transformation theory of the functions, on the lines of the transformation theory of the functions of the third order, and, in view of the complexity of all the series which are involved, I am becoming somewhat skeptical concerning the existence of an exact transformation theory for functions of the fifth order."

The object of this paper is to provide the counterparts of (1.2) for the fifth and seventh order mock theta functions. This is, I believe, the necessary first step in finding the transformation theory whose existence is doubted by Watson. As an example, let us consider

$$f_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)(1+q^2)\cdots(1+q^n)}, \tag{1.3}$$

one of the fifth order mock theta functions [**20**, p. 277]. We shall show in §5 that

$$f_0(q)\prod_{n=1}^{\infty}(1-q^n) = \sum_{j=-\infty}^{\infty}\sum_{n\ge|j|}(-1)^j q^{n(5n+1)/2-j^2}(1-q^{4n+2}). \tag{1.4}$$

Note the resemblance of the expression on the right-hand side of (1.4) to certain identities for modular forms due to Hecke [**12**] and Rogers [**15**] (see §4). Presumably this resemblance can be exploited to obtain the transformation theory alluded to by Watson.

The next three sections describe the necessary background for our work. §5 is a slight digression since we are able to prove certain Hecke type identities directly from our work as well as a formula related to sums of three squares. Also we obtain the new identity

$$\begin{aligned}\left(\sum_{n=0}^{\infty} q^{\binom{n+1}{2}}\right)^3 &= \sum_{n=0}^{\infty}\sum_{j=0}^{2n} \frac{q^{2n^2+2n-\binom{j+1}{2}}\left(1+q^{2n+1}\right)}{\left(1-q^{2n+1}\right)} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{2n} \frac{q^{n+j(4n+1-j)/2}\left(1+q^{2n+1}\right)}{\left(1-q^{2n+1}\right)},\end{aligned} \tag{1.5}$$

from which follows immediately Gauss's classic result that every natural number is the sum of three triangular numbers. §§6 and 7 contain our main results on the mock theta functions.

2. Bailey chains. In [**8**], we presented a comprehensive treatment of Rogers-Ramanujan type identities based on a little known result of W. N. Bailey [**9**, §4]. For the statement of Bailey's Lemma we need the following standard notation:

$$(a;q)_\infty = (a)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \tag{2.1}$$

$$(a;q)_n = (a)_n = (a;q)_\infty/(aq^n;q)_\infty \tag{2.2}$$
$$\left(= (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right) \text{ for } n \text{ a nonnegative integer}\right).$$

BAILEY'S LEMMA. *If for $n \ge 0$ the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are related by*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}, \tag{2.3}$$

then for $n \geq 0$

$$\beta_n' = \sum_{r=0}^{n} \frac{\alpha_r'}{(q)_{n-r}(aq)_{n+r}}, \tag{2.4}$$

where

$$\beta_n' = \frac{1}{(aq/\rho_1)_n(aq/\rho_2)_n} \sum_{j=0}^{n} \frac{(\rho_1)_j(\rho_2)_j(aq/\rho_1\rho_2)_{n-j}(aq/\rho_1\rho_2)^j\beta_j}{(q)_{n-j}} \tag{2.5}$$

and

$$\alpha_r' = \frac{(\rho_1)_r(\rho_2)_r(aq/\rho_1\rho_2)^r\alpha_r}{(aq/\rho_1)_r(aq/\rho_2)_r}. \tag{2.6}$$

The above formulation is not at all the way Bailey stated this result [**9**, §4]. However, formulated as above it turns out to be incredibly powerful in obtaining and understanding Rogers-Ramanujan type identities. Pairs α_n, β_n can be substituted into identities like (3.1) to yield directly Rogers-Ramanujan type identities. The point is that once you find a pair of sequences α_n, β_n that satisfies (2.3) you can produce a new pair α_n', β_n' that satisfies the same identity. Thus an infinite family

$$(\alpha_n, \beta_n) \to (\alpha_n', \beta_n') \to (\alpha_n'', \beta_n'') \to \cdots$$

of such "Bailey pairs" can be obtained merely by iterating Bailey's Lemma. Furthermore, if only the α_n sequence is given, then the β_n sequence is completely determined by (2.3). If only the β_n sequence is given, then (2.3) may be inverted to yield [**5**, Lemma 3]

$$\alpha_n = (1 - aq^{2n}) \sum_{j=0}^{n} \frac{(aq)_{n+j-1}(-1)^{n-j}q^{\binom{n-j}{2}}}{(q)_{n-j}}\beta_j. \tag{2.7}$$

Thus if only the β_n sequence is given, then the α_n sequence is completely determined by (2.7).

