Note. For cusp forms, better estimates for the order of magnitude of the $c(n)$ have been obtained by Kloosterman, Salié, Davenport, Rankin, and Selberg (see [46]). It has been shown that

$$c(n) = O(n^{k-(1/4)+\varepsilon})$$

for every $\varepsilon > 0$, and it has been conjectured that the exponent can be further improved to $k - \frac{1}{2} + \varepsilon$. For the discriminant Δ, Ramanujan conjectured the sharper estimate

$$|\tau(p)| \le 2p^{11/2}$$

for primes p. This was recently proved by P. Deligne [7].

6.16 Modular forms and Dirichlet series

Hecke found a remarkable connection between each modular form with Fourier series

$$f(\tau) = c(0) + \sum_{n=1}^{\infty} c(n)e^{2\pi i n\tau} \tag{49}$$

and the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s} \tag{50}$$

formed with the same coefficients (except for $c(0)$). If $f \in M_{2k}$ then $c(n) = O(n^k)$ if f is a cusp form, and $c(n) = O(n^{2k-1})$ if f is not a cusp form. Therefore, the Dirichlet series in (50) converges absolutely for $\sigma = \text{Re}(s) > k + 1$ if f is a cusp form, and for $\sigma > 2k$ if f is not a cusp form.

Theorem 6.19. *If the coefficients $c(n)$ satisfy the multiplicative property*

$$c(m)c(n) = \sum_{d|(m,n)} d^{2k-1}c\left(\frac{mn}{d^2}\right) \tag{51}$$

the Dirichlet series will have an Euler product representation of the form

$$\varphi(s) = \prod_p \frac{1}{1 - c(p)p^{-s} + p^{2k-1}p^{-2s}}, \tag{52}$$

absolutely convergent with the Dirichlet series.

PROOF. Since the coefficients are multiplicative we have (see [4], Theorem 11.7)

$$\varphi(s) = \prod_p \left\{1 + \sum_{n=1}^{\infty} c(p^n)p^{-ns}\right\} \tag{53}$$

whenever the Dirichlet series converges absolutely. Now (51) implies

$$c(p)c(p^n) = c(p^{n+1}) + p^{2k-1}c(p^{n-1})$$

for each prime p. Using this it is easy to verify the power series identity

$$(1 - c(p)x + p^{2k-1}x^2)\left(1 + \sum_{n=1}^{\infty} c(p^n)x^n\right) = 1$$

for all $|x| < 1$. Taking $x = p^{-s}$, we find that (53) reduces to (52). □

EXAMPLE. For the Ramanujan function we have the Euler product representation

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}$$

for $\sigma > 7$ since $\tau(n) = O(n^6)$.

Hecke also deduced the following analytic properties of $\varphi(s)$.

Theorem 6.20. *Let $\varphi(s)$ be the function defined for $\sigma > k$ by the Dirichlet series (50) associated with a modular form $f(\tau)$ in M_k having the Fourier series (49), where k is an even integer ≥ 4. Then $\varphi(s)$ can be continued analytically beyond the line $\sigma = k$ with the following properties:*

(a) *If $c(0) = 0$, $\varphi(s)$ is an entire function of s.*

(b) *If $c(0) \ne 0$, $\varphi(s)$ is analytic for all s except for a simple pole at $s = k$ with residue*

$$\frac{(-1)^{k/2}c(0)(2\pi)^k}{\Gamma(k)}.$$

(c) *The function φ satisfies the functional equation*

$$(2\pi)^{-s}\Gamma(s)\varphi(s) = (-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)\varphi(k-s).$$

PROOF. From the integral representation for $\Gamma(s)$ we have

$$\Gamma(s)(2n\pi)^{-s} = \int_0^{\infty} e^{-2\pi n y}y^{s-1}\,dy$$

if $\sigma > 0$. Therefore if $\sigma > k$ we can multiply both members by $c(n)$ and sum on n to obtain

$$(2\pi)^{-s}\Gamma(s)\varphi(s) = \int_0^{\infty} \{f(iy) - c(0)\}y^{s-1}\,dy.$$

Since f is a modular form in M_k we have $f(i/y) = (iy)^k f(iy)$ so

$$\begin{aligned}
(2\pi)^{-s}\Gamma(s)\varphi(s) &= \int_1^\infty \{f(iy) - c(0)\}y^{s-1}\,dy + \int_0^1 \left\{(iy)^{-k} f\left(\frac{i}{y}\right) - c(0)\right\}y^{s-1}\,dy \\
&= \int_1^\infty \{f(iy) - c(0)\}y^{s-1}\,dy + i^{-k}\int_1^\infty f(iw)w^{k-s-1}\,dw - \frac{c(0)}{s} \\
&= \int_1^\infty \{f(iy) - c(0)\}y^{s-1}\,dy \\
&\quad + (-1)^{k/2}\int_1^\infty \{f(iw) - c(0)\}w^{k-s-1}\,dw \\
&\quad + (-1)^{k/2}c(0)\int_1^\infty w^{k-s-1}\,dw - \frac{c(0)}{s} \\
&= \int_1^\infty \{f(iy) - c(0)\}(y^s + (-1)^{k/2}y^{k-s})\frac{dy}{y} \\
&\quad - c(0)\left(\frac{1}{s} + \frac{(-1)^{k/2}}{k-s}\right).
\end{aligned}$$

