$$\sum_{m\geq 1}\sum_{n\geq 1}(A_{mn}-B_{mn}) = \sum_{n\geq 1}\sum_{m\geq 1}(A_{mn}-B_{mn}),$$

und $(*)$ ergibt

$$(**)\qquad F(\tau) = \sum_{m\geq 1}\sum_{n\geq 1}B_{mn} - \sum_{n\geq 1}\sum_{m\geq 1}B_{mn}.$$

Da sich die Terme abwechselnd wegheben, hat man sofort

$$\sum_{n\geq 1}B_{mn} = 0.$$

Andererseits erhält man

$$\begin{aligned}\tau\cdot\sum_{m\geq 1}B_{mn} &= \sum_{m\geq 1}\left(\frac{1}{m+(n-1)/\tau} - \frac{1}{m+n/\tau} + \frac{1}{m-n/\tau} - \frac{1}{m-(n-1)/\tau}\right)\\ &= \sum_{m\geq 1}\left(\frac{2(n-1)/\tau}{[(n-1)/\tau]^2-m^2} - \frac{2n/\tau}{[n/\tau]^2-m^2}\right) = \varphi(n-1)-\varphi(n)\ ,\end{aligned}$$

wobei aufgrund der Partialbruchentwicklung des Cotangens (vgl. R. REMMERT, G. SCHUMACHER [2002], Satz 11.2.1)

$$\varphi(\xi) := \begin{cases}\pi\cot(\pi\xi/\tau) - \dfrac{1}{\xi/\tau} & \text{für }\ \xi\neq 0,\\ 0 & \text{für }\ \xi = 0\end{cases}$$

gilt. Nach $(**)$ folgt daher

$$\tau\cdot F(\tau) = -\tau\cdot\sum_{n\geq 1}\sum_{m\geq 1}B_{mn} = -\sum_{n\geq 1}(\varphi(n-1)-\varphi(n)) = -\varphi(0) + \lim_{n\to\infty}\varphi(n).$$

Für $z = x+iy$ gilt

$$\cot z = i\cdot\frac{e^{ix-y}+e^{-ix+y}}{e^{ix-y}-e^{-ix+y}},\quad \text{also}\quad \lim_{y\to-\infty}\cot z = i.$$

Wegen $\operatorname{Im}(1/\tau) < 0$ folgt $\tau\cdot F(\tau) = \pi i$, also die Behauptung. □

Bemerkung. Die Idee dieses Beweises findet man bereits bei G. EISENSTEIN (*Math. Werke I*, 357–478); eine präzise Durchführung der Beweisidee gibt wohl erstmals A. HURWITZ in seiner Dissertation (*Math. Werke I*, 23–26). Man vergleiche R. FUETER [1924], 21–23, J.–P. SERRE [1973], 95–96, und N. KOBLITZ [1993], Proposition III.2.7.

2. Das Transformationsverhalten von η. Nach R. DEDEKIND (*Ges. math. Werke I*, 159–173) definiert man eine holomorphe Funktion $\eta : \mathbb{H}\to\mathbb{C}$ durch

$$(1)\qquad \eta(\tau) := e^{\pi i\tau/12}\cdot\prod_{m=1}^{\infty}(1-e^{2\pi i m\tau}).$$

Diese DEDEKIND*sche η–Funktion* darf nicht mit der in I.6.1(4) eingeführten η–Funktion verwechselt werden! Offenbar gilt

$$\eta(\tau+1) = e^{\pi i/12}\cdot\eta(\tau). \tag{2}$$

Da das Produkt absolut konvergiert, hat man außerdem

$$\eta(\tau) \neq 0 \quad \text{für alle } \tau \in \mathbb{H}. \tag{3}$$

Satz A. *Es gilt*

$$\eta(-1/\tau) = \sqrt{\tau/i}\cdot\eta(\tau) \quad \textit{für alle } \tau\in\mathbb{H}.$$

Dabei ist der Zweig der Wurzel zu wählen, der für positive Argumente selbst positiv wird.