Furthermore, the sequence may be extended to the left as well:

$$\cdots \to (\alpha_n^{(-2)}, \beta_n^{(-2)}) \to (\alpha_n^{(-1)}, \beta_n^{(-1)}) \to (\alpha_n, \beta_n) \to (\alpha_n, \beta_n') \to \cdots.$$

Obviously from (2.6)

$$\alpha_r^{(-1)} = \frac{(aq/\rho_1)_r(aq/\rho_2)_r(\rho_1\rho_2/aq)^r\alpha_r}{(\rho_1)_r(\rho_2)_r}. \tag{2.8}$$

To back up in the chain of β's is a little trickier. The relation is

$$\beta_n^{(-1)} = \frac{1}{(\rho_1)_n(\rho_2)_n} \sum_{j=0}^{n} \frac{(aq/\rho_1)_j(aq/\rho_2)_j(\rho_1\rho_2/aq)_{n-j}(\rho_1\rho_2/aj)^{2n-j}\beta_j}{(q)_{n-j}}. \tag{2.9}$$

To see this, let us define

$$B_n = \left(\frac{aq}{\rho_1}\right)_n \left(\frac{aq}{\rho_2}\right)_n \beta_n, \tag{2.10}$$

$$B_n^{(-1)} = (\rho_1)_n(\rho_2)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n^{(-1)}, \tag{2.11}$$

$$b(t) = \sum_{n=0}^{\infty} B_n t^n \tag{2.12}$$

and

$$b_{-1}(t) = \sum_{n=0}^{\infty} B_n^{(-1)} t^n. \tag{2.13}$$

Then by the q-binomial series [**4**, p. 17, (2.2.1)], we see that (2.5) is equivalent to

$$b(t) = \frac{(aqt/\rho_1\rho_2)_\infty b_{(-1)}(t)}{(t)_\infty}. \tag{2.14}$$

Obviously (2.14) is equivalent to

$$b_{-1}(t) = \frac{(t)_\infty}{(aqt/\rho_1\rho_2)_\infty} b(t), \tag{2.15}$$

and (2.15) yields (2.8) by invocation again of the q-binomial series.

We now have the four identities ((2.3), (2.7)–(2.9)) necessary so that we may start with either sequence of a Bailey pair, obtain the other sequence and then move either direction in the Bailey chain.

3. The role of SCRATCHPAD. How can one gain a foothold in studying the mock theta functions? Our approach was to implement the study of Bailey chains on SCRATCHPAD, IBM's symbolic algebra package. To keep things as simple as possible we considered the case where ρ_1 and $\rho_2 \to \infty$ and $a = 1$. Thus from Bailey's Lemma, we see that

$$\sum_{n=0}^{\infty} q^{n^2}\beta_n = (q)_\infty \sum_{n=0}^{\infty} q^{n^2}\alpha_n, \tag{3.1}$$

where by (2.7)

$$\alpha_n = (1 - q^{2n}) \sum_{j=0}^{n} \frac{(q)_{n+j-1}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_j}{(q)_{n-j}}. \tag{3.2}$$

Suppose we want the left side of (3.1) to coincide with (1.1). Then we must take

$$\beta_n = \frac{1}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}. \tag{3.3}$$

We now define α_n and β_n explicitly in SCRATCHPAD and ask for the first four values of α_n. SCRATCHPAD responds with

$$1$$

$$\frac{-4q}{q+1}$$

$$\frac{4q^3}{q^2+1}$$

$$\frac{-4q^6}{q^3+1}.$$

The pattern suggested is quite clear; namely it is reasonable to conjecture that α_n is $(-1)^n 4q^{n(n+1)/2}/(1+q^n)$ for $n \geq 1$, $\alpha_0 = 1$. The insertion of this conjecture in (3.1) coincides, not surprisingly, with the known fact (1.2).

Suppose now we redefine β_n by

$$\beta_n = \frac{1}{(1+q)(1+q^2)\cdots(1+q^n)}. \tag{3.4}$$

This then makes the left-side of (3.1) identical with the series in (1.3) for the fifth order mock theta function $f_0(q)$. The pattern for the related α_n arising from (3.2) is now somewhat complicated so we ask SCRATCHPAD for the first nine values:

$$\begin{aligned}
&1\\
&q^2 - 3q\\
&q^7 - 2q^6 - q^5 + 2q^4 + 2q^3\\
&q^{15} - 2q^{14} - q^{12} + 4q^{11} - 2q^8 - 2q^6\\
&q^{26} - 2q^{25} + q^{22} + 2q^{21} - 2q^{18} - 2q^{17} + 2q^{13} + 2q^{10}\\
&\vdots\\
&q^{100} - 2q^{99} + 2q^{96} - q^{92} - 2q^{88} + 2q^{84} + 2q^{83}\\
&\qquad -2q^{76} - 2q^{75} + 2q^{67} + 2q^{64} - 2q^{56} - 2q^{51} + 2q^{43} + 2q^{36}.
\end{aligned}$$

A careful look at α_8 (and, in the actual discovery, α_7 and α_9) suggests that a reasonable conjecture for α_n is

$$q^{n(3n+1)/2} \sum_{j=-n}^{n} (-1)^j q^{-j^2} - q^{n(3n-1)/2} \sum_{j=-n+1}^{n-1} (-1)^j q^{-j^2}. \tag{3.5}$$

If we insert this conjectured formula for α_n into (3.1) we obtain (1.4) almost directly. Thus SCRATCHPAD provides a powerful tool for the empirical analysis of Bailey chains. It should be emphasized that identity (1.4) appears nowhere in any of Ramanujan's known writings.

