-plane of convergence the series represents an analytic function of the complex variable s.

11.2 The half-plane of absolute convergence of a Dirichlet series

First we note that if $\sigma \geq a$ we have $|n^s| = n^\sigma \geq n^a$ hence

$$\left|\frac{f(n)}{n^s}\right| \leq \frac{|f(n)|}{n^a}.$$

Therefore, if a Dirichlet series $\sum f(n)n^{-s}$ converges absolutely for $s = a + ib$, then by the comparison test it also converges absolutely for all s with $\sigma \geq a$. This observation implies the following theorem.

Theorem 11.1 *Suppose the series $\sum |f(n)n^{-s}|$ does not converge for all s or diverge for all s. Then there exists a real number σ_a, called the abscissa of absolute convergence, such that the series $\sum f(n)n^{-s}$ converges absolutely if $\sigma > \sigma_a$ but does not converge absolutely if $\sigma < \sigma_a$.*

PROOF. Let D be the set of all real σ such that $\sum |f(n)n^{-s}|$ diverges. D is not empty because the series does not converge for all s, and D is bounded above because the series does not diverge for all s. Therefore D has a least upper bound which we call σ_a. If $\sigma < \sigma_a$ then $\sigma \in D$, otherwise σ would be an upper bound for D smaller than the least upper bound. If $\sigma > \sigma_a$ then $\sigma \notin D$ since σ_a is an upper bound for D. This proves the theorem. □

Note. If $\sum |f(n)n^{-s}|$ converges everywhere we define $\sigma_a = -\infty$. If the series $\sum |f(n)n^{-s}|$ converges nowhere we define $\sigma_a = +\infty$.

EXAMPLE 1 Riemann zeta function. The Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ converges absolutely for $\sigma > 1$. When $s = 1$ the series diverges, so $\sigma_a = 1$. The sum of this series is denoted by $\zeta(s)$ and is called the Riemann zeta function.

EXAMPLE 2 If f is bounded, say $|f(n)| \leq M$ for all $n \geq 1$, then $\sum f(n)n^{-s}$ converges absolutely for $\sigma > 1$, so $\sigma_a \leq 1$. In particular if χ is a Dirichlet character the L-series $L(s, \chi) = \sum \chi(n)n^{-s}$ converges absolutely for $\sigma > 1$.

EXAMPLE 3 The series $\sum n^n n^{-s}$ diverges for every s so $\sigma_a = +\infty$.

EXAMPLE 4 The series $\sum n^{-n} n^{-s}$ converges absolutely for every s so $\sigma_a = -\infty$.

11.3 The function defined by a Dirichlet series

Assume that $\sum f(n)n^{-s}$ converges absolutely for $\sigma > \sigma_a$ and let $F(s)$ denote the sum function

(4) $$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{for } \sigma > \sigma_a.$$

This section derives some properties of $F(s)$. First we prove the following lemma.

Lemma 1 *If* $N \geq 1$ *and* $\sigma \geq c > \sigma_a$ *we have*

$$\left| \sum_{n=N}^{\infty} f(n)n^{-s} \right| \leq N^{-(\sigma-c)} \sum_{n=N}^{\infty} |f(n)| n^{-c}.$$

PROOF. We have

$$\left| \sum_{n=N}^{\infty} f(n)n^{-s} \right| \leq \sum_{n=N}^{\infty} |f(n)| n^{-\sigma} = \sum_{n=N}^{\infty} |f(n)| n^{-c} n^{-(\sigma-c)}$$

$$\leq N^{-(\sigma-c)} \sum_{n=N}^{\infty} |f(n)| n^{-c}.$$ □

The next theorem describes the behavior of $F(s)$ as $\sigma \to +\infty$.

Theorem 11.2 *If* $F(s)$ *is given by* (4), *then*

$$\lim_{\sigma \to +\infty} F(\sigma + it) = f(1)$$

uniformly for $-\infty < t < +\infty$.