Beweis. Für $\tau \in \mathbb{H}$ betrachte man die Funktion $f(\tau) := \eta'(\tau)/\eta(\tau)$. Aus (1) folgert man direkt

$$\begin{aligned} f(\tau) &= \frac{\pi i}{12}\cdot\left(1 - 24\cdot\sum_{m\geq 1} m\cdot\frac{e^{2\pi i m\tau}}{1-e^{2\pi i m\tau}}\right) = \frac{\pi i}{12}\cdot\left(1-24\cdot\sum_{m\geq 1}\sum_{r\geq 1} m\cdot e^{2\pi i r m\tau}\right) \\ &= \frac{\pi i}{12}\cdot\left(1-24\cdot\sum_{n\geq 1}\sigma_1(n)\cdot e^{2\pi i n\tau}\right) = \frac{i}{4\pi}\cdot G_2(\tau)\ , \end{aligned}$$

wenn man Proposition 1 verwendet. Satz 1 übersetzt sich damit in

$$f\left(-\frac{1}{\tau}\right)\cdot\frac{1}{\tau^2} - f(\tau) - \frac{1}{2\tau} = 0. \tag{$*$}$$

Für

$$g(y) := \frac{\eta\,(i/y)}{\eta(iy)\sqrt{y}}, \quad y>0,$$

erhält man dann

$$\frac{g'(y)}{g(y)} = f\left(\frac{i}{y}\right)\cdot\frac{-i}{y^2} - i\cdot f(iy) - \frac{1}{2y} = 0$$

nach ($*$). Es gibt also eine Konstante γ mit $\eta(i/y) = \gamma\cdot\sqrt{y}\cdot\eta(iy)$. Für $y=1$ folgt $\gamma = 1$, also die Behauptung mit dem Identitätssatz. □

Satz B. *Es gilt* $\eta^{24} = \Delta^*$.

Beweis. Mit η ist auch $f := \eta^{24}$ auf $\mathbb{H}$ holomorph. Wegen (1) gilt

$$f(\tau) = e^{2\pi i\tau}\cdot\prod_{m=1}^{\infty}\left(1-e^{2\pi i m\tau}\right)^{24} = e^{2\pi i\tau} + \cdots\ , \tag{$*$}$$

so dass f in eine FOURIER–Reihe entwickelbar ist mit $\alpha_f(0) = 0$ und $\alpha_f(1) = 1$. Gleichung (2) und Satz A zeigen, dass

PROPOSITION 2.8. *Suppose that* $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$. *If* ψ *is a Dirichlet character with modulus* m, *then*

$$f_\psi(z) \in M_k(\Gamma_0(Nm^2), \chi\psi^2).$$

Moreover, if $f(z)$ *is a cusp form, then so is* $f_\psi(z)$.

EXAMPLE 2.9. Suppose that $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_2(\Gamma_0(36))$ and $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_2(\Gamma_0(144))$ are the eta-quotients defined by

$$f(z) := \eta(6z)^4 = q - 4q^7 + 2q^{13} + 8q^{19} - 5q^{25} - \cdots,$$

$$g(z) := \frac{\eta(12z)^{12}}{\eta(6z)^4\eta(24z)^4} = q + 4q^7 + 2q^{13} - 8q^{19} - 5q^{25} + \cdots.$$

If χ_{-4} is the nontrivial Dirichlet character with modulus 4, then

$$g(z) = f_{\chi_{-4}}(z).$$

2.3. The Theta operator

Here we recall and examine the action of Ramanujan's differential operator. Ramanujan's Theta-operator is defined by

$$\Theta\left(\sum_{n=h}^{\infty} a(n)q^n\right) := \sum_{n=h}^{\infty} na(n)q^n. \tag{2.1}$$

REMARK 2.10. It is easy to see that

$$\Theta = q\frac{d}{dq} = \frac{1}{2\pi i}\cdot\frac{d}{dz}.$$

We refer to this operator, which plays many roles, as "Ramanujan's operator" since he first observed that

$$\Theta(E_4) = (E_4E_2 - E_6)/3 \quad \text{and} \quad \Theta(E_6) = (E_6E_2 - E_8)/2. \tag{2.2}$$

We have the following fundamental fact.

PROPOSITION 2.11. *If* $f(z) = \sum_{n=h}^{\infty} a(n)q^n$ *is a weight* k *meromorphic modular form on a congruence subgroup* Γ *of* $\mathrm{SL}_2(\mathbb{Z})$, *then*

$$\Theta(f) = (\tilde{f} + kfE_2)/12, \tag{2.3}$$

where $\tilde{f}$ *is a meromorphic modular form of weight* $k+2$ *on* Γ.

It is natural to seek an explicit description of the $\tilde{f}$ appearing in Proposition 2.11. Here we obtain such a description for meromorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$.

