(4.3) Lemma. *Es gilt:*

a) $B_n \in \mathbb{Q}$ *für alle* $n \in \mathbb{N}_0$.

b) $B_1 = -\frac{1}{2}$ *und* $B_{2n+1} = 0$ *für alle* $n \in \mathbb{N}$.

c) $B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = \frac{-1}{30}$.

Beweis. b) Für $0 < |z| < 2\pi$ gilt

$$2\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} B_{2n+1} z^{2n+1} = g(z) - g(-z) = \frac{z}{e^z - 1} - \frac{-z}{e^{-z}-1} = -z.$$

Daraus folgt die Behauptung durch Koeffizientenvergleich.
a), c) Es gilt

$$B_0 = \lim_{z\to 0} \frac{z}{e^z - 1} = \frac{1}{\frac{d}{dz} e^z\big|_{z=0}} = \frac{1}{e^0} = 1$$

sowie

$$1 = \left(\sum_{n=0}^{\infty} \frac{1}{n!} B_n z^n\right) \cdot \left(\frac{e^z - 1}{z}\right) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} B_n z^n\right) \cdot \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^n\right).$$

Daraus erhält man eine Rekursionsformel für die B_n und $B_n \in \mathbb{Q}$ für alle $n \in \mathbb{N}$. Mit b) folgt

$$\begin{aligned} 1 &= \left(1 - \frac{1}{2}z + \frac{1}{2}B_2 z^2 + \frac{1}{24}B_4 z^4 + \ldots\right) \cdot \left(1 + \frac{1}{2}z + \frac{1}{6}z^2 + \frac{1}{24}z^3 + \frac{1}{120}z^4 + \ldots\right) \\ &= 1 + \left(\frac{1}{2}B_2 - \frac{1}{4} + \frac{1}{6}\right) z^2 + \left(\frac{1}{4}B_2 - \frac{1}{12} + \frac{1}{24}\right) z^3 + \left(\frac{1}{24}B_4 + \frac{1}{12}B_2 - \frac{1}{48} + \frac{1}{120}\right) z^4 + \ldots. \end{aligned}$$

Daraus ergibt sich

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{3} + \frac{1}{2} - \frac{1}{5} = \frac{-1}{30}.$$

□

Als Anwendung erhalten wir den

(4.4) Satz von EULER (1737). *Für jedes* $k \in \mathbb{N}$ *gilt*

$$\zeta(2k) = \sum_{n=1}^{\infty} n^{-2k} = (-1)^{k+1} \frac{(2\pi)^{2k}}{2 \cdot (2k)!} B_{2k} \in \mathbb{Q}\pi^{2k}.$$

Beweis. Nach (4.1) und (4.2) gilt für $0 < |z| < 1$

$$\begin{aligned} \pi z \cot(\pi z) &= 1 + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 - n^2} = 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1}{1 - \frac{z^2}{n^2}}\right) \\ &= 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{\infty} \left(\frac{z^2}{n^2}\right)^k = 1 - 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{z^{2k}}{n^{2k}} \\ &= 1 - 2\sum_{k=1}^{\infty} \zeta(2k) z^{2k}, \end{aligned}$$

Beweis. "(i) $\Rightarrow$ (ii)" Man verwende (5.10).
"(ii) $\Rightarrow$ (i)" Ist χ nicht primitiv, so hat man eine Darstellung $\chi = \chi_0 \cdot \psi$ mit einem primitiven Charakter ψ mod M, wobei $M|N$, $M < N$. Mit $g = \text{ggT}\left(\frac{N}{M}, M\right)$ hat man nach (5.12) b)

$$G\left(\tfrac{N}{M}, \chi\right) = \frac{G_\psi \cdot g \cdot \varphi(N/M)}{\varphi(g)} \neq 0 = \overline{\chi}\left(\tfrac{N}{M}\right) \cdot G_\chi. \qquad \square$$

Im nächsten Schritt untersuchen wir Verallgemeinerungen der BERNOULLIschen Zahlen B_n aus (4.2).

(5.14) Definition. Die BERNOULLI*schen Polynome* sind definiert durch

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k \in \mathbb{Q}[x].$$

Wegen $B_0 = 1$ ist $B_n(x)$ ein normiertes Polynom vom Grad n. Mit den speziellen Werten aus (4.2) folgt

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \quad B_3(x) = x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{2}x.$$

(5.15) Satz. *Es gilt*

a) $B_n(0) = B_n$ *für alle* $n \in \mathbb{N}_0$.

b) $\frac{d}{dx} B_n(x) = n B_{n-1}(x)$ *für alle* $n \in \mathbb{N}$.

c) $\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1}$ *für* $x \in \mathbb{C}$, $|t| < 2\pi$.

d) $B_n(1-x) = (-1)^n B_n(x)$ *für alle* $n \in \mathbb{N}_0$.

e) $B_n(x+1) = B_n(x) + nx^{n-1}$ *für alle* $n \in \mathbb{N}$.

f) $\sum_{k=1}^{N} k^n = \frac{B_{n+1}(N+1) - B_{n+1}(0)}{n+1} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} B_j \frac{N^{n+1-j}}{n+1-j}$ *für alle* $n, N \in \mathbb{N}$.

