(4) $\qquad \zeta(0) = -\frac{1}{2}$

(5) $\qquad \zeta(-2n) = 0 \qquad (n = 1, 2, 3, \ldots)$,

(6) $\qquad \zeta(1-2n) = -\frac{B_{2n}}{2n} \qquad (n = 1, 2, 3, \ldots)$,

wobei die rationalen Zahlen $B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}, \ldots$ *die durch*

(7) $\qquad \dfrac{t}{e^t - 1} = \sum\limits_{k=0}^{\infty} \dfrac{B_k}{k!} t^k \qquad (|t| < 2\pi)$

definierten Bernoullischen Zahlen *sind. Die Werte der Zetafunktion an positiven geraden ganzen Zahlen sind durch*

(8) $\qquad \zeta(2n) = \dfrac{(-1)^{n-1} \, 2^{2n-1} \, B_{2n}}{(2n)!} \, \pi^{2n} \qquad (n = 1, 2, 3, \ldots)$

gegeben.

<u>Beweis</u>: Wir gehen von der Integraldarstellung (3) aus. Seien die Zahlen B_k (k = 0, 1, 2, ...) durch (7) definiert, d.h. wir entwickeln

$$\frac{t}{e^t - 1} = \frac{t}{t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \ldots} = 1 - \frac{t}{2} + \frac{t^2}{12} + 0t^3 - \frac{t^4}{720} + \ldots$$

und definieren B_n als n! mal den Koeffizienten von t^n auf der rechten Seite; aus

$$\frac{t}{e^t - 1} - \frac{-t}{e^{-t} - 1} = -t$$

folgt, daß abgesehen von $B_1 = -\frac{1}{2}$ alle B_n mit n ungerade Null sind. Sei jetzt n > 0 fest und

$$f_n(t) = \sum\limits_{k=0}^{n} (-1)^k \frac{B_k}{k!} t^k = 1 + \frac{t}{2} + \frac{B_2}{2!} t^2 + \ldots + \frac{B_n}{n!} t^n$$

(für n > 1 ist $(-1)^n B_n = B_n$). Dann ist für $\sigma > 1$

$$\Gamma(s) \cdot \zeta(s) = \int\limits_0^{\infty} \frac{te^t}{e^t - 1} e^{-t} t^{s-2} dt$$

(9)
$$= \int\limits_0^{\infty} \left(\frac{te^t}{e^t - 1} - f_n(t) \right) e^{-t} t^{s-2} dt + \int\limits_0^{\infty} f_n(t) e^{-t} t^{s-2} dt$$

$$= I_1(s) + I_2(s) \ ,$$

wobei wir mit I_1 und I_2 die beiden Integrale bezeichnen. Die Funk-

Aufgaben

1. Man beweise elementar die Existenz unendlich vieler Primzahlen
   der Gestalt $4n - 1$ bzw. $4n + 1$, indem man unter der Annahme,
   es gäbe nur endlich viele, die Zahlen

$$4 \prod_{p \equiv 3 \pmod 4} p - 1 \quad \text{bzw.} \quad 4 \prod_{p \equiv 1 \pmod 4} p^2 + 1$$

   bildet und einen Widerspruch ableitet. Für welche anderen arithme-
   tischen Folgen gibt es einen analogen Beweis?

2. Man zeige, daß für jeden Dirichletschen Charakter χ

$$\frac{1}{L(s,\chi)} = \sum_{n=1}^{\infty} \frac{\mu(n)\,\chi(n)}{n^s} \qquad (\sigma > 1)$$

   ist, wobei $\mu(n)$ die in §2 eingeführte Möbiussche Funktion be-
   zeichnet. Kann man aus dem Satz dieses Paragraphen schließen, daß
   die Reihe für $s = 1$ konvergiert?

§7 Werte von Dirichletschen Reihen, insbesondere von L-Reihen, an negativen ganzen Stellen

In §4 haben wir gesehen, daß die Riemannsche Zetafunktion $\zeta(s)$ für
alle geraden Argumente $s > 1$ und für alle ganzzahligen Argumente
$s < 1$ Werte annimmt, die man in geschlossener Gestalt angeben kann;
für $s < 1$ sind diese Werte stets rational und in der Hälfte der
Fälle gleich Null. Ähnliche Eigenschaften gelten für alle Dirichlet-
schen L-Reihen. Da diese Reihen Funktionalgleichungen erfüllen,
braucht man die Werte nur für $s \geq 1$ oder für $s \leq 0$ zu berechnen.
Überraschenderweise stellt sich heraus, daß die Werte an den negati-
ven ganzzahligen Stellen trotz der Nichtkonvergenz der Reihen wesent-
lich einfacher auszurechnen sind als die an positiven ganzzahligen
Stellen. Dies liegt an folgendem Satz, welcher es ermöglicht, unter
sehr allgemeinen Voraussetzungen Werte von Dirichtletschen Reihen für
ganzzahlige negative Argumente zu bestimmen.

SATZ 1: *Sei* $\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *eine Dirichletsche Reihe, die für mindestens einen*
(komplexen) *Wert von* s *konvergiert, und sei* $f(t) = \sum_{n=1}^{\infty} a_n e^{-nt}$ *die ent-*
sprechende Exponentialreihe (welche für alle $t > 0$ konvergiert). *Hat*

$f(t)$ *für* $t \to 0$ *die asymptotische Entwicklung*

(1) $$f(t) \sim b_0 + b_1 t + b_2 t^2 + \ldots \qquad (t \to 0) \ ,$$

so läßt sich $\varphi(s)$ *holomorph in die ganze komplexe Ebene fortsetzen und es gilt*

(2) $$\varphi(-n) = (-1)^n \, n! \, b_n \qquad (n = 0, 1, 2, \ldots) \ .$$

Allgemeiner, wenn $f(t)$ *für* $t \to 0$ *die asymptotische Entwicklung*

(3) $$f(t) \sim \frac{b_{-1}}{t} + b_0 + b_1 t + b_2 t^2 + \ldots$$

besitzt, so hat $\varphi(s)$ *eine meromorphe Fortsetzung, die Funktion* $\varphi(s) - \dfrac{b_{-1}}{s-1}$ *ist ganz, und die Werte* $\varphi(0), \varphi(-1), \ldots$ *werden nach wie vor durch die Formel (2) gegeben.*

Bemerkungen: 1. "Asymptotische Entwicklung" bedeutet, daß für jede natürliche Zahl N die Abschätzung

(4) $$\left| f(t) - \sum_{n < N} b_n \, t^n \right| \le C t^N \qquad (0 < t < t_0)$$

gilt (kurz: $f(t) = \sum\limits_{n < N} b_n t^n + O(t^N)$). Insbesondere ist (1) erfüllt, falls $f(t)$ im Punkt $t = 0$ analytisch ist und die Taylor-Entwicklung $\sum\limits_{n=0}^{\infty} b_n \, t^n$ hat; i.a. wird aber nicht verlangt, daß die Reihe $\sum\limits_{n=0}^{\infty} b_n \, t^n$ konvergiert, noch, falls sie das tut, daß ihr Wert gleich $f(t)$ ist.