PROOF. Since $F(s) = f(1) + \sum_{n=2}^{\infty} f(n)n^{-s}$ we need only prove that the second term tends to 0 as $\sigma \to +\infty$. Choose $c > \sigma_a$. Then for $\sigma \geq c$ the lemma implies

$$\left| \sum_{n=2}^{\infty} \frac{f(n)}{n^s} \right| \leq 2^{-(\sigma-c)} \sum_{n=2}^{\infty} |f(n)| n^{-c} = \frac{A}{2^\sigma}$$

where A is independent of σ and t. Since $A/2^\sigma \to 0$ as $\sigma \to +\infty$ this proves the theorem. □

EXAMPLES $\zeta(\sigma + it) \to 1$ and $L(\sigma + it, \chi) \to 1$ as $\sigma \to +\infty$.

We prove next that all the coefficients are uniquely determined by the sum function.

Theorem 11.3 Uniqueness theorem. *Given two Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and } G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

both absolutely convergent for $\sigma > \sigma_a$. If $F(s) = G(s)$ for each s in an infinite sequence $\{s_k\}$ such that $\sigma_k \to +\infty$ as $k \to \infty$, then $f(n) = g(n)$ for every n.

PROOF. Let $h(n) = f(n) - g(n)$ and let $H(s) = F(s) - G(s)$. Then $H(s_k) = 0$ for each k. To prove that $h(n) = 0$ for all n we assume that $h(n) \neq 0$ for some n and obtain a contradiction.

Let N be the smallest integer for which $h(n) \neq 0$. Then

$$H(s) = \sum_{n=N}^{\infty} \frac{h(n)}{n^s} = \frac{h(N)}{N^s} + \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}.$$

Hence

$$h(N) = N^s H(s) - N^s \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}.$$

Putting $s = s_k$ we have $H(s_k) = 0$ hence

$$h(N) = -N^{s_k} \sum_{n=N+1}^{\infty} h(n) n^{-s_k}.$$

Choose k so that $\sigma_k > c$ where $c > \sigma_a$. Then Lemma 1 implies

$$|h(N)| \leq N^{\sigma_k}(N+1)^{-(\sigma_k - c)} \sum_{n=N+1}^{\infty} |h(n)| n^{-c} = \left(\frac{N}{N+1}\right)^{\sigma_k} A$$

where A is independent of k. Letting $k \to \infty$ we find $(N/(N+1))^{\sigma_k} \to 0$ so $h(N) = 0$, a contradiction. □

The uniqueness theorem implies the existence of a half-plane in which a Dirichlet series does not vanish (unless, of course, the series vanishes identically).

Theorem 11.4 *Let $F(s) = \sum f(n) n^{-s}$ and assume that $F(s) \neq 0$ for some s with $\sigma > \sigma_a$. Then there is a half-plane $\sigma > c \geq \sigma_a$ in which $F(s)$ is never zero.*

PROOF. Assume no such half-plane exists. Then for every $k = 1, 2, \ldots$ there is a point s_k with $\sigma_k > k$ such that $F(s_k) = 0$. Since $\sigma_k \to +\infty$ as $k \to \infty$ the uniqueness theorem shows that $f(n) = 0$ for all n, contradicting the hypothesis that $F(s) \neq 0$ for some s. □

11.4 Multiplication of Dirichlet series

The next theorem relates products of Dirichlet series with the Dirichlet convolution of their coefficients.

Theorem 11.5 *Given two functions $F(s)$ and $G(s)$ represented by Dirichlet series,*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{for } \sigma > a,$$

and

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \quad \text{for } \sigma > b.$$

Then in the half-plane where both series converge absolutely we have

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \tag{5}$$

where $h = f * g$, the Dirichlet convolution of f and g:

$$h(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

Conversely, if $F(s)G(s) = \sum \alpha(n)n^{-s}$ for all s in a sequence $\{s_k\}$ with $\sigma_k \to +\infty$ as $k \to \infty$ then $\alpha = f * g$.