We require a specific sequence of modular functions $j_m(z)$. To define this sequence, let

$$j_0(z) := 1 \quad \text{and} \quad j_1(z) := j(z) - 744. \tag{2.4}$$

If $m \geq 2$, then define $j_m(z)$ by

$$j_m(z) := j_1(z) \mid T_0(m), \tag{2.5}$$

where $T_0(m) := mT_{m,0}$ is the normalized mth weight zero Hecke operator. Observe that this operator may be described as follows.

$$g(z) \mid T_0(m) = \sum_{\substack{d|m \\ ad=m}} \sum_{b=0}^{d-1} g\left(\frac{az+b}{d}\right). \tag{2.6}$$

REMARK 2.12. Although we defined the Hecke operators for modular forms, they are also well defined for meromorphic modular forms. One simply extends their definition in the obvious way to take into account a possible pole at infinity.

PROPOSITION 2.13. *If m is a nonnegative integer, then $j_m(z)$ is a monic polynomial in $j(z)$ of degree m with coefficients in $\mathbb{Z}$.*

PROOF. Since $j(z)$ is holomorphic on $\mathcal{H}$, (2.6) implies that $j_m(z)$ is also holomorphic on $\mathcal{H}$. Since each modular function is a rational function in $j(z)$, a function which is a bijection between $\mathfrak{F} = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ and $\mathbb{C}$, it follows that $j_m(z)$ must be a polynomial in $j(z)$. Otherwise, $j_m(z)$ would have a pole. Since the q-expansion of $j_m(z)$ has the form

$$j_m(z) = q^{-m} + \sum_{n=1}^{\infty} c_m(n)q^n \in q^{-m}\mathbb{Z}[[q]],$$

it must be that $j_m(z)$ is a monic polynomial in $j(z)$ of degree m. That this polynomial has integer coefficients follows from the fact that $j_m(z)$ has integer coefficients. □

Here we list the first few $j_m(z)$:

$$j_0(z) = 1,$$

$$j_1(z) = j(z) - 744 = q^{-1} + 196884q + \cdots,$$

$$j_2(z) = j(z)^2 - 1488j(z) + 159768 = q^{-2} + 42987520q + \cdots,$$

$$j_3(z) = j(z)^3 - 2232j(z)^2 + 1069956j(z) - 36866976 = q^{-3} + 2592899910q + \cdots.$$

Asai, Kaneko and Ninomiya [**AKN**] proved the following beautiful theorem regarding the polynomials $j_m(z)$. To state their result, for each $\tau \in \mathcal{H}$ define $H_\tau(z)$ by

$$H_\tau(z) := \sum_{n=0}^{\infty} j_n(\tau)q^n. \tag{2.7}$$

THEOREM 2.14. *If $\tau \in \mathcal{H}$, then*

$$H_\tau(z) = \sum_{n=0}^{\infty} j_n(\tau)q^n = \frac{E_4(z)^2E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z)-j(\tau)}.$$

Notice that if $\tau \in \mathcal{H}$, then $H_\tau(z)$ is a weight 2 meromorphic modular form. To illustrate the utility of Theorem 2.14, we mention that it can be used to prove that

$$j(\tau) - j(z) = p^{-1}\exp\left(-\sum_{n=1}^{\infty} j_n(z)\cdot\frac{p^n}{n}\right),$$

where $p = e^{2\pi i\tau}$. This identity is equivalent to the famous denominator formula for the Monster Lie algebra

$$j(\tau) - j(z) = p^{-1} \prod_{m>0 \text{ and } n\in\mathbb{Z}} (1 - p^m q^n)^{c(mn)}, \tag{2.8}$$

where the exponents $c(n)$ are defined as the coefficients of

$$j_1(z) = j(z) - 744 = \sum_{n=-1}^{\infty} c(n)q^n.$$

EXAMPLE 2.15. For $\tau = i$ and ω, we have the following beautiful formulas:

$$H_\omega(z) = \frac{E_6(z)}{E_4(z)} = \sum_{n=0}^{\infty} j_n(\omega)q^n,$$

$$H_i(z) = \frac{E_8(z)}{E_6(z)} = \sum_{n=0}^{\infty} j_n(i)q^n.$$

In [**BKO**], Bruinier, Kohnen and the author used these forms to obtain an explicit description of the action of the Θ-operator on meromorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$. To state this result, first define a rational number e_τ, for each $\tau \in \mathfrak{F}$, by

$$e_\tau := \begin{cases} 1/2 & \text{if } \tau = i, \\ 1/3 & \text{if } \tau = \omega, \\ 1 & \text{otherwise.} \end{cases} \tag{2.9}$$

Recall that $\mathfrak{F}$ does not contain the cusp infinity.