4. Extensions of Shanks' formulas. Once the truncated theta series are observed arising in (3.5), one immediately recalls the two elegant papers by D. Shanks on truncated theta series [**16**, **17**]. Presumably, if Shanks' results can be embedded in the hierarchy of q-hypergeometric function identities, then adequate generalizations of his results should allow us to derive (3.5) and related formulae for the other fifth order mock theta functions. It turns out that if we move to the left one place in the Bailey chain for the fifth order mock theta functions, our work is greatly simplified. The following two lemmas provide us adequate q-hypergeometric series machinery for the appropriate new Bailey pairs given in Theorems 3 and 4.

LEMMA 1. *The sequences A_n, B_n form a Bailey pair where*

$$B_n = \frac{(-1)^n (b)_n q^{-n(n-1)/2} a^{-n}}{(q)_n (bq)_n}, \tag{4.1}$$

$$(4.2) \qquad A_n = \frac{(-1)^n (aq)_{n-1}(1-aq^{2n}) q^{n(n-1)/2}}{(bq)_n} \sum_{j=0}^{n} \frac{(b)_j q^{j(1-n)} a^{-j}}{(q)_j}.$$

PROOF. All that is necessary is to evaluate (2.7) with β_n replaced by B_n:

$$A_n = (1-aq^{2n}) \sum_{j=0}^{n} \frac{(aq)_{n+j-1}(-1)^{n-j} q^{\binom{n-j}{2}} B_j}{(q)_{n-j}}$$

$$= \frac{(-1)^n (aq)_{n-1}(1-aq^{2n}) q^{\binom{n}{2}}}{(q)_n} \sum_{j=0}^{n} \frac{(aq^n)_j (q)_n (b)_j a^{-j} q^{j(1-n)}}{(q)_j (q)_{n-j} (bq)_j}$$

$$= \frac{(-1)^n (aq)_{n-1}(1-aq^{2n}) q^{\binom{n}{2}}}{(q)_n} \sum_{j=0}^{n} \frac{(q^{-n})_j (b)_j (aq^n)_j q^{j-\binom{j}{2}} a^{-j}}{(q)_j (bq)_j}$$

$$= \frac{(-1)^n (aq)_{n-1}(1-aq^{2n}) q^{\binom{n}{2}}}{(q)_n} \lim_{t \to 0} \sum_{j=0}^{n} \frac{(q^{-n})_j (b)_j (aq^n)_j (q/at)^j}{(q)_j (bq)_j (t^{-1})_j}$$

$$= \frac{(-1)^n (aq)_{n-1}(1-aq^{2n}) q^{\binom{n}{2}}}{(q)_n} \cdot \frac{(q)_n}{(bq)_n} \sum_{j=0}^{n} \frac{(q^{-n})_j (b)_j a^{-j} q^{j(1-n)}}{(q)_j (q^{-n})_j}$$

(by [**22**, p. 175, (10.2)] with $a = q^{-n}, c = aq^n, e = bq, f = t^{-1}, p = q$)

$$= \frac{(-1)^n (aq)_{n-1}(1-aq^{2n}) q^{\binom{n}{2}}}{(bq)_n} \sum_{j=0}^{n} \frac{(b)_j a^{-j} q^{j(1-n)}}{(q)_j},$$

as desired. □

LEMMA 2.

$$(4.3) \quad 1 + \sum_{j=1}^{n} \frac{(a)_j (1-aq^{2j})(b)_j a^j q^{j^2} b^{-j}}{(a)_j (1-a)(aq/b)_j} = \frac{(aq)_n}{(aq/b)_n} \sum_{j=0}^{n} \frac{(b)_j (aq^{n+1}/b)^j}{(q)_j}.$$

PROOF. This result is merely a limiting case of Watson's q-analog of Whipple's theorem [**18**, p. 100, (3.4.1.5)]. In fact, (4.3) follows immediately from the substitutions $g = q^{-n}, e = b, c = aq^{n+1}, f \to \infty, d \to \infty$. □

We remark that if in (4.3) we take $a = 1$ and let $b \to \infty$, we obtain Shank's finite version of Euler's pentagonal number theorem [**16**, p. 747, (2)]. Shank's finite version of Gauss's theorem [**17**, p. 609, (3′)] follows by replacing q by q^2 in (4.3) and then setting $a = 1, b = q$.