Although this last relation was proved under the assumption that $\sigma > k$, the right member is meaningful for all complex s. This gives the analytic continuation of $\varphi(s)$ beyond the line $\sigma = k$ and also verifies (a) and (b). Moreover, replacing s by $k - s$ leaves the right member unchanged except for a factor $(-1)^{k/2}$ so we also obtain (c). □

Hecke also proved a converse to Theorem 6.20 to the effect that every Dirichlet series φ which satisfies a functional equation of the type in (c), together with some analytic and growth conditions, necessarily arises from a modular form in M_k. For details, see [15].

Exercises for Chapter 6

Exercises 1 through 6 deal with arithmetical functions f satisfying a relation of the form

$$f(m)f(n) = \sum_{d|(m,n)} \alpha(d) f\left(\frac{mn}{d^2}\right) \tag{54}$$

for all positive integers m and n, where α is a given completely multiplicative function (that is, $\alpha(1) = 1$ and $\alpha(mn) = \alpha(m)\alpha(n)$ for all m and n). An arithmetical function satisfying (54) will be called *α-multiplicative*. We write $f = 0$ if $f(n) = 0$ for all n.

1. Assume f is α-multiplicative and $f \neq 0$. Prove that $f(1) = 1$. Also prove that cf is α-multiplicative if, and only if, $c = 0$ or $c = 1$.

2. If f and g are α-multiplicative, prove that $f + g$ is α-multiplicative if, and only if, $f = 0$ or $g = 0$.

3. Let $f_1, \ldots, f_k$ be k distinct nonzero α-multiplicative functions. If a linear combination

$$f = \sum_{i=1}^k c_i f_i$$

is also α-multiplicative, prove that:
(a) The functions $f_1, \ldots, f_k$ are linearly independent.
(b) Either all the c_i are 0 or else exactly one of the c_i is 1 and the others are 0. Hence either $f = 0$ or $f = f_i$ for some i. In other words, linear combinations of α-multiplicative functions are never α-multiplicative except for trivial cases.

4. If f is α-multiplicative, prove that

$$\alpha(n)f(m) = \sum_{d|n} \mu(d) f(mnd) f\left(\frac{n}{d}\right).$$

5. If f is multiplicative, prove that f is α-multiplicative if, and only if,

$$f(p^{k+1}) = f(p)f(p^k) - \alpha(p)f(p^{k-1}) \tag{55}$$

for all primes p and all integers $k \geq 1$.

6. The recursion relation (55) shows that $f(p^n)$ is a polynomial in $f(p)$, say

$$f(p^n) = Q_n(f(p)).$$

The sequence $\{Q_n(x)\}$ is determined by the relations

$$Q_1(x) = x,\ Q_2(x) = x^2 - \alpha(p),\ Q_{r+1}(x) = xQ_r(x) - \alpha(p)Q_{r-1}(x) \quad \text{for } r \geq 2.$$

Show that

$$Q_n(2\alpha(p)^{1/2}x) = \alpha(p)^{n/2}U_n(x),$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind, defined by the relations

$$U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_{r+1}(x) = 2xU_r(x) - U_{r-1}(x) \quad \text{for } r \geq 1.$$

7. Let $E_{2k}(\tau) = \frac{1}{2}G_{2k}(\tau)/\zeta(2k)$. If $x = e^{2\pi i\tau}$ verify that the Fourier expansion of $E_{2k}(\tau)$ has the following form for $k = 2, 3, 4, 5, 6$, and 7:

$$E_4(\tau) = 1 + 240\sum_{n=1}^\infty \sigma_3(n)x^n,$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^\infty \sigma_5(n)x^n,$$

$$E_8(\tau) = 1 + 480\sum_{n=1}^\infty \sigma_7(n)x^n,$$

$$E_{10}(\tau) = 1 - 264\sum_{n=1}^\infty \sigma_9(n)x^n,$$

$$E_{12}(\tau) = 1 + \frac{65520}{691}\sum_{n=1}^\infty \sigma_{11}(n)x^n,$$

$$E_{14}(\tau) = 1 - 24\sum_{n=1}^\infty \sigma_{13}(n)x^n.$$