Beweis. a) ~~Die Aussage folgt direkt aus (5.11).~~ ? Einsetzen $B_n(0) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} 0^k = \binom{n}{0} B_n \cdot 0^0 = B_n$.
b) Es gilt

$$\begin{aligned} \tfrac{d}{dx} B_n(x) &= \sum_{k=1}^{n} k \binom{n}{k} B_{n-k} x^{k-1} \\ &= n \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-1-(k-1)} x^{k-1} = n B_{n-1}(x). \end{aligned}$$

c) Mit (4.2) erhält man die für $x \in \mathbb{C}$, $|t| < 2\pi$ absolut konvergenten Potenzreihen

$$\begin{aligned} e^{xt} \frac{t}{e^t - 1} &= \sum_{k=0}^{\infty} \frac{1}{k!} (xt)^k \cdot \sum_{j=0}^{\infty} \frac{B_j}{j!} t^j \\ &= \sum_{k \geq 0, j \geq 0} \frac{1}{k! j!} B_j x^k t^{k+j} \\ &= \sum_{n=0}^{\infty} \cdot \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \end{aligned}$$

d) Mit c) erhält man für $x \in \mathbb{C}$, $|t| < 2\pi$:

$$\sum_{n=0}^{\infty} B_n(1-x)\frac{t^n}{n!} = \frac{t\,e^{(1-x)t}}{e^t - 1} = \frac{t\,e^{-xt}}{1-e^{-t}} = \frac{(-t)\,e^{x(-t)}}{e^{-t}-1} = \sum_{n=0}^{\infty}(-1)^n B_n(x)\frac{t^n}{n!}.$$

Ein Koeffizientenvergleich liefert

$$B_n(1-x) = (-1)^n B_n(x) \quad \text{für alle } n \in \mathbb{N}_0.$$

e) Analog ergibt sich

$$\sum_{n=0}^{\infty} B_n(x+1)\frac{t^n}{n!} = \frac{t\,e^{(x+1)t}}{e^t-1} = \frac{t\,e^{xt}}{e^t-1} + te^{xt} = B_0(x) + \sum_{n=1}^{\infty}\left(B_n(x) + nx^{n-1}\right)\frac{t^n}{n!}.$$

Ein Koeffizientenvergleich liefert die Behauptung.

f) Mit e), d) und a) erhält man

$$\begin{aligned}\sum_{k=1}^{N} k^n &= \frac{1}{n+1}\sum_{k=1}^{N}\left(B_{n+1}(k+1) - B_{n+1}(k)\right) = \frac{1}{n+1}\left(B_{n+1}(N+1) - B_{n+1}(0)\right) \\ &= \frac{1}{n+1}\left((-1)^{n+1}B_{n+1}(-N) - B_{n+1}\right) = \frac{1}{n+1}\left(\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}B_{n+1-k}N^k - B_{n+1}\right) \\ &= \sum_{j=0}^{n}(-1)^j\binom{n}{j}B_j\frac{N^{n+1-j}}{n+1-j} - \frac{B_{n+1}}{n+1}\left(1+(-1)^n\right).\end{aligned}$$

Nach (4.2) gilt $B_{n+1}\cdot(1+(-1)^n) = 0$ für alle $n \in \mathbb{N}$, so dass die Behauptung folgt. □

Im Zusammenhang mit DIRICHLETschen Charakteren notieren wir die

(5.16) Definition. Ist χ ein DIRICHLETscher Charakter mod$N$, so definiert man die *verallgemeinerten* BERNOULLI*schen Zahlen* $B_{n,\chi}$ bezüglich χ durch

$$\sum_{k=1}^{N-1}\frac{\chi(k)te^{kt}}{e^{Nt}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!} \quad \text{für } |t| < \frac{2\pi}{N}.$$

Den Zusammenhang mit den BERNOULLIschen Polynomen beschreibt das

(5.17) Lemma. *Ist* χ *ein* DIRICHLET*scher Charakter* mod$N$, *so gilt*

$$B_{n,\chi} = N^{n-1}\sum_{k=1}^{N-1}\chi(k)B_n\left(\frac{k}{N}\right)$$

und

5.15c) $$B_{n,\chi} \in \mathbb{Q}(\chi(j), j \bmod N).$$

Beweis. Aus ~~(5.15e)~~ ergibt sich

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!} &= \sum_{k=1}^{N-1}\frac{\chi(k)}{N}\cdot\frac{Nte^{tN\cdot k/N}}{e^{Nt}-1} = \sum_{k=1}^{N-1}\frac{\chi(k)}{N}\sum_{n=0}^{\infty}B_n\left(\frac{k}{N}\right)\frac{(tN)^n}{n!} \\ &= \sum_{n=0}^{\infty}N^{n-1}\sum_{k=1}^{N-1}\chi(k)B_n\left(\frac{k}{N}\right)\frac{t^n}{n!}.\end{aligned}$$

Ein Koeffizientenvergleich liefert die Behauptung. □

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INTRODUCTION TO BERNOULLI AND EULER POLYNOMIALS

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Abstract. In this lecture note we develop the theory of Bernoulli and Euler polynomials in an elementary way so that middle school students can understand most part of the theory.

1. Basic properties of Bernoulli and Euler polynomials

Definition 1.1. The Bernoulli numbers $B_0, B_1, B_2, \cdots$ are given by $B_0 = 1$ and the recursion

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0,\ \text{i.e.}\ B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k \quad (n = 1, 2, 3, \cdots). \tag{1.1}$$

The Euler numbers $E_0, E_1, E_2, \cdots$ are defined by $E_0 = 1$ and the recursion

$$\sum_{\substack{k=0\\ 2\mid n-k}}^{n}\binom{n}{k}E_k = 0, \quad \text{i.e.}\quad E_n = -\sum_{\substack{k=0\\ 2\mid n-k}}^{n-1}\binom{n}{k}E_k \quad (n = 1, 2, 3, \cdots). \tag{1.2}$$

By induction, all the Bernoulli numbers are rationals and all the Euler numbers are integers. Below we list values of B_n and E_n with $n \le 10$.