THEOREM 3. *The sequences A_n, B_n form a Bailey pair where*

$$(4.4) \qquad B_n = \frac{(-1)^n (b)_n q^{-n(n-1)/2} a^{-n}}{(q)_n (bq)_n},$$

$$A_n = \frac{(-1)^n (aq)_{n-1}(b)_n a^{-n} q^{-\binom{n}{2}}(1-aq^{2n})}{(bq)_n (q)_n} + \frac{(-1)^n q^{\binom{n}{2}} b^{n-1}(aq/b)_{n-1}(1-aq^{2n})}{(bq)_n} \cdot \left(1 + \sum_{j=1}^{n-1} \frac{(aq)_{j-1}(1-aq^{2j})(b)_j a^{-j} q^{-j^2} b^{-j}}{(q)_j (aq/b)_j}\right). \tag{4.5}$$

PROOF. Since the B_n sequence given by (4.4) is the same as the one in Lemma 1, equation (4.1), we see that we only need to identify the expression in (4.2) with the right-hand side of (4.5). In Lemma 2 replace a by a^{-1}, b by b^{-1}, q by q^{-1} and n by $n-1$; hence

$$1 + \sum_{j=1}^{n-1} \frac{(aq)_{j-1}(1-aq^{2j})(b)_j a^{-j} b^{-j} q^{-j^2}}{(q)_j (aq/b)_j} = \frac{b^{1-n}(aq)_{n-1}}{(aq/b)_{n-1}} \sum_{j=0}^{n-1} \frac{(b)_j a^{-j} q^{j(1-n)}}{(q)_j}. \tag{4.6}$$

Now let us examine the sum in (4.2). We split off the term $j = n$ which yields the first summand on the right-hand side of (4.5); the remainder of the sum in (4.2) is identical with the sum of the right-hand side of (4.6). Hence multiplying both sides of (4.6) by $b^{n-1}(aq/b)_{n-1}/(aq)_{n-1}$ and substituting the resulting left-hand side into (4.2), we obtain the second term in (4.5). □

THEOREM 4. *The sequences A'_n, B'_n form a Bailey pair where*

$$B'_n = \frac{1}{(bq)_n}, \tag{4.7}$$

$$A'_n = \frac{(-1)^n (aq)_{n-1}(b)_n q^{\binom{n+1}{2}}(1-aq^{2n})}{(bq)_n (q)_n} + \frac{(-1)^n a^n q^{n(3n-1)/2} b^{n-1}(aq/b)_{n-1}(1-aq^{2n})}{(bq)_n} \cdot \left(1 + \sum_{j=1}^{n-1} \frac{(aq)_{j-1}(1-aq^{2j})(b)_j a^{-j} q^{-j^2} b^{-j}}{(q)_j (aq/b)_j}\right). \tag{4.8}$$

PROOF. We apply Bailey's Lemma (with $\rho_1, \rho_2 \to \infty$) to the Bailey pair in Theorem 3 and the resulting $A'_n = a^n q^{n^2} A_n$ which is precisely (4.8). Now by (2.5)

$$B'_n = \sum_{j=0}^{n} \frac{a^j q^{j^2}}{(q)_{n-j}} \frac{(-1)^j (b)_j q^{-j(j-1)/2} a^{-j}}{(q)_j (bq)_j} = \frac{1}{(q)_n} \sum_{j=0}^{n} \frac{(q^{-n})_j (b)_j}{(q)_j (bq)_j} q^{(n+1)j} = \frac{1}{(bq)_n} \quad \text{by } [\mathbf{18}, \text{p. } 97, (3.3.2.6)], \tag{4.9}$$

as desired. □

5. Hecke modular form identities. E. Hecke [**12**] made an extensive study of double theta type series involving an indefinite quadratic form. For example, he showed that

$$(q)_\infty^2 = \sum_{m=-\infty}^{\infty} \sum_{n \geq 2|m|} (-1)^{n+m} q^{\binom{n+1}{2} - m(3m-1)/2}. \tag{5.1}$$

Actually as D. Bressoud points out [**10**], identity (5.1) was discovered originally by L. J. Rogers [**15**]. Subsequent studies of these types of identities have been made by Kac and Peterson [**13**], Bressoud [**10**], and the author [**7**].

The Bailey pair in Theorem 3 leads naturally to Theorem 5, an infinite product expansion which implies several Hecke modular form type results.

THEOREM 5. *Let A_n be defined by* (4.5); *then*

$$\frac{(q)_\infty (bq/y)_\infty (aq)_\infty}{(bq)_\infty (q/y)_\infty (aq/y)_\infty} = \sum_{n=0}^{\infty} \frac{(y)_n (-a/y)^n q^{\binom{n+1}{2}} A_n}{(aq/y)_n}. \tag{5.2}$$

PROOF. We take the Bailey pair from Theorem 3 and substitute into (2.4) with $\rho_1 = y$, $\rho_2 \to \infty$, $n \to \infty$. This yields

$$\frac{1}{(aq/y)_\infty} \sum_{n=0}^{\infty} \frac{(y)_n (b)_n}{(q)_n (bq)_n} \left(\frac{q}{y}\right)^n = \frac{1}{(aq)_\infty} \sum_{r=0}^{\infty} \frac{(y)_r}{(aq/y)_r} \left(-\frac{a}{y}\right)^r q^{\binom{r+1}{2}} A_r. \tag{5.3}$$

Now

$$\sum_{n=0}^{\infty} \frac{(y)_n (b)_n (q/y)^n}{(q)_n (bq)_n} = \frac{(q)_\infty (bq/y)_\infty}{(bq)_\infty (q/y)_\infty} \tag{5.4}$$

by [**18**, p. 97, (3.3.2.5)]. Substituting the right-hand side of (5.4) into (5.3) we obtain (5.2). □

To apply (5.2) with ease to Hecke modular form identities, we prove three lemmas.