THEOREM 2.16. *If $f(z) = \sum_{n=h}^{\infty} a(n)q^n$ is a nonzero weight k meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ for which $a(h) = 1$, then*

$$\frac{\Theta(f)}{f} = \frac{kE_2}{12} - f_\Theta,$$

where f_Θ is defined by

$$f_\Theta := \sum_{\tau\in\mathfrak{F}} e_\tau \mathrm{ord}_\tau(f) H_\tau(z).$$

REMARK 2.17. This theorem has been generalized to certain genus zero congruence subgroups by Ahlgren [**A4**], and to the so-called Hecke subgroups of $\mathrm{SL}_2(\mathbb{R})$ by Choie and Kohnen [**CK**].

Before proceeding to the proof, we first recall the following straightforward fact which explains the connection between Theorem 2.16 and the combinatorial properties of the "logarithmic derivative" of the infinite product expansion of a modular form (see Proposition 2.1 of [**BKO**]). This proposition plays an important role in the proof of Theorem 2.16.

PROPOSITION 2.18. *Let $f(z) = \sum_{n=h}^{\infty} a(n)q^n$ be a meromorphic function in a neighborhood of $q = 0$, and suppose that $a(h) = 1$. Then there are uniquely determined complex numbers $c(n)$ for which*

$$f(z) = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},$$

where the product converges in a small neighborhood of $q = 0$. *Moreover, the following identity is true*

$$\frac{\Theta(f)}{f} = h - \sum_{n=1}^{\infty} \sum_{d|n} c(d) d q^n.$$

SKETCH OF THE PROOF OF THEOREM 2.16. We cut off $\mathfrak{F}$ by a horizontal line

$$\mathcal{L} := \left\{ iC - t \ : \ -\frac{1}{2} \leq t \leq \frac{1}{2} \right\},$$

where $C > 0$ is chosen so large that all poles and zeros of $f(z)$, apart from those at the cusp at infinity, are contained in

$$\{z \in \mathcal{H} \ : \ \mathrm{Im}\,(z) < C\} \cap \mathfrak{F}.$$

For simplicity, suppose that $f(z)$ has no zeros or poles on the boundary $\partial\mathfrak{F}$ except possibly at i or ω, and let γ be the closed path with positive orientation consisting of $\mathcal{L}$ and γ_1, where γ_1 is the part of $\partial\mathfrak{F}$ below $\mathcal{L}$ modified in the usual way (i.e. using arcs of radius r around i and ω as in the standard proof of Theorem 1.29, the "$k/12$-valence formula").

We integrate

$$\frac{1}{2\pi i} \frac{f'(z)}{f(z)} j_n(z)$$

along γ. By the Residue Theorem, taking into account that $j_n(z)$ is holomorphic on $\mathcal{H}$, this integral is equal to

$$\sum_{\tau \in \mathfrak{F} - \{\omega, i\}} \mathrm{ord}_\tau(f) j_n(\tau).$$

On the other hand, the integral can be evaluated separately along the different pieces of γ. If we let r tend to zero, then we find that

(2.10)

$$\sum_{\tau \in \mathfrak{F} - \{\omega, i\}} \mathrm{ord}_\tau(f) j_n(\tau) = \\ -\frac{1}{3}\mathrm{ord}_\omega(f) j_n(\omega) - \frac{1}{2}\mathrm{ord}_i(f) j_n(i) + \frac{1}{2\pi i}\int_\rho \frac{F'(q)}{F(q)} J_n(q) dq - \frac{k}{2\pi i}\int_\sigma \frac{j_n(z)}{z} dz.$$

Here $F(q) = f(z)$ and $J_n(q) := j_n(z)$. Furthermore, ρ is a small circle around $q = 0$ with negative orientation and not containing any pole or zero of $F(q)$ except possibly 0, and σ is the part of the unit circle in the upper half-plane that connects ω and i, with positive orientation.