2. Wie aus dem Beweis ersichtlich sein wird, gilt der Satz auch für allgemeine Dirichletsche Reihen $\varphi(s) = \sum\limits_{n=0}^{\infty} \dfrac{a_n}{\lambda_n^s}$ (falls sie für mindestens ein s absolut konvergieren), wenn man $f(t) = \sum\limits_{n=0}^{\infty} a_n \, e^{-\lambda_n t}$ setzt.

Beweis des Satzes: Es ist klar, daß die Reihe für $f(t)$ für alle positiven Werte von t absolut konvergiert, da die a_n höchstens polynomial wachsen und die e^{-nt} exponentiell abfallen. Wegen Gleichung (16) von §3 (bzw. Gleichung (18) von §3 im Falle von allgemeinen Dirichletschen Reihen) gilt im Bereich der absoluten Konvergenz die Formel

$$\Gamma(s)\varphi(s) = \int_0^{\infty} f(t) t^{s-1} \, dt \ .$$

Wir zerlegen das Integral als $I_1(s) + I_2(s)$ mit

$$I_1(s) = \int_0^1 f(t)t^{s-1}\,dt \ , \quad I_2(s) = \int_1^\infty f(t)t^{s-1}\,dt \ .$$

Da $f(t)$ für $t \to \infty$ exponentiell abfällt (nämlich $f(t) = O(e^{-t})$ bzw. $f(t) = O(e^{-\lambda_0 t})$), konvergiert das zweite Integral für alle s, und zwar absolut und gleichmäßig auf kompakten Mengen, es stellt also eine ganze Funktion von s dar. Weiter gilt

$$\int_0^1 \left(\sum_{n<N} b_n t^n \right) t^{s-1}\,dt = \sum_{n<N} b_n \left. \frac{t^{n+s}}{n+s} \right|_0^1 = \sum_{n<N} \frac{b_n}{n+s} \qquad (\sigma > 1) \ ,$$

also

$$I_1(s) = \sum_{n<N} \frac{b_n}{n+s} + \int_0^1 \left(f(t) - \sum_{n<N} b_n t^n \right) t^{s-1}\,dt \qquad (\sigma > 1) \ ,$$

wobei das Integral wegen (4) für alle s mit $\mathrm{Re}(s) > -N$ absolut und auf kompakten Mengen gleichmäßig konvergiert, also in diesem Gebiet eine holomorphe Funktion darstellt. Die Funktion $\Gamma(s)\varphi(s) - \sum_{n<N} \frac{b_n}{n+s}$ hat also eine holomorphe Fortsetzung in die Halbebene $\mathrm{Re}(s) > -N$. Da N beliebig groß gewählt werden kann, folgt, daß $\Gamma(s)\varphi(s)$ eine meromorphe Fortsetzung auf ganz $\mathbb{C}$ hat, die bis auf (eventuelle) einfache Pole bei $s = -n$ ($n = -1, 0, 1, 2, \ldots$) mit Residuum b_n holomorph ist. Da die Funktion $1/\Gamma(s)$ ganz ist und für $s = 0$, $s = -1$, $s = -2$, ... verschwindet, ist $\varphi(s)$ bis auf einen einfachen Pol bei $s = 1$ mit dem Residuum b_{-1} holomorph. Durch Vergleich der Residuen von $\Gamma(s)\varphi(s)$ und $\Gamma(s)$ (vgl. Aufgabe 3, §3) erhält man die Werte (2).

Als erstes Beispiel nehmen wir $\varphi(s) = \zeta(s)$, die Riemannsche Zetafunktion. Hier ist $a_n = 1$, also

$$f(t) = \sum_{n=1}^\infty e^{-nt} = \frac{1}{e^t - 1}$$

mit der asymptotischen Entwicklung (hier sogar konvergent für $t < 2\pi$)

$$f(t) \sim \frac{1}{t} + \sum_{n=0}^\infty \frac{B_{n+1}}{(n+1)!}\, t^n \ ,$$

und der Satz gibt sofort die holomorphe Fortsetzung von $\zeta(s) - \frac{1}{s-1}$

auf die ganze Ebene sowie die in §4 erhaltenen Werte

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} = \begin{cases} -\frac{1}{2} & (n = 0) \ , \\ -\frac{B_{n+1}}{n+1} & (n \geq 1, \ n \ \text{ungerade}) \ , \\ 0 & (n \geq 2, \ n \ \text{gerade}) \ . \end{cases}$$

Wir betrachten jetzt einen Dirichletschen Charakter χ (mod N) und setzen $a_n = \chi(n)$, $\varphi(s) = L(s,\chi)$ in Satz 1. Wegen der Periodizität der Koeffizienten gilt dann

$$f(t) = \sum_{n=1}^{\infty} \chi(n) \ e^{-nt}$$

$$= \sum_{m=1}^{N} \chi(m) (e^{-mt} + e^{-(m+N)t} + e^{-(m+2N)t} + \ldots)$$

$$= \sum_{m=1}^{N} \chi(m) \ \frac{e^{-mt}}{1-e^{-Nt}} \ .$$

Die Funktion e^{-mt} hat für $t \to 0$ die asymptotische Entwicklung $\sum_{k=0}^{\infty} \frac{(-m)^k}{k!} t^k$ und die Funktion $\frac{1}{1-e^{-Nt}}$ hat die Entwicklung $\sum_{r=0}^{\infty} \frac{(-1)^r B_r}{r!} (Nt)^{r-1}$, wobei die B_r Bernoullische Zahlen sind (s. (4.7)). Es gilt also

$$f(t) \sim \sum_{m=1}^{N} \chi(m) \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^{k+r} m^k N^{r-1} B_r}{r! \ k!} \ t^{r+k-1} \ ,$$

d.h. eine asymptotische Entwicklung der Gestalt (3) mit

(5) $$b_n = \sum_{m=1}^{N} \chi(m) \sum_{\substack{k,r \geq 0 \\ k+r=n+1}} \frac{(-1)^{k+r} B_r m^k N^{r-1}}{k! \ r!} \qquad (n \geq -1) \ .$$

Für $n = -1$ reduziert sich diese Summe auf

$$b_{-1} = \frac{1}{N} \sum_{m=1}^{N} \chi(m) \ ,$$

was für $\chi \neq \chi_0$ verschwindet (§5, Satz 2); nach Satz 1 läßt sich also $L(s,\chi)$ in diesem Fall zu einer für alle s holomorphen Funktion fortsetzen. Für $\chi = \chi_0$ ist

$$b_{-1} = \frac{1}{N} \sum_{\substack{m=1 \\ (m,N)=1}}^{N} 1 = \frac{\phi(N)}{N}$$

APPENDIX. THE MELLIN TRANSFORM
AND RELATED ANALYTIC TECHNIQUES

D. ZAGIER

1. THE GENERALIZED MELLIN TRANSFORMATION

The Mellin transformation is a basic tool for analyzing the behavior of many important functions in mathematics and mathematical physics, such as the zeta functions occurring in number theory and in connection with various spectral problems. We describe it first in its simplest form and then explain how this basic definition can be extended to a much wider class of functions, important for many applications.