LEMMA 6. *Let $A_n(a, b, q)$ denote the A_n in* (4.5). *Then*

$$A_n(aq, b, q) = \frac{(-1)^n (1 - aq^{2n+1}) q^{\binom{n}{2}} b^n (aq/b)_n}{(1 - aq)(bq)_n} \cdot \left(1 + \sum_{j=1}^{n} \frac{(aq)_{j-1} (1 - aq^{2j}) (b)_j a^{-j} q^{-j^2} b^{-j}}{(q)_j (aq/b)_j}\right). \tag{5.5}$$

PROOF. By (4.2)

$$\begin{aligned} A_n(aq, b, q) &= \frac{(-1)^n (aq^2)_{n-1} (1 - aq^{2n+1}) q^{\binom{n}{2}}}{(bq)_n} \cdot \sum_{j=0}^{n} \frac{(b)_j q^{-jn} a^{-j}}{(q)_j} \\ &= \frac{(-1)^n (aq)_n (1 - aq^{2n+1}) q^{\binom{n}{2}}}{(1 - aq)(bq)_n} \frac{b^n (aq/b)_n}{(aq)_n} \end{aligned}$$

(equation continues)

$$\cdot\left(1+\sum_{j=1}^{n}\frac{(aq)_{j-1}(1-aq^{2j})(b)_j a^{-j}q^{-j^2}b^{-j}}{(q)_j(aq/b)_j}\right)$$

(by (4.6) with n replaced by $n+1$)

$$=\frac{(-1)^n(1-aq^{2n+1})q^{\binom{n}{2}}b^n(aq/b)_n}{(1-aq)(bq)_n}$$

$$\cdot\left(1+\sum_{j=1}^{n}\frac{(aq)_{j-1}(1-aq^{2j})(b)_j a^{-j}q^{-j^2}b^{-j}}{(q)_j(aq/b)_j}\right),$$

as desired. □

LEMMA 7. *Let $A_n(a, b, q)$ denote the A_n in* (4.5). *Then*

$$A_n(1,-1,q)=q^{\binom{n+1}{2}}\sum_{j=-n}^{n}(-1)^j q^{-j^2}-q^{\binom{n}{2}}\sum_{j=-n+1}^{n-1}(-1)^j q^{-j^2}, \tag{5.6}$$

$$A_n(q,-1,q)=\frac{(1-q^{2n+1})q^{\binom{n}{2}}}{(1-q)}\left(1+2\sum_{j=1}^{n}(-1)^j q^{-j^2}\right), \tag{5.7}$$

$$A_n(1,q^{-1},q^2)=(-1)^n\left(q^{n^2+2n}\sum_{j=0}^{2n}q^{-\binom{j+1}{2}}-q^{n^2-2n}\sum_{j=0}^{2n-2}q^{-\binom{j+1}{2}}\right), \tag{5.8}$$

$$A_n(q^2,q,q^2)=\frac{(-1)^n(1+q^{2n+1})q^{n^2}}{1+q}\sum_{j=0}^{2n}q^{-\binom{j+1}{2}}, \tag{5.9}$$

$$A_n(1,0,q)=q^{n^2+n}\sum_{j=-n}^{n}(-1)^j q^{-j(3j+1)/2}-q^{n^2-n}\sum_{j=-n+1}^{n-1}(-1)^j q^{-j(3j+1)/2}, \tag{5.10}$$

$$A_n(q,0,q)=\frac{q^{n^2}(1-q^{2n+1})}{(1-q)}\sum_{j=-n}^{n}(-1)^j q^{-j(3j+1)/2}. \tag{5.11}$$

PROOF. By (4.5),

$$A_n(1,-1,q)=(-1)^n 2q^{-\binom{n}{2}}-q^{\binom{n}{2}}(1-q^n)\left(1+2\sum_{j=1}^{n-1}(-1)^j q^{-j^2}\right)$$

$$=q^{\binom{n+1}{2}}\sum_{j=-n}^{n}(-1)^j q^{-j^2}-q^{\binom{n}{2}}\sum_{j=-n+1}^{n-1}(-1)^j q^{-j^2},$$

as asserted in (5.6).

By (5.5),

$$A_n(q,-1,q)=\frac{(1-q^{2n+1})q^{\binom{n}{2}}}{(1-q)}\left(1+2\sum_{j=1}^{n}(-1)^j q^{-j^2}\right),$$

as asserted in (5.7).