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$-1/2$	$1/6$	0	$-1/30$	0	$1/42$	0	$-1/30$	0	$5/66$
E_n	1	0	-1	0	5	0	-61	0	1385	0	-50521

Definition 1.2. For $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$, the nth Bernoulli polynomial $B_n(x)$ and the nth Euler polynomial $E_n(x)$ are defined as follows:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad \text{and} \quad E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}. \tag{1.3}$$

Clearly both $B_n(x)$ and $E_n(x)$ are monic polynomials with rational coefficients. Note that $B_n(0) = B_n$ and $E_n(1/2) = E_n/2^n$.

Here we list $B_n(x)$ and $E_n(x)$ for $n \le 5$.

n	0	1	2	3	4	5
$B_n(x)$	1	$x - \frac{1}{2}$	$x^2 - x + \frac{1}{6}$	$x^3 - \frac{3}{2}x^2 + \frac{x}{2}$	$x^4 - 2x^3 + x^2 - \frac{1}{30}$	$x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{x}{6}$
$E_n(x)$	1	$x - \frac{1}{2}$	$x^2 - x$	$x^3 - \frac{3}{2}x^2 + \frac{1}{6}$	$x^4 - 2x^3 + \frac{2}{3}x$	$x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^2 - \frac{1}{2}$

Lemma 1.1. *Let $k, l \in \mathbb{N}$ and $k \ge l$. Then*

$$\binom{x}{k}\binom{k}{l} = \binom{x}{l}\binom{x-l}{k-l}.$$

Proof. Clearly

$$\begin{aligned}\binom{x}{l}\binom{x-l}{k-l} =& \frac{\prod_{0 \le i < l}(x-i)}{l!} \cdot \frac{\prod_{0 \le j < k-l}(x-l-j)}{(k-l)!} \\ =& \frac{\prod_{0 \le r < k}(x-r)}{k!} \cdot \frac{k!}{l!(k-l)!} = \binom{x}{k}\binom{k}{l}.\end{aligned}$$

This ends the proof. □

Lemma 2.2. *Let $n \in \mathbb{N}$, and $\delta_{n,m}$ be 1 or 0 according as $m = n$ or not. Then*

$$B_n(1) - B_n(0) = \delta_{n,1} \quad \textit{and} \quad E_n(1) + E_n(0) = 2\delta_{n,0}. \tag{1.3}$$

Proof. By Definition 1.2,

$$B_n(1) - B_n(0) = \sum_{k=0}^{n} \binom{n}{k} B_k - B_n = \sum_{0 \le k < n} \binom{n}{k} B_k = \delta_{n,1}$$

and

$$E_n(1)+E_n(0)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(\left(1-\frac{1}{2}\right)^{n-k}+\left(0-\frac{1}{2}\right)^{n-k}\right)$$
$$=\frac{1}{2^{n-1}}\sum_{\substack{k=0\\ 2|n-k}}^{n}\binom{n}{k}E_k=2\delta_{n,0}.$$

We are done. □

Theorem 1.1. *Let $n\in\mathbb{N}$. Then we have*

$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k},\ \textit{and}\ \ E_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k}. \tag{1.4}$$

Also,

$$B_n(x+1)-B_n(x)=nx^{n-1}\quad \textit{and}\ \ E_n(x+1)+E_n(x)=2x^n. \tag{1.5}$$

Proof. By the binomial theorem and Lemma 1.1,

$$B_n(x+y)=\sum_{l=0}^{n}\binom{n}{l}B_l(x+y)^{n-l}=\sum_{l=0}^{n}\binom{n}{l}B_l\sum_{k=l}^{n}\binom{n-l}{k-l}x^{k-l}y^{n-k}$$
$$=\sum_{0\le l\le k\le n}\binom{n}{l}\binom{n-l}{k-l}B_lx^{k-l}y^{n-k}=\sum_{0\le l\le k\le n}\binom{n}{k}\binom{k}{l}B_lx^{k-l}y^{n-k}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}B_lx^{k-l}y^{n-k}=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k}.$$

Similarly,

$$E_n(x+y)=\sum_{l=0}^{n}\binom{n}{l}\frac{E_l}{2^l}\left(x+y-\frac{1}{2}\right)^{n-l}$$
$$=\sum_{l=0}^{n}\binom{n}{l}\frac{E_l}{2^l}\sum_{k=l}^{n}\binom{n-l}{k-l}\left(x-\frac{1}{2}\right)^{k-l}y^{n-k}$$
$$=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}\frac{E_l}{2^l}\left(x-\frac{1}{2}\right)^{k-l}y^{n-k}=\sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k}.$$

In view of the above and Lemma 1.2,

$$B_n(x+1)-B_n(x)=\sum_{k=0}^{n}\binom{n}{k}\left(B_k(1)-B_k(0)\right)x^{n-k}=nx^{n-1}$$

and

$$E_n(x+1)+E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\left(E_k(1)+E_k(0)\right)x^{n-k}=2x^n.$$

This concludes the proof. □

Theorem 1.2. *Let $n\in\mathbb{N}$. Then we have the recursion*

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)=(n+1)x^k \quad \textit{and} \quad \sum_{k=0}^{n}\binom{n}{k}E_k(x)+E_n(x)=2x^n. \tag{1.6}$$