By Proposition 2.18, we have that

$$\frac{qF'(q)}{F(q)} = \frac{\Theta(f)}{f} = h - \sum_{n=1}^{\infty} \sum_{d|n} c(d) d q^n,$$

where h is the order of F at $q = 0$. Therefore we find that

$$\frac{1}{2\pi i}\int_\rho \frac{F'(q)}{F(q)} J_n(q) dq = \sum_{d|n} c(d) d. \tag{2.11}$$

Since (2.10) holds for

$$f(z) = \Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

and since $\Delta(z)$ has no zeros in $\mathcal{H}$, (2.10) implies that

$$\frac{1}{2\pi i} \int_{\sigma} \frac{j_n(z)}{z} dz = 2\sigma_1(n). \tag{2.12}$$

To prove that

$$\frac{\Theta(f)}{f} = \frac{kE_2}{12} - f_\Theta,$$

where f_Θ has the claimed form, one now simply argues coefficient by coefficient using (2.10), (2.11), (2.12), and the fact that

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

□

2.4. Further operators

Here we briefly recall further important operators on spaces of integer weight modular forms. Let k be a positive integer. Recall from Section 1.2 that the group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on functions $f : \mathcal{H} \to \mathbb{C}$ by the operator

$$(f|_k\gamma)(z) = (\det \gamma)^{k/2}(cz + d)^{-k} f(\gamma z), \tag{2.13}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$.

Definition 2.19. For a prime divisor p of N with $\mathrm{ord}_p(N) = \ell$, let $Q_p := p^\ell$. Define the *Atkin-Lehner operator* $|_k W(Q_p)$ on $M_k(\Gamma_0(N))$ by any matrix

$$W(Q_p) := \begin{pmatrix} Q_p\alpha & \beta \\ N\gamma & Q_p\delta \end{pmatrix} \in M_2(\mathbb{Z})$$

with determinant Q_p, where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. Furthermore, define the *Fricke involution* $|_k W(N)$ on $M_k(\Gamma_0(N))$ by the matrix

$$W(N) := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

Remark 2.20. It is straightforward to verify that these operators are well defined on $M_k(\Gamma_0(N))$; this follows from the fact that $W(Q_p)$ is unique up to left multiplication by elements of $\Gamma_0(N)$.

It is straightforward to verify the following fact.

Proposition 2.21. *The operators $|_k W(Q_p)$, for primes $p \mid N$, and $|_k W(N)$ are involutions on $M_k(\Gamma_0(N))$. Furthermore, these operators commute with all of the Hecke operators $T_{n,k}$ for which $\gcd(n, N) = 1$.*

(i.e., as a Lerch-like sum or as a mock Eisenstein series), the first two values being

$$F_2 = q + 2q^2 + q^3 + 2q^4 - q^5 + 3q^6 - \cdots ,$$
$$F_4 = 7q + 26q^2 + 7q^3 + 26q^4 - 91q^5 + \cdots .$$

Then the function

$$f(\tau) = \frac{F_2(q) - 12E_2(\tau)}{\eta(\tau)} = q^{-1/24}\left(1 - 35q - 130q^2 - 273q^3 - 595q^4 - \cdots\right),$$

where $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$ is the usual quasimodular Eisenstein series of weight 2, is a mock modular form of weight $\frac{3}{2}$ on the full modular group with shadow $\eta(\tau)$, and for each integer $n > 0$ the sum of $12F_{2n+2}(\tau)$ and $24^n\binom{2n}{n}^{-1}[f,\eta]_n$ (where $[f,g]_n$ denotes the n-th Rankin–Cohen bracket, here in weight $(\frac{3}{2},\frac{1}{2})$), is a modular form of weight $2n+2$ on $\mathrm{SL}(2,\mathbb{Z})$. In a different direction, the Eichler integral $\widetilde{f} = \sum_{n=1}^{\infty} n^{-k+1}a(n)\,q^n$ of a classical cusp form $f = \sum a(n)\,q^n$ of weight k is a mock modular form of weight $2-k$, but of a somewhat generalized kind in which the "shadow" is allowed to be a weakly holomorphic modular form. (This latter fact was observed independently by K.-H. Fricke in Bonn.) Yet another example—actually the oldest—is the generating function of class numbers of imaginary quadratic fields (more precisely, of Hurwitz–Kronecker class numbers), which was shown in [23] to be a mock modular form of weight $\frac{3}{2}$ and level 4 with shadow $\sum q^{n^2}$, although the notion had not yet been formulated at that time.

7. APPLICATIONS

Since the appearance of Zwegers's thesis, Kathrin Bringmann and Ken Ono and their collaborators have developed the theory further and given a number of beautiful applications, a sampling of which we describe in this final section.