By (4.5),

$$A_n(1, q^{-1}, q^2) = \frac{(-1)^n(1+q^{2n})q^{n-n^2}(1-q^{-1})}{(1-q^{2n-1})} + \frac{(-1)^n q^{(n-1)^2}(1-q^{4n})}{(1-q)}$$

$$\cdot\left(1 + \sum_{j=1}^{n-1} \frac{(1+q^{2j})(1-q^{-1})(1-q)q^{j-2j^2}}{(1-q^{2j-1})(1-q^{2j+1})}\right)$$

$$= \frac{(-1)^n(1+q^{2n})q^{n-n^2}(1-q^{-1})}{(1-q^{2n-1})}$$

$$+ \frac{(-1)^n q^{(n-1)^2}(1-q^{4n})}{(1-q)} \sum_{j=-n+1}^{n-1} \frac{(1-q^{-1})(1-q)q^{j-2j^2}}{(1-q^{2j-1})(1-q^{2j+1})}$$

$$= \frac{(-1)^n(1+q^{2n})q^{n-n^2}(1-q^{-1})}{(1-q^{2n-1})} + \frac{(-1)^n q^{(n-1)^2}(1-q^{4n})(1-q^{-1})}{q^{-1}(1-q^2)}$$

$$\cdot \sum_{j=-n+1}^{n-1} q^{j-2j^2}\left(\frac{q^{-1}}{1-q^{2j-1}} - \frac{q}{1-q^{2j+1}}\right)$$

$$= \frac{(-1)^n(1+q^{2n})q^{n-n^2}(1-q^{-1})}{(1-q^{2n-1})}$$

$$+ \frac{(-1)^n q^{(n-1)^2}(1-q^{4n})}{(1+q)} \sum_{j=-n+1}^{n-1} \frac{q^{j-2j^2}(1+q)}{(1-q^{2j+1})}$$

(where we have replaced j by $-j$ in the first sum)

$$= \frac{(-1)^n(1+q^{2n})q^{n-n^2}(1-q^{-1})}{(1-q^{2n-1})} - \frac{(-1)^n q^{(n-1)^2-n-2n^2}(1-q^{4n})}{(1-q^{-2n+1})}$$

$$+(-1)^n q^{(n-1)^2}(1-q^{4n}) \sum_{j=0}^{n-1}\left(\frac{q^{j-2j^2}}{1-q^{2j+1}} + \frac{q^{-j-1-2(j+1)^2}}{1-q^{-2j-1}}\right)$$

$$= (-1)^n q^{-n^2-n}(1+q^{2n})$$

$$-(-1)^n q^{(n-1)^2}(1-q^{4n}) \sum_{j=0}^{n-1} \frac{q^{-2j^2-3j-2}(1-q^{4j+2})}{(1-q^{2j+1})}$$

$$= (-1)^n q^{-n^2-n}(1+q^{2n})$$

$$-(-1)^n q^{n^2-2n}(1-q^{4n}) \sum_{j=0}^{n-1}\left(q^{-\binom{2j+2}{2}} + q^{-\binom{2j+1}{2}}\right)$$

$$= (-1)^n q^{-n^2-n}(1+q^{2n}) - (-1)^n q^{n^2-2n}(1-q^{4n}) \sum_{j=0}^{2n-1} q^{-\binom{j+1}{2}}$$

$$= (-1)^n q^{n^2+2n} \sum_{j=0}^{2n} q^{-\binom{j+1}{2}} - (-1)^n q^{n^2-2n} \sum_{j=0}^{2n-2} q^{-\binom{j+1}{2}},$$

as asserted in (5.8).

Next by (5.5)

$$A_n(q^2, q, q^2) = \frac{(-1)^n(1+q^{2n+1})q^{n^2}}{1+q}\left(1 + \sum_{j=1}^{n}(1+q^{2j})q^{-2j^2-j}\right)$$
$$= \frac{(-1)^n(1+q^{2n+1})q^{n^2}}{1+q}\sum_{j=0}^{2n} q^{-\binom{j+1}{2}},$$

as asserted in (5.9).

By (4.5),

$$\begin{aligned}A_n(1,0,q) &= (-1)^n q^{-\binom{n}{2}}(1+q^n) \\ &\quad - q^{n^2-n}(1-q^{2n})\left(1+\sum_{j=1}^{n-1}(1+q^j)(-1)^j q^{-j(3j+1)/2}\right) \\ &= (-1)^n q^{-\binom{n}{2}}(1+q^n) - q^{n^2-n}(1-q^{2n})\sum_{j=-n+1}^{n-1}(-1)^j q^{-j(3j+1)/2} \\ &= q^{n^2+n}\sum_{j=-n}^{n}(-1)^j q^{-j(3j+1)/2} - q^{n^2-n}\sum_{j=-n+1}^{n-1}(-1)^j q^{-j(3j+1)/2},\end{aligned}$$

as asserted in (5.10).