Also,

$$B_n(1-x)=(-1)^nB_n(x) \quad \textit{and} \quad E_n(1-x)=(-1)^nE_n(x). \tag{1.7}$$

Proof. By Theorem 1.1,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)=&\sum_{k=0}^{n+1}\binom{n+1}{k}B_k(x)1^{n+1-k}-B_{n+1}(x)\\ =&B_{n+1}(x+1)-B_n(x)=(n+1)x^n\end{aligned}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}E_k(x)+E_n(x)=\sum_{k=0}^{n}\binom{n}{k}E_k(x)1^{n-k}+E_n(x)=E_n(x+1)+E_n(x)=2x^n.$$

In view of the above and Theorem 1.1,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n+1}{k}\left(B_k(1-x)-(-1)^kB_k(x)\right)\\ =&\sum_{k=0}^{n}\binom{n+1}{k}B_k(1-x)+(-1)^n\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)(-1)^{n+1-k}\\ =&(n+1)(1-x)^n+(-1)^n\left(B_{n+1}(x-1)-B_{n+1}(x)\right)\\ =&(-1)^n\left((n+1)(x-1)^n-\left(B_{n+1}((x-1)+1)-B_{n+1}(x-1)\right)\right)=0.\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\left(E_k(1-x)-(-1)^kE_k(x)\right)+E_n(1-x)-(-1)^nE_n(x)\\ =&\sum_{k=0}^{n}\binom{n}{k}E_k(1-x)+E_n(1-x)-(-1)^n\left(\sum_{k=0}^{n}\binom{n}{k}E_k(x)(-1)^{n-k}+E_n(x)\right)\\ =&2(1-x)^n-(-1)^n\left(E_n(x-1)+E_n(x-1+1)\right)=0.\end{aligned}$$

On the basis of these two recursions, (1.7) follows by induction. □

Corollary 1.1. *Let $n > 1$ be an integer.*

(i) *When n is odd, we have $B_n(1/2) = E_n = 0$, and $B_n = 0$ if $n > 1$.*

(ii) *If n is even, then $E_n(0) = 0$.*

Proof. When n is odd, taking $x = 1/2$ in (1.7) we find that $B_n(1/2) = E_n(1/2) = 0$. Recall that $E_n = 2^n E_n(1/2)$.

By (1.7), $B_n(1) = (-1)^n B_n(0)$ and $E_n(1) = (-1)^n E_n(0)$. This, together with (1.3), shows that $B_n = 0$ if $n > 1$ and $2 \nmid n$, and that $E_n(0) = 0$ if $2 \mid n$. $\square$

2. On the sums $\sum_{r=0}^{n-1} r^k$ and $\sum_{r=0}^{n-1} (-1)^r r^k$

For $k \in \mathbb{N} = \{0, 12, \cdots\}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$, we set

$$S_k(n) = \sum_{r=0}^{n-1} r^k \quad \text{and} \quad T_k(n) = \sum_{r=0}^{n-1} (-1)^r r^k. \tag{2.1}$$

It is well known that

$$S_0(n) = n, \ S_1(n) = \frac{n(n-1)}{2} \text{ and } S_2(n) = \frac{n(n-1)(2n-1)}{6}.$$

In 1713 J. Bernoulli introduced the Bernoulli numbers, and used them to express $S_k(n)$ as a polynomial in n with degree $k+1$. Later Euler introduced the Euler numbers to study the sum $T_k(n)$.

Theorem 2.1. *Let k and n be positive integers. Then*

$$S_k(n) = \frac{B_{k+1}(n) - B_{k+1}}{k+1} = \frac{n^{k+1}}{k+1} - \frac{n^k}{2} + \sum_{\substack{1<l\le k \\ 2\mid l}} \binom{k}{l-1} \frac{B_l}{l} n^{k-l+1} \tag{2.2}$$

and

$$T_k(n) = \frac{E_k(0) - (-1)^n E_k(n)}{2} = 2^{k+1} S_k\left(\left[\frac{n+1}{2}\right]\right) - S_k(n). \tag{2.3}$$

Proof. By Theorem 2.1, $B_{k+1}(x+1)-B_{k+1}(x)=(k+1)x^k$. Therefore

$$\begin{aligned}(k+1)S_k(n)=&\sum_{r=0}^{n-1}(B_{k+1}(r+1)-B_{k+1}(r))\\ =&B_{k+1}(n)-B_{k+1}=\sum_{l=0}^{k}\binom{k+1}{l}B_l n^{k+1-l}\\ =&n^{k+1}-(k+1)\frac{n^k}{2}+(k+1)\sum_{1<l\le k}\binom{k}{l-1}\frac{B_l}{l}n^{k-l+1}.\end{aligned}$$

By Corollary 1.1 $B_l=0$ for $l=3,5,\cdots$, so (2.2) follows.

In view of Theorem 2.1, $E_k(x+1)+E_k(x)=2x^k$. Thus

$$\begin{aligned}2T_k(n)=&\sum_{r=0}^{n-1}(-1)^r(E_k(r)+E_k(r+1))\\ =&\sum_{r=0}^{n-1}\big((-1)^rE_k(r)-(-1)^{r+1}E_k(r+1)\big)=E_k(0)-(-1)^nE_k(n).\end{aligned}$$

We also have

$$T_k(n)=2\sum_{\substack{r=0\\ 2|r}}^{n-1}r^k-\sum_{r=0}^{n-1}r^k=2^{k+1}\sum_{j=0}^{[(n-1)/2]}j^k-S_k(n)=2^{k+1}S_k\left(\left[\frac{n+1}{2}\right]\right)-S_k(n).$$