Let $\varphi(t)$ be a function on the positive real axis $t > 0$ which is reasonably smooth (actually, continuous or even piecewise continuous would be enough) and decays rapidly at both 0 and ∞, i.e., the function $t^A \varphi(t)$ is bounded on $\mathbb{R}_+$ for any $A \in \mathbb{R}$. Then the integral

$$\widetilde{\varphi}(s) \; = \; \int_0^\infty \varphi(t)\, t^{s-1}\, dt \tag{1}$$

converges for any complex value of s and defines a holomorphic function of s called the *Mellin transform* of $\varphi(s)$. The following small table, in which α denotes a complex number and λ a positive real number, shows how $\widetilde{\varphi}(s)$ changes when $\varphi(t)$ is modified in various simple ways:

$$\begin{array}{ccccc} \varphi(\lambda t) & t^\alpha \varphi(t) & \varphi(t^\lambda) & \varphi(t^{-1}) & \varphi'(t) \\ \lambda^{-s}\widetilde{\varphi}(s) & \widetilde{\varphi}(s+\alpha) & \lambda^{-s}\widetilde{\varphi}(\lambda^{-1}s) & \widetilde{\varphi}(-s) & (1-s)\widetilde{\varphi}(s-1) \end{array} \;. \tag{2}$$

We also mention, although we will not use it in the sequel, that the function $\varphi(t)$ can be recovered from its Mellin transform by the *inverse Mellin transformation* formula

$$\varphi(t) \; = \; \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \widetilde{\varphi}(s)\, t^{-s}\, ds \,,$$

where C is any real number. (That this is independent of C follows from Cauchy's formula.)

However, most functions which we encounter in practise are not very small at both zero and infinity. If we assume that $\varphi(t)$ is of rapid decay at infinity but grows like t^{-A} for some real number A as $t \to 0$, then the integral (1) converges and defines a holomorphic function only in the right half-plane $\Re(s) > A$. Similarly, if $\varphi(t)$ is of rapid decay at zero but grows like t^{-B} at infinity for some real number B, then $\widetilde{\varphi}(s)$ makes sense and is holomorphic only in the left half-plane $\Re(s) < B$, while if $\varphi(t)$ has polynomial growth at both ends, say like t^{-A} at 0 and like t^{-B} at ∞ with $A < B$, then $\widetilde{\varphi}(s)$ is holomorphic only in the strip $A < \Re(s) < B$. But it turns out that in many cases the function $\widetilde{\varphi}(t)$ has a meromorphic extension to a larger half-plane or strip than the one in which the original integral (1) converges, or even to the whole complex plane. Moreover, this extended Mellin transform can sometimes be defined even in cases where $A > B$, in which case the integral (1) does not converge for any value of s at all.

1

Let us start with the frequently occurring case where $\varphi(t)$ is of rapid decay at infinity and is C^∞ at zero, i.e., it has an asymptotic expansion $\varphi(t) \sim \sum_{n=0}^{\infty} a_n t^n$ as $t \to 0$. (Recall that this means that the difference $\varphi(t) - \sum_{n=0}^{N-1} a_n t^n$ is $O(t^N)$ as $t \to 0$ for any integer $N \geq 0$; it is not required that the series $\sum a_n t^n$ be convergent for any positive t.) Then for s with $\Re(s) > 0$ and any positive integer N the integral (1) converges and can be decomposed as follows:

$$\widetilde{\varphi}(s) \;=\; \int_0^1 \varphi(t)\, t^{s-1}\, dt \;+\; \int_1^\infty \varphi(t)\, t^{s-1}\, dt$$

$$=\; \int_0^1 \left(\varphi(t) - \sum_{n=0}^{N-1} a_n t^n \right) t^{s-1}\, dt \;+\; \sum_{n=0}^{N-1} \frac{a_n}{n+s} \;+\; \int_1^\infty \varphi(t)\, t^{s-1}\, dt \,.$$

The first integral on the right converges in the larger half-plane $\Re(s) > -N$ and the second for all $s \in \mathbb{C}$, so we deduce that $\widetilde{\varphi}(s)$ has a meromorphic continuation to $\Re(s) > -N$ with simple poles of residue a_n at $s = -n$ ($n = 0, \ldots, N-1$) and no other singularities. Since this holds for every n, it follows that the Mellin transform $\widetilde{\varphi}(s)$ in fact has a meromorphic continuation to all of $\mathbb{C}$ with simple poles of residue a_n at $s = -n$ ($n = 0, 1, 2, \ldots$) and no other poles. The same argument shows that, more generally, if $\varphi(t)$ is of rapid decay at infinity and has an asymptotic expansion

$$\varphi(t) \;\sim\; \sum_{j=1}^{\infty} a_j\, t^{\alpha_j} \qquad (t \to 0) \tag{3}$$

as t tends to zero, where the α_j are real numbers tending to $+\infty$ as $j \to \infty$ or complex numbers with real parts tending to infinity, then the function $\widetilde{\varphi}(s)$ defined by the integral (1) for $\Re(s) > -\min_j \Re(\alpha_j)$ has a meromorphic extension to all of $\mathbb{C}$ with simple poles of residue a_j at $s = -\alpha_j$ ($j = 1, 2, \ldots$) and no other poles. Yet more generally, we can allow terms of the form $t^\alpha (\log t)^m$ with $\lambda \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$ in the asymptotic expansion of $\varphi(t)$ at $t = 0$ and each such term contributes a pole with principal part $(-1)^m m!/(s+\alpha)^{m+1}$ at $s = -\alpha$, because $\int_0^1 t^{\alpha+s-1}(\log t)^m\, dt = (\partial/\partial\alpha)^m \int_0^1 t^{\alpha+s-1}\, dt = (-1)^m m!/(\alpha+s)^{m+1}$ for $\Re(s+\alpha) > 0$.