Finally by (5.5)

$$A_n(q,0,q) = \frac{q^{n^2}(1-q^{2n+1})}{(1-q)}\left(1+\sum_{j=1}^{n}(1+q^j)q^{-j^2-\binom{j+1}{2}}(-1)^j\right)$$
$$= \frac{q^{n^2}(1-q^{2n+1})}{(1-q)}\sum_{j=-n}^{n}(-1)^j q^{-j(3j+1)/2},$$

which is (5.11). □

LEMMA 8. *Let $A_n = A_n(a, b, q)$ be defined by* (4.5). *Then*

$$\frac{(q)_\infty (aq)_\infty}{(bq)_\infty} = \sum_{n=0}^{\infty} a^n q^{n^2} A_n, \tag{5.12}$$

$$\frac{(q)_\infty^2(-bq)_\infty}{(-q)_\infty^2(bq)_\infty} = 1 + 2\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{1+q^n}A_n(1,b,q), \tag{5.13}$$

$$\frac{(q^2;q^2)_\infty^2(bq;q^2)_\infty}{(q;q^2)_\infty^2(bq^2;q^2)_\infty} = \sum_{n=0}^{\infty}(-1)^n q^{n^2}A_n(1,b,q^2). \tag{5.14}$$

PROOF. These identities are immediate from Theorem 5 under the substitutions $y \to \infty$; $a = 1, y = -1$; finally replace q by q^2, then set $a = 1, y = q$. □

It is now possible to combine Lemmas 7 and 8 to obtain 18 different identities of is Theorem 4 and the observation that $A'_n(a, b, q) = a^n q^{n^2} A_n(a, b, q)$. Thus all our formulae for the $A_n(a, b, q)$ can be applied in evaluating $A'_n(a, b, q)$.

We begin with (6.1), and let $B'_n = B'_n(b, q)$.

$$\begin{aligned} f_0(q) &= \sum_{n=0}^{\infty} q^{n^2} B'_n(-1, q) \quad (\text{by } (4.7)) \\ &= \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} q^{n^2} A'_n(1, -1, q) \quad (\text{by } (3.1)) \\ &= \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} q^{2n^2} A_n(1, -1, q) \quad (\text{by } (2.6)) \\ &= \frac{1}{(q)_\infty} \left(\sum_{\substack{n=0 \\ |j| \le n}}^{\infty} q^{n(5n+1)/2 - j^2} (-1)^j - \sum_{\substack{n=0 \\ |j| < n}}^{\infty} q^{n(5n-1)/2 - j^2} (-1)^j \right) \quad (\text{by } (5.6)) \\ &= \frac{1}{(q)_\infty} \sum_{\substack{n=0 \\ |j| \le n}}^{\infty} q^{n(5n+1)/2 - j^2} (-1)^j (1 - q^{4n+2}), \end{aligned}$$

which is (6.1).

Next

$$\begin{aligned} F_0(q) &= \sum_{n=0}^{\infty} q^{2n^2} B'_n(q^{-1}, q^2) \quad (\text{by } (4.7)) \\ &= \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} q^{2n^2} A'_n(1, q^{-1}, q^2) \quad (\text{by } (3.1)) \\ &= \frac{1}{(q^2; q^2)} \sum_{n=0}^{\infty} q^{4n^2} A_n(1, q^{-1}, q^2) \quad (\text{by } (2.6)) \\ &= \frac{1}{(q^2; q^2)_\infty} \left(\sum_{n=0}^{\infty} \sum_{j=0}^{2n} (-1)^n q^{5n^2 + 2n - \binom{j+1}{2}} \right. \\ &\qquad \left. - \sum_{n=0}^{\infty} \sum_{j=0}^{2n-2} (-1)^n q^{5n^2 - 2n - \binom{j+1}{2}} \right) \quad (\text{by } (5.8)) \\ &= \frac{1}{(q^2; q^2)} \sum_{n=0}^{\infty} \sum_{j=0}^{2n} (-1)^n q^{5n^2 + 2n - \binom{j+1}{2}} (1 + q^{6n+3}), \end{aligned}$$

as asserted in (6.2).

$$\begin{aligned} 1 + 2\psi_0(q) &= \sum_{n=0}^{\infty} (-1; q)_n q^{\binom{n+1}{2}} B'_n(0, q) \\ &= \frac{(-q)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{\binom{n+1}{2}} A'_n(1, 0, q)}{(-q; q)_n} \\ &\qquad (\text{by } (2.4), n \to \infty, \rho_1 \to \infty, \rho_2 = -1, a = 1) \end{aligned}$$

(equation continues)

tends to 0 as $\mathrm{Im}(z) \to \infty$, so $f \equiv 0$ by Liouville's theorem.
(4) Replace x by $-x$ in the integral.
(5) Let $g(x) = \frac{1}{\cosh \pi x}$. We first compute the Fourier transform $\mathcal{F}g$ of g: Using Cauchy's formula we get

$$\left(\int_{\mathbf{R}} - \int_{\mathbf{R}+i}\right) \frac{e^{2\pi i z x}}{\cosh \pi x}\, dx = 2\pi i \operatorname*{Res}_{x=i/2} \frac{e^{2\pi i z x}}{\cosh \pi x} = 2e^{-\pi z},$$

but

$$\int_{\mathbf{R}+i} \frac{e^{2\pi i z x}}{\cosh \pi x}\, dx = \int_{\mathbf{R}} \frac{e^{2\pi i z (x+i)}}{\cosh \pi (x+i)}\, dx = -e^{-2\pi z} \int_{\mathbf{R}} \frac{e^{2\pi i z x}}{\cosh \pi x}\, dx,$$

so we find

$$(\mathcal{F}g)(z) := \int_{\mathbf{R}} \frac{e^{2\pi i z x}}{\cosh \pi x}\, dx = \frac{2e^{-\pi z}}{1 + e^{-2\pi z}} = g(z).$$

We see that g is its own Fourier transform! (Note the unusual plus sign in the definition of the Fourier transform).