This ends our proof. □

Example 2.1 As $B_4(x)=x^4-2x^3+x^2-1/30$, we have

$$S_3(n)=\frac{B_4(n)-B_4}{4}=\frac{n^4-2n^3+n^2}{4}=\frac{n^2(n-1)^2}{4}=S_1(n)^2.$$

Similarly,

$$S_4(n)=\frac{B_5(n)-B_5}{5}=\frac{B_5(n)}{5}=\frac{n^5}{5}-\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}.$$

Since $E_3(x)=x^3-(3/2)x^2+1/6$ and $E_4(x)=x^4-2x^3+(2/3)x$, we have

$$\begin{aligned}T_3(n)=&\frac{E_3(0)-(-1)^nE_3(n)}{2}=\frac{1}{12}-\frac{(-1)^n}{2}\left(n^3-\frac{3}{2}n^2+\frac{1}{6}\right)\\ =&\frac{1-(-1)^n}{12}-(-1)^n\frac{n^2}{4}(2n-3)\end{aligned}$$

and

$$T_4(n)=\frac{E_4(0)-(-1)^nE_4(n)}{2}=(-1)^{n-1}\left(n^4-2n^3+\frac{2}{3}n\right).$$

Corollary 2.1. *For any $k \in \mathbb{N}$ we have*

$$E_k(x) = \frac{2^{k+1}}{k+1}\left(B_{k+1}\left(\frac{x+1}{2}\right) - B_{k+1}\left(\frac{x}{2}\right)\right). \tag{2.4}$$

Proof. Whenever $n \in \{2, 4, 6, \cdots\}$ we have

$$\begin{aligned}\frac{E_k(0) - E_k(n)}{2} =& T_k(n) = \frac{2^{k+1}S_k(n/2) - S_k(n)}{k+1} \\ =& \frac{2^{k+1}B_{k+1}(n/2) - B_{k+1}(n) + (1 - 2^{k+1})B_{k+1}}{k+1}.\end{aligned}$$

Since both sides are polynomials in n, it follows that

$$\frac{E_k(0) - E_k(x)}{2} = \frac{2^{k+1}B_{k+1}(x/2) - B_{k+1}(x) + (1 - 2^{k+1})B_{k+1}}{k+1}. \tag{$*$}$$

If $n \in \{1, 3, 5, \cdots\}$, then

$$\begin{aligned}\frac{E_k(0) + E_k(n)}{2} =& T_k(n) = \frac{2^{k+1}S_k((n+1)/2) - S_k(n)}{k+1} \\ =& \frac{2^{k+1}B_{k+1}((n+1)/2) - B_{k+1}(n) + (1 - 2^{k+1})B_{k+1}}{k+1}.\end{aligned}$$

So

$$\frac{E_k(0) + E_k(x)}{2} = \frac{2^{k+1}B_{k+1}((x+1)/2) - B_{k+1}(x) + (1 - 2^{k+1})B_{k+1}}{k+1}. \tag{$\star$}$$

($\star$) minus ($*$) yields (2.4) immediately. □

3. Raabe's theorem and its applications

The following theorem of Raabe plays important roles in the theory of Bernoulli polynomials.

Theorem 3.1. *Let $m > 0$ and $n \geq 0$ be integers. Then*

$$\sum_{r=0}^{m-1} B_n\left(\frac{x+r}{m}\right) = m^{1-n}B_n(x). \tag{3.1}$$

Proof. For any $k=0,1,2,\cdots$ we have

$$\begin{aligned}(k+1)\sum_{r=0}^{m-1}(x+r)^k=&\sum_{r=0}^{m-1}\left(B_{k+1}(x+r+1)-B_{k+1}(x+r)\right)\\ =&B_{k+1}(x+m)-B_{k+1}(x)=\sum_{l=0}^{k}\binom{k+l}{l}B_l(x)m^{k+1-l}.\end{aligned}$$

This, together with Lemma 1.1 and the recursion for Bernoulli numbers, yields that

$$\begin{aligned}\sum_{r=0}^{m-1}B_n\left(\frac{x+r}{m}\right)=&\sum_{r=0}^{m-1}\sum_{k=0}^{n}\binom{n}{k}\frac{B_{n-k}}{m^k}(x+r)^k\\ =&\sum_{k=0}^{n}\binom{n}{k}\frac{B_{n-k}}{k+1}\sum_{l=0}^{k}\binom{k+1}{l}B_l(x)m^{1-l}\\ =&\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}B_{n-k}\sum_{l=0}^{k}\binom{k+1}{l}B_l(x)m^{1-l}\\ =&\frac{1}{n+1}\sum_{0\le l\le k\le n}\binom{n+1}{l}\binom{n+1-l}{k+1-l}m^{1-l}B_l(x)B_{n-k}\\ =&\frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}m^{1-l}B_l(x)\sum_{k=l}^{n}\binom{n+1-l}{n-k}B_{n-k}\\ =&\frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}m^{1-l}B_l(x)\delta_{l,n}=m^{1-n}B_n(x).\end{aligned}$$

This completes the proof. □

Corollary 3.1. *For $n\in\mathbb{N}$ we have*

$$E_n(x)=\frac{2}{n+1}\left(B_{n+1}(x)-2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right).\qquad(3.2)$$

Proof. Applying Theorem 3.1 with $m=2$, we obtain that

$$B_{n+1}\left(\frac{x}{2}\right)+B_{n+1}\left(\frac{x+1}{2}\right)=\frac{B_{n+1}(x)}{2^n}.$$

On the other hand, by Corollary 2.1,

$$B_{n+1}\left(\frac{x+1}{2}\right)-B_{n+1}\left(\frac{x}{2}\right)=\frac{n+1}{2^{n+1}}E_n(x).$$

The first equation minus the second one yields that

$$2B_{n+1}\left(\frac{x}{2}\right) = \frac{2B_{n+1}(x) - (n+1)E_n(x)}{2^{n+1}}.$$

which is equivalent to (3.2). □

From Theorem 3.1 we can deduce the following result.