By exactly the same considerations, or by replacing $\varphi(t)$ by $\varphi(t^{-1})$, we find that if $\varphi(t)$ is of rapid decay (faster than any power of t) as $t \to 0$ but has an asymptotic expansion of the form

$$\varphi(t) \;\sim\; \sum_{k=1}^{\infty} b_k\, t^{\beta_k} \qquad (t \to \infty) \tag{4}$$

at infinity, where now the exponents β_k are complex numbers whose real parts tend to $-\infty$, then the function $\widetilde{\varphi}(s)$, originally defined by (1) in a left half-plane $\Re(s) < -\max_k \Re(\beta_k)$, extends meromorphically to the whole complex s-plane with simple poles of residue $-b_k$ at $s = -\beta_k$ and no other poles. (More generally, again as before, we can allow terms $b_k t^{\beta_k}(\log t)^{n_k}$ in (3) which then produce poles with principal parts $(-1)^{n_k+1} n_k!\, b_k/(s+\beta_k)^{n_k+1}$ at $s = -\beta_k$.)

Now we can use these ideas to define $\widetilde{\varphi}(s)$ for functions which are not small either at 0 or at ∞, even when the integral (1) does not converge for any value of s. We simply assume that $\varphi(t)$ is a smooth (or continuous) function on $(0, \infty)$ which has asymptotic expansions of the forms (3) and (4) at zero and infinity, respectively. (Again, we could allow more general terms with powers of $\log t$ in the expansions, as already explained, but the corresponding modifications are easy and for simplicity of expression we will assume expansions purely in powers of t.) For convenience we assume that the numbering s such that $\Re(\alpha_1) \leq \Re(\alpha_2) \leq \cdots$ and $\Re(\beta_1) \geq \Re(\beta_2) \geq \cdots$. Then, for

2

any $T > 0$ (formerly we took $T = 1$, but the extra freedom of being able to choose any value of T will be very useful later) we define two "half-Mellin transforms" $\widetilde{\varphi}_{\leq T}(s)$ and $\widetilde{\varphi}_{\geq T}(s)$ by

$$\widetilde{\varphi}_{\leq T}(s) \;=\; \int_0^T \varphi(t)\, t^{s-1}\, dt \qquad \big(\Re(s) \,>\, -\Re(\alpha_1)\big)\,,$$

$$\widetilde{\varphi}_{\geq T}(s) \;=\; \int_T^\infty \varphi(t)\, t^{s-1}\, dt \qquad \big(\Re(s) \,<\, -\Re(\beta_1)\big)\,.$$

Just as before, we see that for each integer $J \geq 1$ the function $\widetilde{\varphi}_{\leq T}(s)$ extends by the formula

$$\widetilde{\varphi}_{\leq T}(s) \;=\; \int_0^T \left(\varphi(t) - \sum_{j=1}^J a_j t^{\alpha_j} \right) t^{s-1}\, dt \;+\; \sum_{j=1}^J \frac{a_j}{s + \alpha_j}\, T^{s+\alpha_j}$$

to the half-plane $\Re(s) > -\Re(\alpha_{J+1})$ and hence, letting $J \to \infty$, that $\widetilde{\varphi}_{\leq T}(s)$ is a meromorphic function of s with simple poles of residue a_j at $s = -\alpha_j$ $(j = 1, 2, \dots)$ and no other poles. Similarly, $\widetilde{\varphi}_{\geq T}(s)$ extends to a meromorphic function whose only poles are simple ones of residue $-b_k$ at $s = -\beta_k$. We now define

$$\widetilde{\varphi}(s) \;=\; \widetilde{\varphi}_{\leq T}(s) \;+\; \widetilde{\varphi}_{\geq T}(s)\,. \tag{$4\tfrac{1}{2}$}$$

This is a meromorphic function of s and is independent of the choice of T, since the effect of changing T to T' is simply to add the everywhere holomorphic function $\int_T^{T'} \varphi(t)\, t^{s-1}\, dt$ to $\widetilde{\varphi}_{\leq T}(s)$ and subtract the same function from $\widetilde{\varphi}_{\geq T}(s)$, not affecting the sum of their analytic continuations.

In summary, if $\varphi(t)$ is a function of t with asymptotic expansions as a sum of powers of t (or of powers of t multiplied by integral powers of $\log t$) at both zero and infinity, then we can define in a canonical way a Mellin transform $\widetilde{\varphi}(s)$ which is meromorphic in the entire s-plane and whose poles reflect directly the coefficients in the asymptotic expansions of $\varphi(t)$. This definition is consistent with and has the same properties (2) as the original definition (1). We end this section by giving two simple examples, while Sections 2 and 3 will give further applications of the method.

Example 1. Let $\varphi(t) = t^\alpha$, where α is a complex number. Then φ has an asymptotic expansion (3) at 0 with a single term $\alpha_1 = \alpha$, $a_1 = 1$, and an asymptotic expansion (4) at ∞ with a single term $\beta_1 = \alpha$, $b_1 = 1$. We immediately find that $\widetilde{\varphi}_{\leq T}(s) = T^{s+\alpha}/(s + \alpha)$ for $\Re(s + \alpha) > 0$ and $\widetilde{\varphi}_{\geq T}(s) = -T^{s+\alpha}/(s + \alpha)$ for $\Re(s + \alpha) < 0$, so that, although the original Mellin transform integral (1) does not converge for any value of s, the function $\widetilde{\varphi}(s)$ defined as the sum of the meromorphic continuations of $\widetilde{\varphi}_{\leq T}(s)$ and $\widetilde{\varphi}_{\geq T}(s)$ makes sense, is independent of T, and in fact is identically zero. More generally, we find that $\widetilde{\varphi}(s) \equiv 0$ whenever $\varphi(t)$ is a finite linear combination of functions of the form $t^\alpha \log^m t$ with $\alpha \in \mathbb{C}$, $m \in \mathbb{Z}_{\geq 0}$. (These are exactly the functions whose images $\varphi_\lambda(t) = \varphi(\lambda t)$ under the action of the multiplicative group $\mathbb{R}_+$ span a finite-dimensional space.) In particular, we see that the generalized Mellin transformation is no longer injective.

Example 2. Let $\varphi(t) = e^{-t}$. Here the integral (1) converges for $\Re(s) > 0$ and defines Euler's gamma-function $\Gamma(s)$. From the fact that $\varphi(t)$ is of rapid decay at infinity and has the asymptotic (here even convergent) expansion $\sum_{n=0}^\infty (-t)^n/n!$ at zero, we deduce that $\Gamma(s) = \widetilde{\varphi}(s)$ has a meromorphic continuation to all s with a simple pole of residue $(-1)^n/n!$ at $s = -n$ $(n = 0, 1, \dots)$ and no other poles. Of course, in this special case these well-known properties can also be deduced from the functional equation $\Gamma(s + 1) = s\Gamma(s)$ (proved for $\Re(s) > 0$ by integration by parts in the integral defining $\Gamma(s)$), N applications of which gives the meromorphic extension $\Gamma(s) = s^{-1}(s + 1)^{-1} \cdots (s + N - 1)^{-1} \Gamma(s + N)$ of $\Gamma(s)$ to the half-plane $\Re(s) > -N$.