Let $f_\tau(x) = e^{\pi i \tau x^2}$, $\tau \in \mathcal{H}$. The Fourier transform of f_τ is given by

$$\mathcal{F} f_\tau = \frac{1}{\sqrt{-i\tau}} f_{-\frac{1}{\tau}}.$$

We now see

$$\begin{aligned} \int_{\mathbf{R}} \frac{e^{\pi i \tau x^2 + 2\pi i z x}}{\cosh \pi x}\, dx &= \mathcal{F}(f_\tau \cdot g)(z) = (\mathcal{F} f_\tau) * (\mathcal{F} g)(z) \\ &= \frac{1}{\sqrt{-i\tau}} f_{-\frac{1}{\tau}} * g(z) = \frac{1}{\sqrt{-i\tau}} \int_{\mathbf{R}} \frac{e^{\pi i \frac{-1}{\tau}(z-x)^2}}{\cosh \pi x}\, dx. \end{aligned}$$

This identity holds for $z \in \mathbf{R}$. Since both sides are analytic functions of z, the identity holds for all $z \in \mathbf{C}$. If we replace z by iz we get the desired result.

We may also prove the identity of part (5) by using (1) and (2) to show that $z \mapsto \frac{1}{\sqrt{-i\tau}} e^{\pi i z^2/\tau} h(\frac{z}{\tau}; -\frac{1}{\tau})$ also satisfies the two equations (1) and (2). By uniqueness we get the equation.
(6) Using (1) and (2) we can show that the right hand side, considered as a function of z, also satisfies (1) and (2). The equation now follows from (3). □

1.3 Lerch sums

In this section we will study the function

$$\sum_{n \in \mathbf{Z}} \frac{(-1)^n e^{\pi i (n^2+n)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}} \qquad (\tau \in \mathcal{H}, v \in \mathbf{C}, u \in \mathbf{C} \setminus (\mathbf{Z}\tau + \mathbf{Z})).$$

This function was also studied by Lerch. The original paper [15] is in Czech and is not very easy to obtain. See [14] for an abstract in German. We will prove elliptic and modular transformation properties of this function in Proposition 1.4 and 1.5 respectively. These results are equivalent to the results found by Lerch.

It is more convenient to normalize the above sum by dividing by the classical Jacobi theta function ϑ. (Lerch did this too.) We will first give, without proof, some standard properties of ϑ. For the theory of ϑ-functions see [19].

Proposition 1.3 *For $z \in \mathbf{C}$ and $\tau \in \mathcal{H}$ define*

$$\vartheta(z) = \vartheta(z;\tau) := \sum_{\nu \in \frac{1}{2}+\mathbf{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu (z+\frac{1}{2})}.$$

Then ϑ satisfies:

(1) $\vartheta(z+1) = -\vartheta(z)$.

(2) $\vartheta(z+\tau) = -e^{-\pi i \tau - 2\pi i z}\vartheta(z)$.

(3) *Up to a multiplicative constant, $z \mapsto \vartheta(z)$ is the unique holomorphic function satisfying (1) and (2).*

(4) $\vartheta(-z) = -\vartheta(z)$.

(5) *The zeros of ϑ are the points $z = n\tau + m$, with $n, m \in \mathbf{Z}$. These are simple zeros.*

(6) $\vartheta(z;\tau+1) = e^{\frac{\pi i}{4}}\vartheta(z;\tau)$.

(7) $\vartheta(\frac{z}{\tau}; -\frac{1}{\tau}) = -i\sqrt{-i\tau}\, e^{\pi i z^2/\tau}\vartheta(z;\tau)$.

(8) $\vartheta(z;\tau) = -iq^{\frac{1}{8}}\zeta^{-\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)(1-\zeta q^{n-1})(1-\zeta^{-1}q^n)$, *with $q = e^{2\pi i\tau}$, $\zeta = e^{2\pi i z}$. This is the Jacobi triple product identity.*

(9) $\vartheta'(0;\tau+1) = e^{\frac{\pi i}{4}}\vartheta'(0;\tau)$ *and* $\vartheta'(0;-\frac{1}{\tau}) = (-i\tau)^{3/2}\ \vartheta'(0;\tau)$.

(10) $\vartheta'(0;\tau) = -2\pi\eta(\tau)^3$, *with η as in the introduction.*

We now turn to the normalized version of Lerch's function:

Proposition 1.4 *For $u, v \in \mathbf{C} \setminus (\mathbf{Z}\tau + \mathbf{Z})$ and $\tau \in \mathcal{H}$, define*

$$\mu(u,v) = \mu(u,v;\tau) := \frac{e^{\pi i u}}{\vartheta(v;\tau)} \sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i (n^2+n)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}}.$$

Then μ satisfies:

(1) $\mu(u+1,v) = -\mu(u,v)$,