Theorem 3.2. *Let $n \in \mathbb{N}$. Then*

$$B_n\left(\frac{1}{2}\right) = \left(2^{1-n} - 1\right) B_n. \tag{3.3}$$

When $2 \mid n$, we have

$$B_n\left(\frac{1}{3}\right) = B_n\left(\frac{2}{3}\right) = (3^{1-n} - 1)\frac{B_n}{2}, \tag{3.4}$$

$$B_n\left(\frac{1}{4}\right) = B_n\left(\frac{3}{4}\right) = 2^{-n}(2^{1-n} - 1)B_n, \tag{3.5}$$

$$B_n\left(\frac{1}{6}\right) = B_n\left(\frac{5}{6}\right) = (2^{1-n} - 1)(3^{1-n} - 1)\frac{B_n}{2}. \tag{3.6}$$

Proof. Taking $x = 0$ and $m = 2$ in (3.1), we find that

$$B_n\left(\frac{0}{2}\right) + B_n\left(\frac{1}{2}\right) = 2^{1-n}B_n(0), \quad \text{i.e. } B_n\left(\frac{1}{2}\right) = (2^{1-n} - 1)B_n.$$

Now we let n be even. Note that $B_n(1-x) = (-1)^n B_n(x) = B_n(x)$. (3.1) in the case $x = 0$ and $m = 3$, yields that

$$B_n(0) + B_n\left(\frac{1}{3}\right) + B_n\left(\frac{2}{3}\right) = 3^{1-n}B_n,$$

which is equivalent to (3.4). Taking $x = 1/2$ and $m = 2$ in (3.1), we get that

$$B_n\left(\frac{1/2+0}{2}\right) + B_n\left(\frac{1/2+1}{2}\right) = 2^{1-n}B_n\left(\frac{1}{2}\right).$$

So

$$B_n\left(\frac{1}{4}\right) = B_n\left(\frac{3}{4}\right) = 2^{-n}B_n\left(\frac{1}{2}\right) = 2^{-n}(2^{1-n} - 1)B_n.$$

(3.1) in the case $x = 1/3$ and $m = 2$, gives that

$$B_n\left(\frac{1/3+0}{2}\right) + B_n\left(\frac{1/3+1}{2}\right) = 2^{1-n}B_n\left(\frac{1}{3}\right);$$

therefore

$$B_n\left(\frac{5}{6}\right) = B_n\left(\frac{1}{6}\right) = 2^{1-n}B_n\left(\frac{1}{3}\right) - B_n\left(\frac{1}{3}\right) = (2^{1-n}-1)(3^{1-n}-1)\frac{B_n}{2}.$$

This completes the proof. □

Corollary 3.2. *Let $n \in \mathbb{N}$. Then*

$$E_n(0) = 2(1-2^{n+1})\frac{B_{n+1}}{n+1}. \tag{3.7}$$

If n is odd, then

$$E_n\left(\frac{1}{3}\right) = -E_n\left(\frac{2}{3}\right) = (2^{n+1}-1)(3^{-n}-1)\frac{B_{n+1}}{n+1}. \tag{3.8}$$

Proof. Taking $x = 0$ in (3.2) we obtain (3.7).

Now let n be odd. Then $E_n(2/3) = (-1)^n E_n(1/3) = -E_n(1/3)$. By Corollary 3.1 and Theorem 3.2, we have

$$\begin{aligned}E_n\left(\frac{1}{3}\right) =& \frac{2}{n+1}\left(B_{n+1}\left(\frac{1}{3}\right) - 2^{n+1}B_{n+1}\left(\frac{1}{6}\right)\right)\\ =&\frac{2}{n+1}\left(\frac{3^{-n}-1}{2}B_{n+1} - 2^{n+1}(2^{-n}-1)(3^{-n}-1)\frac{B_{n+1}}{2}\right)\\ =&(2^{n+1}-1)(3^{-n}-1)\frac{B_{n+1}}{n+1}.\end{aligned}$$

Theorem 3.3. *Let $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}$. If $2 \mid m$ then*

$$\sum_{r=0}^{m-1}(-1)^r B_{n+1}\left(\frac{x+r}{m}\right) = -\frac{n+1}{2m^n}E_n(x); \tag{3.9}$$

if $2 \nmid m$ then

$$\sum_{r=0}^{m-1}(-1)^r E_n\left(\frac{x+r}{m}\right) = \frac{E_n(x)}{m^n}. \tag{3.10}$$

Proof. We use Corollary 2.1 and Theorem 3.1. When m is even, we have

$$\begin{aligned}&\sum_{r=0}^{m-1}(-1)^r B_{n+1}\left(\frac{x+r}{m}\right)\\ &=\sum_{s=0}^{m/2-1} B_{n+1}\left(\frac{x+2s}{m}\right)-\sum_{s=0}^{m/2-1} B_{n+1}\left(\frac{x+1+2s}{m}\right)\\ &=\sum_{s=0}^{m/2-1} B_{n+1}\left(\frac{x/2+s}{m/2}\right)-\sum_{s=0}^{m/2-1} B_{n+1}\left(\frac{(x+1)/2+s}{m/2}\right)\\ &=\left(\frac{m}{2}\right)^{-n} B_{n+1}\left(\frac{x}{2}\right)-\left(\frac{m}{2}\right)^{-n} B_{n+1}\left(\frac{x+1}{2}\right)\\ &=\left(\frac{2}{m}\right)^{n}\left(B_{n+1}\left(\frac{x}{2}\right)-B_{n+1}\left(\frac{x+1}{2}\right)\right)=-\frac{n+1}{2m^n}E_n(x).\end{aligned}$$