Proof. For any s for which both series converge absolutely we have

$$F(s)G(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \sum_{m=1}^{\infty} g(m)m^{-s} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(n)g(m)(mn)^{-s}.$$

Because of absolute convergence we can multiply these series together and rearrange the terms in any way we please without altering the sum. Collect together those terms for which mn is constant, say $mn = k$. The possible values of k are $1, 2, \ldots$, hence

$$F(s)G(s) = \sum_{k=1}^{\infty} \left(\sum_{mn=k} f(n)g(m) \right) k^{-s} = \sum_{k=1}^{\infty} h(k)k^{-s}$$

where $h(k) = \sum_{mn=k} f(n)g(m) = (f * g)(k)$. This proves the first assertion, and the second follows from the uniqueness theorem. □

Example 1 Both series $\sum n^{-s}$ and $\sum \mu(n)n^{-s}$ converge absolutely for $\sigma > 1$. Taking $f(n) = 1$ and $g(n) = \mu(n)$ in (5) we find $h(n) = [1/n]$, so

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 \quad \text{if } \sigma > 1.$$

In particular, this shows that $\zeta(s) \neq 0$ for $\sigma > 1$ and that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad \text{if } \sigma > 1.$$

EXAMPLE 2 More generally, assume $f(1) \neq 0$ and let $g = f^{-1}$, the Dirichlet inverse of f. Then in any half-plane where both series $F(s) = \sum f(n)n^{-s}$ and $G(s) = \sum g(n)n^{-s}$ converge absolutely we have $F(s) \neq 0$ and $G(s) = 1/F(s)$.

EXAMPLE 3 Assume $F(s) = \sum f(n)n^{-s}$ converges absolutely for $\sigma > \sigma_a$. If f is completely multiplicative we have $f^{-1}(n) = \mu(n)f(n)$. Since $|f^{-1}(n)| \leq |f(n)|$ the series $\sum \mu(n)f(n)n^{-s}$ also converges absolutely for $\sigma > \sigma_a$ and we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)f(n)}{n^s} = \frac{1}{F(s)} \quad \text{if } \sigma > \sigma_a.$$

In particular for every Dirichlet character χ we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} = \frac{1}{L(s, \chi)} \quad \text{if } \sigma > 1.$$

EXAMPLE 4 Take $f(n) = 1$ and $g(n) = \varphi(n)$, Euler's totient. Since $\varphi(n) \leq n$ the series $\sum \varphi(n)n^{-s}$ converges absolutely for $\sigma > 2$. Also, $h(n) = \sum_{d|n} \varphi(d) = n$ so (5) gives us

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s} = \zeta(s-1) \quad \text{if } \sigma > 2.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \quad \text{if } \sigma > 2.$$

EXAMPLE 5 Take $f(n) = 1$ and $g(n) = n^\alpha$. Then $h(n) = \sum_{d|n} d^\alpha = \sigma_\alpha(n)$, and (5) gives us

$$\zeta(s)\zeta(s-\alpha) = \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s} \quad \text{if } \sigma > \max\{1, 1 + \text{Re}(\alpha)\}.$$

EXAMPLE 6 Take $f(n) = 1$ and $g(n) = \lambda(n)$, Liouville's function. Then

$$h(n) = \sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n = m^2 \text{ for some } m, \\ 0 & \text{otherwise,} \end{cases}$$

so (5) gives us

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \sum_{\substack{n=1 \\ n=\text{square}}}^{\infty} \frac{1}{n^s} = \sum_{m=1}^{\infty} \frac{1}{m^{2s}} = \zeta(2s).$$

Hence

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \quad \text{if } \sigma > 1.$$

11.5 Euler products

The next theorem, discovered by Euler in 1737, is sometimes called the analytic version of the fundamental theorem of arithmetic.