From the first of the properties listed in (2), we find the following formula, which we will use many times:

$$\varphi(t) \;=\; e^{-\lambda t} \quad \Rightarrow \quad \widetilde{\varphi}(s) \;=\; \Gamma(s)\, \lambda^{-s} \qquad (\lambda > 0)\,. \tag{5}$$

3

2. Dirichlet series and their special values

In this section we look at functions $\varphi(t)$ for which the Mellin transform defined in Section 1 is related to a Dirichlet series. The key formula is (5), because it allows us to convert Dirichlet series into exponential series, which are much simpler.

Example 3. Define $\varphi(t)$ for $t > 0$ by $\varphi(t) = 1/(e^t - 1)$. This function is of rapid decay at infinity and has an asymptotic expansion (actually convergent for $t < 2\pi$)

$$\frac{1}{e^t - 1} = \frac{1}{t + \dfrac{t^2}{2} + \dfrac{t^3}{6} + \cdots} = \sum_{r=0}^{\infty} \frac{B_r}{r!} t^{r-1} \tag{6}$$

with certain rational coefficients $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, ... called *Bernoulli numbers*. From the results of Section 1 we know that the Mellin transform $\widetilde{\varphi}(s)$, originally defined for $\Re(s) > 1$ by the integral (1), has a meromorphic continuation to all s with simple poles of residue $B_r/r!$ at $s = 1 - r$ $(r = 0, 1, 2, \dots)$. On the other hand, since $e^t > 1$ for $t > 0$, we can expand $\varphi(t)$ as a geometric series $e^{-t} + e^{-2t} + e^{-3t} + \cdots$, so (5) gives (first in the region of convergence) $\widetilde{\varphi}(s) = \Gamma(s)\zeta(s)$, where

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \qquad (\Re(s) > 1) \tag{7}$$

is the Riemann zeta function. Since $\Gamma(s)$, as we have seen is also meromorphic, with simple poles of residue $(-1)^n/n!$ at non-positive integral arguments $s = -n$ and no other poles, and since $\Gamma(s)$ (as is well-known and easily proved) never vanishes, we deduce that $\zeta(s)$ has a meromorphic continuation to all s with a unique simple pole of residue $1/\Gamma(1) = 1$ at $s = 1$ and that its values at non-positive integral arguments are rational numbers expressible in terms of the Bernoulli numbers:

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \qquad (n = 0, 1, 2, \dots). \tag{8}$$

Example 4. To approach $\zeta(s)$ in another way, we choose for $\varphi(t)$ the *theta function*

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \qquad (t > 0). \tag{9}$$

(The factor π in the exponent has been included for later convenience.) We can write this out as

$$\vartheta(t) = 1 + 2e^{-\pi t} + 2e^{-4\pi t} + \cdots, \tag{10}$$

and since the generalized Mellin transform of the function 1 is identically $\tilde{0}$ by Example 1, we deduce from (5) that $\widetilde{\varphi}(s) = 2\zeta^*(2s)$, where

$$\zeta^*(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s). \tag{11}$$

To obtain the analytic properties of $\zeta(s)$ from the results of Section 1, we need the asymptotics of $\vartheta(t)$ at zero and infinity. They follow immediately from the following famous result, due to Jacobi:

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This time we find the expansions

$$f_3(\varepsilon) \;=\; \frac{2}{3}\,\varepsilon^{-1} + C'\,\varepsilon^{1/2} + \mathrm{O}(e^{-\pi/\varepsilon})\,, \qquad f_4(\varepsilon) \;=\; \frac{\pi^2}{45}\,\varepsilon^{-2} - \frac{2}{3}\varepsilon^{-1/2} + \mathrm{O}(\sqrt{\varepsilon}\,e^{-\pi/\varepsilon})$$

for ε small, the non-exponentially small terms which depend on T again cancel, and taking $T = 1/\lambda$ to make the error as small as possible we obtain (34). The details are left to the reader.

As a final comment, we mention that the constants C and C' occurring in (33) and (34) can be evaluated rapidly by the same method as in **B**, with $\vartheta(t)^2$ instead of $\vartheta(t)^2\vartheta(\lambda t)$, using the fact that the Dirichlet series $4\zeta(s)L_4(s)$ is the Mellin transform of $\vartheta(t)^2$. Their numeical values are

$$C \;=\; 0.91596559417721901505460351493238411077414937428167 2134\cdots\,,$$
$$C' \;=\; 1.43774554488764350693200343638999965018484034037993 3997\cdots\,.$$

4. Asymptotics of sums of the form $\sum f(nt)$

In this section we describe an extremely useful, and not sufficiently well known, asymptotic formula for functions given by expansions of the form

$$g(t) \;=\; f(t) + f(2t) + f(3t) + \cdots\,, \tag{37}$$

where $f : (0, \infty) \to \mathbb{C}$ is a smooth function of sufficiently rapid decay to ensure the convergence of the series (say $f(t) = \mathrm{O}(t^{-1-\varepsilon})$ as $t \to \infty$) and having a known asymptotic expansion at $t = 0$. In the simplest situation, we assume that f has a power series expansion (which may be only asymptotic rather than convergent, i.e. f need only be differentiable rather than analytic at the origin)

$$f(t) \;\sim\; \sum_{n=0}^{\infty} b_n\, t^n \qquad (t \to 0)\,. \tag{38}$$

First, let us try to guess what the answer should be. On the one hand we can argue à la Euler, simply substituting the expansion (38) into (37) and interchanging the order of summation, without worrying about convergence problems. This gives

$$g(t) \;\sim\; \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} b_n\,(mt)^n \;=\; \sum_{n=0}^{\infty} b_n \left(\sum_{m=1}^{\infty} m^n\right) t^n \;=\; \sum_{n=0}^{\infty} b_n\, \zeta(-n)\, t^n\,. \tag{39}$$

Of course this calculation is meaningless, since not only is the interchange of summation not permitted, but each of the interior sums $\sum_m m^n$ is divergent. Nevertheless, we know from Section 2, and Euler knew non-rigorously in 1749, that the numbers $\zeta(-n)$ do make sense and are certain rational numbers given by equation (8), so that at least the final expression in (39) makes sense as a formal power series. Alternatively, we can proceed à la Riemann and consider (37) for t small as $1/t$ times an approximation to the integral

$$I_f \;=\; \int_0^{\infty} f(t)\, dt\,. \tag{40}$$

This suggests instead that the correct asymptotic expansion of $g(t)$ near 0 should be given by

$$g(t) \;\sim\; \frac{I_f}{t} \qquad (t \to 0)\,, \tag{41}$$

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and indeed this formula is true, by the very definition of Riemann integrals as limits of sums, if we interpret the symbol of asymptotic equality "$\sim$" in its weak sense, as saying simply that the ratio of the expressions on its left and right tends to 1 as $t \to 0$. However, we are using "$\sim$" in its strong sense, where $g(t) \sim \sum_\lambda a_\lambda t^\lambda$ means that the difference between $g(t)$ and the finite sum $\sum_{\lambda < C} a_\lambda t^\lambda$ is $O(t^C)$ as $t \to 0$ for any value of C, no matter how large. In this stronger sense, neither (39) nor (41) gives the correct asymptotic development of g. Remarkably enough, however, their sum does give the right answer:

Proposition 3. *Let f be a C^∞ function on the positive real line which has the asymptotic development (38) at the origin and, rogether with all its derivatives, is of rapid decay at infinity. Then the function $g(t)$ defined by (37) has the asymptotic development*

$$g(t) \ \sim \ \frac{I_f}{t} \ + \ \sum_{n=0}^{\infty} b_n \frac{B_{n+1}}{n+1} (-t)^n \tag{42}$$

as $t \to 0$, where I_f is defined by equation (40).

Remark. The relationship of Proposition 3 to the discussion of the Mellin transform in §§1–3 is very easy to describe. Suppose for simplicity that $f(t)$ is of rapid decay as $t \to \infty$. Then (38) implies that the Mellin transform $\tilde{f}(s)$, defined for $\Re(s) > 0$ as $\int_0^\infty f(t)t^{s-1}dt$, has a meromorphic continuation to all s with simple poles of reside b_n at $s = -n$ ($n = 0, 1, 2, \dots$) and with no other singularities. But the transformation rules (2) of the Mellin transform imply that the Mellin transform $\tilde{g}(s)$ of $g(t)$ equals $\zeta(s)\tilde{f}(s)$, so it has (at most) simple poles of residue $\tilde{f}(1) = I_f$ at $s = 1$ and residue $\zeta(-n)b_n$ at $s = -n$ ($n = 0, 1, 2, \dots$). Since $g(t)$ is small at infinity, it follows that, if $g(t)$ has an asymptotic expansion as $t \to 0$ of the form $\sum_j a_j t^{\alpha_j}$ (or even $\sum_j a_j t^{\alpha_j} (\log t)^{m_j}$), then this expansion is necessarily given by eq. (42).

Proof. We begin by describing the Euler-Maclaurin summation formula. To state it, we need the Bernoulli polynomials $B_n(x)$, which can be described by the generating function (22), by the explicit formula (23) in terms of Bernoulli numbers, or, most beautiful, by the property that $\int_a^{a+1} B_n(x)dx = a^n$ for every a. (It is easy to see that there is only one polynomial $B_n(x)$ with this property for each n.) From any of these definitions we can deduce without difficulty that $B_n'(x) = nB_{n-1}(x)$ and that $B_n(x+1) - B_n(x) = nx^{n-1}$.

Now let f be a smooth function on the positive reals. Integration by parts and the fact that $B_1(0) = \frac{1}{2} = -B_1(0)$ but $B_{n+1}(1) = B_{n+1}(0) = B_{n+1}$ for $n \geq 2$ gives

$$\int_0^1 f^{(n)}(x) \frac{B_n(x)}{n!} \, dx \ = \ - \int_0^1 f^{(n+1)}(x) \frac{B_{n+1}(x)}{(n+1)!} \, dx + \begin{cases} \frac{1}{2}[f(0) + f(1)] & \text{if } n = 0, \\ \frac{B_{n+1}}{(n+1)!}[f^{(n)}(1) - f^{(n)}(0)] & \text{if } n \geq 1 \end{cases}$$

and hence, by induction on N,

$$\int_0^1 f(x)\, dx \ = \ \frac{1}{2}[f(0) + f(1)] + \sum_{n=1}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!}[f^{(n)}(1) - f^{(n)}(0)] + (-1)^N \int_0^1 f^{(N)}(x) \frac{B_N(x)}{N!} \, dx$$

for every integer $N \geq 1$. Replacing $f(x)$ by $f(x+m-1)$ and summing over $m = 1, 2, \dots, M$ gives

$$\int_0^M f(x)\, dx \ = \ \frac{f(0)}{2} + \sum_{m=1}^{M-1} f(m) + \frac{f(M)}{2} + \sum_{n=1}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!}[f^{(n)}(M) - f^{(n)}(0)]$$

$$+ (-1)^N \int_0^M f^{(N)}(x) \frac{\bar{B}_N(x)}{N!} \, dx \tag{43}$$

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where $\bar{B}_N(x) = B_N(x - [x])$. This is the Euler-Maclaurin summation formula.

Now assume that f and each of its derivatives is of rapid decay at infinity, so that $\int_0^\infty |f^{(N)}(x)|\,dx$ converges. Since $\bar{B}_N(x)$ is periodic and hence bounded, we can let $M \to \infty$ in (43) to get

$$\sum_{m=1}^\infty f(m) = \int_0^\infty f(x)\,dx + \sum_{n=0}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} f^{(n)}(0) - (-1)^N \int_0^\infty f^{(N)}(x) \frac{\bar{B}_N(x)}{N!}\,dx\,.$$

Replacing $f(x)$ by $f(tx)$ and then x by x/t with $t > 0$ changes this formula to

$$\sum_{m=1}^\infty f(mt) = \frac{1}{t} \int_0^\infty f(x)\,dx + \sum_{n=0}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} f^{(n)}(0)\,t^n + (-t)^{N-1} \int_0^\infty f^{(N)}(x) \frac{\bar{B}_N(x/t)}{N!}\,dx\,.$$

The last integral is bounded as $t \to 0$ with N fixed for the same reason as before, so the final term is $O_N(t^{N-1})$. Substituting $f^{(n)}(0) = n!\,b_n$ from (38), we get the desired asymptotic formula (42). □

Before giving examples of Proposition 3, we mention three extensions to more general sums.

1. First, instead of (37) we can look at shifted sums of the form $g(t) = \sum_{m=0}^\infty f((m+a)t)$, where $a > 0$. Here the "Riemann Ansatz" and the "Euler Ansatz" predict $I_f t^{-1}$ and $\sum_{n=0}^\infty b_n \zeta(-n, a)t^n$ for the asymptotic expansion of g, and again the correct answer is the sum of these two:

$$\sum_{m=0}^\infty f((m+a)t) \sim \frac{I_f}{t} + \sum_{n=0}^\infty b_n \frac{B_{n+1}(a)}{n+1}\,t^n \tag{44}$$

(cf. equation (23)). The proof is similar to that of Proposition 3 and will be omitted. By taking rational values of a in (44) and forming suitable linear combinations, we can use this formula to give the asymptotic development of $\sum_{m \geq 1} \chi(m)f(mt)$ as $t \to 0$ for any periodic function $\chi(m)$, such as a Dirichlet character.