If m is odd, then

$$\begin{aligned}&\frac{n+1}{2^{n+1}}\sum_{r=0}^{m-1}(-1)^r E_n\left(\frac{x+r}{m}\right)\\ &=\sum_{r=0}^{m-1}(-1)^r\left(B_{n+1}\left(\frac{(x+r)/m+1}{2}\right)-B_{n+1}\left(\frac{(x+r)/m}{2}\right)\right)\\ &=-\sum_{r=0}^{m-1}\left((-1)^r B_{n+1}\left(\frac{x+r}{2m}\right)+(-1)^{r+m}B_{n+1}\left(\frac{x+r+m}{2m}\right)\right)\\ &=-\sum_{r=0}^{2m-1}(-1)^r B_{n+1}\left(\frac{x+r}{2m}\right)=\frac{n+1}{2(2m)^n}E_n(x).\end{aligned}$$

So (3.10) also holds. □

4. Number-theoretic properties of Bernoulli numbers and Bernoulli polynomials

Let p be a prime. A rational a/b with $a, b \in \mathbb{Z}$ and $(b, p) = 1$, will be called a p-integer. We let $\mathbb{Z}_p$ denote the set of all p-integers. For $x, y \in \mathbb{Z}_p$ and $n \in \mathbb{N}$, by $x \equiv y \pmod{p^n}$ we mean that $x - y \in p^n\mathbb{Z}_p$.

Lemma 4.1. *Let k be a positive integer and p be a prime. Then $pB_k \in \mathbb{Z}_p$ and*

$$\frac{S_k(p)-pB_k}{k}\equiv\frac{p}{2}pB_{k-1}\pmod{p}. \tag{4.1}$$

Furthermore, if $p > 3$ *then*

$$\frac{S_k(p) - pB_k}{k} \equiv \frac{p}{2} pB_{k-1} \pmod{p^2}. \tag{4.2}$$

Proof. By Theorem 2.1,

$$\begin{aligned} S_k(p) =& \frac{B_{k+1}(p) - B_{k+1}}{k+1} = \frac{1}{k+1} \sum_{j=1}^{k+1} \binom{k+1}{j} p^j B_{k+1-j} \\ =& \frac{1}{k+1} \sum_{l=0}^{k} \binom{k+1}{l+1} p^{l+1} B_{k-l} = pB_k + \sum_{l=1}^{k} \binom{k}{l} \frac{p^l}{l+1} pB_{k-l}. \end{aligned}$$

Clearly $p^l \geq (1+1)^l \geq l+1$ and hence $p^l/(l+1) \in \mathbb{Z}_p$. So $pB_k \in \mathbb{Z}_p$ by induction on k.

Observe that

$$\begin{aligned} \frac{S_k(p) - pB_k}{k} =& \frac{1}{k} \sum_{l=1}^{k} \binom{k}{l} \frac{p^l}{l+1} pB_{k-l} = \sum_{l=1}^{k} \binom{k-1}{l-1} \frac{p^l}{l(l+1)} pB_{k-l} \\ =& \frac{p}{2} pB_{k-1} + p \sum_{1 < l \leq k} \binom{k-1}{l-1} \frac{p^{l-1}}{l(l+1)} pB_{k-l}. \end{aligned}$$

Obviously $p^{2-1}/(2 \cdot 3) = p/6 \in \mathbb{Z}_p$, and $p/6 \in p\mathbb{Z}_p$ if $p > 3$. When $l \in \{3, 4, \cdots\}$, we have $p^{l-1} \geq (1+1)^{l-1} \geq 1 + (l-1) + 1 = l + 1$, and

$$p^{l-2} \geq (1+4)^{l-2} \geq 1 + 4(l-2) \geq l + 1$$

providing $p \geq 5$. Thus, if $l \in \{3, 4, \cdots\}$, then $p^{l-1}/(l(l+1)) = p^{l-1}/(l+1) - p^{l-1}/l \in \mathbb{Z}_p$, moreover $p^{l-1}/(l(l+1)) \in p\mathbb{Z}_p$ providing $p > 3$. In view of the above, (4.1) holds, and (4.2) is also valid if $p > 3$. $\square$

Theorem 4.1 (von Staudt-Clausen). *We have*

$$B_k + \sum_{p-1 \mid k} \frac{1}{p} \in \mathbb{Z} \quad \text{for } k = 2, 4, 6, \cdots. \tag{4.3}$$

Proof. Let $k > 0$ be an even integer. Recall that $B_{k-1} = 0$ if $k > 2$. So, by Lemma 4.1, we have

$$S_k(p) - pB_k \equiv \delta_{k,2}p^2B_1 \equiv 0 \pmod p.$$

If $p - 1 \mid k$, then by Fermat's little theorem

$$S_k(p) = \sum_{r=1}^{p-1} r^k \equiv \sum_{r=1}^{p-1} 1 \equiv -1 \pmod p$$

and hence $B_k + 1/p \in \mathbb{Z}_p$. If $p-1 \nmid k$, then there is a $g \in \mathbb{Z}$ such that $g^k \not\equiv 1 \pmod p$, as $(g^k - 1)S_k(p) = \sum_{r=1}^{p-1}(gr)^k - \sum_{r=1}^{p-1} r^k \equiv 0 \pmod p$ we have $p \mid S_k(p)$ and hence $B_k \in \mathbb{Z}_p$.