Theorem 11.6 *Let f be a multiplicative arithmetical function such that the series $\sum f(n)$ is absolutely convergent. Then the sum of the series can be expressed as an absolutely convergent infinite product,*

$$(6) \qquad \sum_{n=1}^{\infty} f(n) = \prod_{p} \{1 + f(p) + f(p^2) + \cdots\}$$

extended over all primes. If f is completely multiplicative, the product simplifies and we have

$$(7) \qquad \sum_{n=1}^{\infty} f(n) = \prod_{p} \frac{1}{1 - f(p)}.$$

Note. In each case the product is called the *Euler product* of the series.

PROOF. Consider the finite product

$$P(x) = \prod_{p \le x} \{1 + f(p) + f(p^2) + \cdots\}$$

extended over all primes $p \le x$. Since this is the product of a finite number of absolutely convergent series we can multiply the series and rearrange the terms in any fashion without altering the sum. A typical term is of the form

$$f(p_1^{a_1})f(p_2^{a_2}) \cdots f(p_r^{a_r}) = f(p_1^{a_1}p_2^{a_2} \cdots p_r^{a_r})$$

since f is multiplicative. By the fundamental theorem of arithmetic we can write

$$P(x) = \sum_{n \in A} f(n)$$

where A consists of those n having all their prime factors $\le x$. Therefore

$$\sum_{n=1}^{\infty} f(n) - P(x) = \sum_{n \in B} f(n),$$

where B is the set of n having at least one prime factor $> x$. Therefore

$$\left| \sum_{n=1}^{\infty} f(n) - P(x) \right| \le \sum_{n \in B} |f(n)| \le \sum_{n > x} |f(n)|.$$

As $x \to \infty$ the last sum on the right $\to 0$ since $\sum |f(n)|$ is convergent. Hence $P(x) \to \sum f(n)$ as $x \to \infty$.

Now an infinite product of the form $\prod(1 + a_n)$ converges absolutely whenever the corresponding series $\sum a_n$ converges absolutely. In this case we have

$$\sum_{p \le x} |f(p) + f(p^2) + \cdots| \le \sum_{p \le x} (|f(p)| + |f(p^2)| + \cdots) \le \sum_{n=2}^{\infty} |f(n)|.$$

Since all the partial sums are bounded, the series of positive terms

$$\sum_p |f(p) + f(p^2) + \cdots|$$

converges, and this implies absolute convergence of the product in (6).

Finally, when f is completely multiplicative we have $f(p^n) = f(p)^n$ and each series on the right of (6) is a convergent geometric series with sum $(1 - f(p))^{-1}$. □

Applying Theorem 11.6 to absolutely convergent Dirichlet series we immediately obtain:

Theorem 11.7 *Assume $\sum f(n)n^{-s}$ converges absolutely for $\sigma > \sigma_a$. If f is multiplicative we have*

$$\text{(8)} \qquad \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left\{1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right\} \quad \textit{if } \sigma > \sigma_a,$$

and if f is completely multiplicative we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \frac{1}{1 - f(p)p^{-s}} \quad \textit{if } \sigma > \sigma_a.$$

It should be noted that the general term of the product in (8) is the Bell series $f_p(x)$ of the function f with $x = p^{-s}$. (See Section 2.16.)

Examples Taking $f(n) = 1$, $\mu(n)$, $\varphi(n)$, $\sigma_\alpha(n)$, $\lambda(n)$ and $\chi(n)$, respectively, we obtain the following Euler products:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}} \quad \text{if } \sigma > 1.$$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p (1 - p^{-s}) \quad \text{if } \sigma > 1.$$

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_p \frac{1 - p^{-s}}{1 - p^{1-s}} \quad \text{if } \sigma > 2.$$

$$\zeta(s)\zeta(s-\alpha) = \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s} = \prod_p \frac{1}{(1 - p^{-s})(1 - p^{\alpha-s})} \quad \text{if } \sigma > \max\{1, 1 + \text{Re}(\alpha)\},$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \frac{1}{1 + p^{-s}} \quad \text{if } \sigma > 1,$$

$$L(s. \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}} \quad \text{if } \sigma > 1.$$