2. Next, we can allow non-integral exponents of t in (38). If the expansion of $f(t)$ at the origin contains terms $b_\lambda t^\lambda$ with arbitrary real numbers $\lambda > -1$ (or complex numbers with real part greater than -1), then the formula for g need only be modified by adding the corresponding terms $b_\lambda \zeta(-\lambda)t^\lambda$. Terms with $\lambda < -1$ are not interesting since they can simply be subtracted from $f(t)$, since the sum $\sum_m b_m(mt)\lambda$ then converges absolutely. The limiting case $\lambda = -1$ is of interest because it occurs in various applications. Here the answer (proved most easily by taking one function, like $t^{-1}e^{-t}$, which has a $1/t$ singularity at the origin and for which $\sum f(mt)$ can be computed exactly) is

$$f(t) \sim \sum_{\lambda \geq -1} b_\lambda t^\lambda \quad \Rightarrow \quad \sum_{m=1}^\infty f(mt) \sim \frac{1}{t}\left(b_{-1} \log \frac{1}{t} + I_f^*\right) + \sum_{\lambda > -1}^\infty b_\lambda\,\zeta(-\lambda)\,(-t)^\lambda, \tag{45}$$

where $I_f^* = \int_0^\infty (f(t) - b_{-1}e^{-t}/t)dt$.

3. Finally, we can also allow terms of the form $t^\lambda(\log t)^n$ in the expansion of $f(t)$, the corresponding contribution to $g(t)$ being simply the nth derivative with respect to λ of $\zeta(-\lambda)t^\lambda$, e.g. a term $t^\lambda \log \dfrac{1}{t}$ in the expansion of $f(t)$ at 0 leads to a term $t^\lambda\left(\zeta(-\lambda) \log \dfrac{1}{t} + \zeta'(-\lambda)\right)$ in the expansion at 0 of $g(t)$. In particular, using the known value $\zeta'(0) = -\frac{1}{2}\log(2\pi)$ we find

$$f(t) \sim b \log \frac{1}{t} + \sum_{n=0}^\infty b_n t^n \quad \Rightarrow \quad \sum_{m=1}^\infty f(mt) \sim \frac{I_f}{t} - \frac{b}{2} \log \frac{2\pi}{t} + \sum_{n=0}^\infty b_n\,\zeta(-n)\,t^n\,. \tag{46}$$

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We end by giving four examples—two easy and two harder—to illustrate how these asymptotic formulas work.

Example 1. Take $f(t) = e^{-\lambda t}$ with $\lambda > 0$. This function is smooth, small at infinity, and has an expansion (38) at $t = 0$ with $b_n = (-\lambda)^n/n!$. The integral I_f equals $1/\lambda$. Hence (42) gives $g(t) \sim \dfrac{1}{\lambda t} + \sum\limits_{n=0}^{\infty} \dfrac{B_{n+1}}{(n+1)!} (\lambda t)^n$ as the asymptotic expansion of $g(t) = \sum\limits_{m=1}^{\infty} f(mt) = \dfrac{1}{e^{\lambda t} - 1}$, in accordance with the definition (6) of the Bernoulli numbers.

Example 2. Now take $f(t) = e^{-\lambda t^2}$ with $\lambda > 0$. This function is again smooth and small at infinity, and has an expansion (38) at $t = 0$ with $b_{2n} = (-\lambda)^n/n!$ and $b_n = 0$ for n odd. Since all Bernoulli numbers with odd indices > 1 vanish, the asymptotic expansion in (42) breaks off after two terms, and we find $g(t) = I_f/t - 1/2 + \mathrm{O}(t^N)$ for all N, with $I_f = \sqrt{\pi/4\lambda}$. In this case, of course, we know much more, because $g(t)$ is simply $\frac{1}{2}\big(\vartheta(\lambda t^2/\pi) - 1\big)$ with $\vartheta(t)$ as in eq. (9), and therefore equation (12) gives the much more precise statement $g(t) = I_f/t - 1/2 + \mathrm{O}\big(t^{-1} e^{-\pi^2/\lambda t^2}\big)$. The same applies to any function $f(t)$ whose expansion at $t = 0$ has only even powers of t. For such a function, the expansion (42) collapses to $g(t) \sim I_f/t - b_0/2$, but this is always just a weakening of the Poisson summation formula (13), because we can extend $f(t)$ by $f(t) = f(|t|)$ to a smooth even function on the real line and use (14) to get the exact formula

$$2g(t) + f(0) = \sum_{n \in \mathbb{Z}} f(nt) = \frac{1}{t} \sum_{n \in \zeta} \widehat{f}(nt) = \frac{2I_f}{t} + \frac{2}{t} \sum_{n=1}^{\infty} \widehat{f}(nt),$$

and the smoothness of f implies that the function $\widehat{f}(y)$ decays at infinity more rapidly than any negative power of y. The right way to think of Proposition 3 is therefore as a replacement for the Poisson summation formula when one is confronted with a sum over only positive integers rather than a sum over all of $\mathbb{Z}$. Such sums are very much harder to study than sums over all integers—just think of the special values of the Riemann zeta function, where the numbers $\zeta(2k)$ can be obtained in closed form because they can be written as $\frac{1}{2}\sum'_{n \in \mathbb{Z}} n^{-k}$, while the numbers $\zeta(2k + 1)$, which cannot be reduced to sums over all of $\mathbb{Z}$ in this way, are not known exactly.

Example 3. As our next example, we consider the function

$$g_k(q) = \sum_{n=1}^{\infty} \sigma_{k-1}(n) \, q^n \qquad (0 < q < 1),$$

where k is an integer greater than 1 and $\sigma_{k-1}(n)$ denotes the sum of the $(k-1)$st powers of the divisors of a natural number n. In the theory of modular forms, it is shown that if k is even and larger than 2, then g_k satisfies the functional equation

$$-\frac{B_k}{2k} + g_k\big(e^{-2\pi t}\big) = \frac{(-1)^{k/2}}{t^k}\left(-\frac{B_k}{2k} + g_k\big(e^{-2\pi/t}\big)\right) \qquad (k = 4,\, 6,\, 8, \dots;\ t > 0).$$

In particular, $g_k(e^{-2\pi t})$ has the terminating asymptotic expansion

$$g_k\big(e^{-2\pi t}\big) = -(-1)^{k/2}\frac{B_k}{2k}\, t^{-k} + \frac{B_k}{2k} + \mathrm{O}(t^N) \qquad (\forall N > 0) \tag{47}$$

as $t \to 0$ in these cases. Let us see how Proposition 3 permits us to recover this asymptotic formula using without knowing the modularity, and at the same time tells us why (47) fails for $k = 2$ or k odd and what replaces it in those cases.