By the above, $B_k + \sum_{p-1|k} p^{-1} \in \mathbb{Z}_q$ for any prime q. So $B_k + \sum_{p-1|k} p^{-1} \in \mathbb{Z}$. We are done. □

Theorem 4.2 (Beeger, 1913)**.** *Let $p > 3$ be a prime. Then*

$$(p-1)! \equiv pB_{p-1} - p \pmod{p^2}. \tag{4.4}$$

Proof. Wilson's theorem asserts that $w_p = ((p-1)! + 1)/p \in \mathbb{Z}$. For any integer $a \not\equiv 0 \pmod p$ let $q_p(a)$ denote the Fermat quotient $(a^{p-1} - 1)/p$. Then

$$(pw_p - 1)^{p-1} = \prod_{r=1}^{p-1} r^{p-1} = \prod_{r=1}^{p-1}(1 + pq_p(r)) \equiv 1 + p\sum_{r=1}^{p-1} q_p(r) \pmod{p^2}$$

and hence

$$1 - (p-1)pw_p \equiv (pw_p - 1)^{p-1} \equiv 1 + \sum_{r=1}^{p-1}(r^{p-1} - 1) = S_{p-1}(p) - p + 2.$$

By Theorem 4.2, $S_{p-1}(p) \equiv pB_{p-1} \pmod{p^2}$. So $(p-1)! = pw_p - 1 \equiv pB_{p-1} - p \pmod{p^2}$. □

Theorem 4.3. *Let p be a prime and $n > 0$ be an even integer.*

(i) (E. Kummer) *If $p-1 \nmid n$, then $B_n/n \in \mathbb{Z}_p$, moreover $B_m/m \equiv B_n/n \pmod p$ whenever $m \equiv n \pmod{p-1}$.*

(ii) (L. Carlitz) *If $p \neq 2$ and $p - 1 \mid n$ then $(B_n + p^{-1} - 1)/n \in \mathbb{Z}_p$.*

Theorem 4.4 (Voronoi, 1889). *Let $n>0$ be even, $p\in\mathbb{Z}^+$, $m\in\mathbb{Z}$ and $(p,m)=1$. Then*

$$(m^n-1)\,B_n \equiv nm^{n-1}\sum_{j=1}^{p-1} j^{n-1}\left[\frac{jm}{p}\right] \pmod p. \tag{4.5}$$

Theorem 4.5 (L. Euler). *We have*

$$\tan x=\sum_{m=1}^{\infty}(-1)^{m-1}\frac{2^{2m}(2^{2m}-1)B_{2m}}{(2m)!}x^{2m-1}\quad \text{for } x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^{2m}}=(-1)^{m-1}\frac{(2\pi)^{2m}}{2(2m)!}B_{2m}\quad \text{for } m=1,2,3,\cdots.$$

Theorem 4.6 (Kummer, 1847). *Let $p>3$ be a prime such that p does not divide the numerator of $B_2,B_4,\cdots,B_{p-3}$. Then $x^p+y^p=z^p$ has no integer solutions with $p\nmid xyz$.*

Theorem 4.7 (A. Granville and Z. W. Sun, 1996). *Let p be an odd prime relatively prime to a fixed $q\in\{5,8,10,12\}$. Then we can determine $B_{p-1}(a/q)-B_{p-1}$ mod p (with $1\le a\le q$ and $(a,q)=1$) as follows:*

$$B_{p-1}\left(\frac{a}{5}\right)-B_{p-1}\equiv\frac{5}{4}\left(\left(\frac{ap}{5}\right)\frac{1}{p}F_{p-(\frac{5}{p})}+\frac{5^{p-1}-1}{p}\right) \pmod p;$$

$$B_{p-1}\left(\frac{a}{8}\right)-B_{p-1}\equiv\left(\frac{2}{ap}\right)\frac{2}{p}P_{p-(\frac{2}{p})}+4\cdot\frac{2^{p-1}-1}{p} \pmod p;$$

$$B_{p-1}\left(\frac{a}{10}\right)-B_{p-1}\equiv\frac{15}{4}\left(\frac{ap}{5}\right)\frac{1}{p}F_{p-(\frac{5}{p})}+\frac{5}{4}\cdot\frac{5^{p-1}-1}{p}+\frac{2(2^{p-1}-1)}{p} \pmod p;$$

$$B_{p-1}\left(\frac{a}{12}\right)-B_{p-1}\equiv\left(\frac{3}{a}\right)\frac{3}{p}S_{p-(\frac{3}{p})}+\frac{3(2^{p-1}-1)}{p}+\frac{3}{2}\cdot\frac{3^{p-1}-1}{p} \pmod p;$$

where $(-)$ is the Jacobi symbol, and we define the following second-order linear recurrence sequences:

$$F_0=0,\ F_1=1,\ \text{and } F_{n+2}=F_{n+1}+F_n\quad \text{for all } n\ge 0$$

$$P_0=0,\ P_1=1,\ \text{and } P_{n+2}=2P_{n+1}+P_n\quad \text{for all } n\ge 0$$

$$S_0=0,\ S_1=1,\ \text{and } S_{n+2}=4S_{n+1}-S_n\quad \text{for all } n\ge 0.$$