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We first note that

$$g_k(q) \;=\; \sum_{n=1}^{\infty}\Big(\sum_{m|n} m^{k-1}\Big) q^n \;=\; \sum_{m=1}^{\infty} m^{k-1}\big(q^m + q^{2m} + \cdots\big) \;=\; \sum_{m=1}^{\infty} m^{k-1}\,\frac{q^m}{1-q^m}$$

and hence that g_k can be written after a change of variables in the form

$$g_k\big(e^{-t}\big) \;=\; \frac{1}{t^{k-1}}\sum_{m=1}^{\infty} f_k(mt)\,, \qquad f_k(t) \;=\; \frac{t^{k-1}}{e^t - 1}\,.$$

The function $f = f_k$ satisfies the hypotheses of Proposition 3, with integral

$$I_f \;=\; \int_0^{\infty} \frac{t^{k-1}}{e^t - 1}\, dt \;=\; \int_0^{\infty} t^{k-1}\big(e^{-t} + e^{-2t} + \cdots\big)\, dt \;=\; (k-1)!\,\zeta(k)$$

and with Taylor expansion $f(t) = \sum\limits_{r=0}^{\infty} \dfrac{B_r}{r!} t^{r+k-2}$ at zero. Proposition 3 therefore gives

$$g_k\big(e^{-t}\big) \;\sim\; \frac{(k-1)!\,\zeta(k)}{t^k} \;+\; \sum_{r=0}^{\infty}(-1)^{r+k}\frac{B_r}{r!}\,\frac{B_{r+k-1}}{r+k-1}\,t^{r-1} \qquad (t \to 0). \tag{48}$$

If k is even and ≥ 4, then all products $B_r B_{r+k-1}$ with $r \neq 1$ vanish, since r and $r+k-1$ have opposite parity and all odd-index Bernoulli numbers except B_1 are zero. Thereofre only two terms of (48) survive, and replacing t by $2\pi t$ and using the well-known formula for $\zeta(k)$ in terms of B_k, we recover (47). If $k = 2$, then the argument is the same except that now the $r = 0$ term also gives a non-zero contribution, so that we find

$$g_2\big(e^{-2\pi t}\big) \;=\; \frac{1}{24 t^2} \;-\; \frac{1}{4\pi t} \;+\; \frac{1}{24} \;+\; \mathrm{O}(t^N) \qquad (\forall N > 0),$$

instead of (47), in accordance with the known near-modularity property of g_2. Finally, if k is odd then we still get an explicit asymptotic formula with rational coefficients, but it now no longer terminates. Thus for $k = 3$ we find the expansion

$$g_3\big(e^{-t}\big) \;\sim\; \frac{2\,\zeta(3)}{t^3} \;-\; \frac{1}{12 t} \;+\; \frac{t}{1440} \;+\; \frac{t^3}{181440} \;+\; \frac{t^5}{7257600} \;+\; \frac{t^7}{159667200} \;+\; \cdots$$

as $t \to 0$, even though g_k has no modularity properties in this case. We leave it as an exercise to the reader to calculate the corresponding expansion when $k = 1$, where one has to use equation (45) instead of equation (42).

Example 4. As our final example, consider the function

$$P(q) \;=\; \prod_{m=1}^{\infty} \frac{1}{1-q^m} \qquad (|q| < 1)\,,$$

the generating power series of the partition function. To study the behavior of the partition function, we need to know how $P(q)$ blows up as q approaches 1 from below (or, more generally, any root of unity from within the unit circle). Here again the known modularity properties of the function $P(q)$ imply the non-trivial functional equation

$$e^{\pi t/12}\, P\big(e^{-2\pi t}\big) \;=\; \sqrt{t}\, e^{\pi/12 t}\, P\big(e^{-2\pi/t}\big) \qquad (t > 0),$$

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from which one immediately obtains the asymptotic expansion

$$\log P\big(e^{-2\pi t}\big) \;=\; \frac{\pi}{12t} \;+\; \frac{1}{2}\log t \;-\; \frac{\pi}{12}t \;+\; \mathrm{O}(t^N) \qquad (\forall N > 0)\,. \tag{49}$$

To obtain this formula without knowing anything about the modularity of P, we observe that $\log P\big(e^{-t}\big)$ has the form $\sum_{m=1}^{\infty} f(mt)$ with $f(t) = -\log(1 - e^{-t})$. This function is small at infinity, has integral $I_f = \zeta(2)$ (as one sees by integrating by parts once and then calculating as in Example 2), and has an asymptotic development

$$f(t) \;\sim\; \log\frac{1}{t} \;-\; \sum_{n=1}^{\infty} \frac{B_n}{n \cdot n!}\,t^n\,,$$

as one sees easily by differentiating once. Hence equation (47) applies and gives

$$\log P\big(e^{-t}\big) \;\sim\; \frac{\zeta(2)}{t} \;+\; \frac{1}{2}\log\frac{t}{2\pi} \;-\; \sum_{n=1}^{\infty}(-1)^n \frac{B_n}{n \cdot n!}\frac{B_{n+1}}{n+1}\,t^n\,.$$

Again all terms except for the first in the sum on the right vanish because n and $n+1$ have opposite parity, and replacing t by $2\pi t$ and using $\zeta(2) = \pi^2/6$ we recover (49). The same method using (44) with rational values of a lets us compute the exact asymptotics of $P(q)$ as q approaches any root of unity, recovering precisely the same result as that given by the modularity. Moreover, just as in the case of Example 2 for k odd, the method applies even when modularity fails. For example, if we define

$$P_2(q) \;=\; \prod_{m=1}^{\infty} \frac{1}{(1 - q^m)^m} \qquad (|q| < 1)\,,$$

a generating function that occurs in connection with the theory of plane partitions, then an analysis like the one just given for $P(q)$, but now with $f(t) = -t\log(1 - e^{-t})$, produces the complete asymptotic expansion

$$P_2(e^{-t}) \;=\; c\,t^{1/12}\,e^{\zeta(3)/t^2}\left(1 \;-\; \frac{t^2}{2880} \;-\; \frac{17t^4}{12902400} \;-\; \cdots\right) \qquad (t \to 0)$$

with $c = e^{\zeta'(-1)} = 0.847536694\cdots$, and using (46) one can get the expansion of $P_2(\alpha e^{-t})$ for any root of unity α. Furthermore, for reasons similar to those which applied to g_k with k even, one finds that the corresponding expansions for the logarithm of $P_3(q) = \prod(1 - q^n)^{-n^2}$ when q tends to a root of unity are terminating, even though there are no modularity properties in this case.

Warning. Equations numbering have to be checked, e.g., two eqs. (5), one now changed to $(4\frac{1}{2})$.