Define the *rank* of a partition to be its largest part minus the number of its parts, and for $n, t \in \mathbb{N}$ and $r \in \mathbb{Z}/t\mathbb{Z}$ let $N(r,t;n)$ denote the number of partitions of n with rank congruent to r modulo t. The rank was introduced by Dyson to explain in a natural way the first two of Ramanujan's famous congruences

$$p(5\ell+4) \equiv 0 \pmod 5, \quad p(7\ell+5) \equiv 0 \pmod 7, \quad p(11\ell+6) \equiv 0 \pmod{11}$$

for the partition function $p(n)$: he conjectured (and Atkin and Swinnerton-Dyer later proved) that the ranks of the partitions of an integer congruent to 4 (mod 5) or to 5 (mod 7) are equidistributed modulo 5 or 7, respectively, so that $N(r,5;5\ell+4) = \frac{1}{5}\,p(5\ell+4)$, $N(r,7;7\ell+5) = \frac{1}{7}\,p(7\ell+5)$. (He also conjectured the existence of a further invariant, which he dubbed the "crank," which would explain Ramanujan's third congruence in the same way; this invariant was constructed later by Garvan and

Andrews.) The generating function that counts the number of partitions of given size and rank is given by

$$\mathcal{R}(w;q) \;:=\; \sum_{\lambda} w^{\mathrm{rank}(\lambda)}\, q^{\|\lambda\|} \;=\; \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{m=1}^{n}(1-wq^m)(1-w^{-1}q^m)}\,,$$

where the first sum is over all partitions and $\|\lambda\| = n$ means that λ is a partition of n. Clearly knowing the functions $n \mapsto N(r,t;n)$ for all $r \pmod t$ is equivalent to knowing the specializations of $\mathcal{R}(w;q)$ to all t-th roots of unity $w = e^{2\pi i a/t}$. For $w=-1$, the function $\mathcal{R}(w;q)$ specializes to $f(q)$, the first of Ramanujan's mock theta functions, which is $q^{1/24}$ times a mock modular form of weight $\frac{1}{2}$. Bringmann and Ono [6] generalize this to other roots of unity:

THEOREM 7.1. — *If $\xi \neq 1$ is a root of unity, then $q^{-1/24}\mathcal{R}(\xi;q)$ is a mock modular form of weight $\frac{1}{2}$ with shadow proportional to $\bigl(\xi^{1/2}-\xi^{-1/2}\bigr)\sum_{n\in\mathbb{Z}}\bigl(\frac{12}{n}\bigr)\, n\, \xi^{n/2}\, q^{n^2/24}$.*

Remarks. 1. Note that the choice of square root of ξ in the formula for the shadow does not matter, since n in the non-vanishing terms of the sum is odd.

2. In fact Bringmann and Ono prove the theorem only if the order of ξ is odd. (If it is even, they prove a weaker result showing the modularity only for a group of in general infinite index in $\mathrm{SL}(2,\mathbb{Z})$.) Also, both the formulation and the proof of the theorem in [6] are considerably more complicated than the ones given here.

Proof. The proof is based on the following identity of Gordon and McIntosh [8]:

$$\mathcal{R}(\xi;q) \;=\; \frac{1-\xi}{\prod_{n\geq 1}(1-q^n)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n\, q^{(3n^2+n)/2}}{1-q^n\xi}\,.$$

Using the identity $\frac{1}{1-x} = \frac{1+x+x^2}{1-x^3}$ we can rewrite this as

$$\frac{q^{-1/24}\,\mathcal{R}(e^{2\pi i\alpha};q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}} \;=\; \frac{\eta(3\tau)^3/\eta(\tau)}{\theta(3\alpha;3\tau)} \;+\; e^{-2\pi i\alpha}\mu(3\alpha,-\tau;3\tau) \;+\; e^{2\pi i\alpha}\,\mu(3\alpha,\tau;3\tau)$$

with $\theta(v;\tau)$ and $\mu(u,v;\tau)$ as in §2. The first term on the right is a weakly holomorphic modular form of weight $\frac{1}{2}$ and the other two terms are mock modular forms of weight $\frac{1}{2}$, with shadow proportional to $\sum_{n=1}^{\infty}\bigl(\frac{12}{n}\bigr)\, n\, q^{n^2/12}\, \sin(\pi n\alpha)$, by Theorem 2.1. $\square$

As a corollary of Theorem 7.1 we see that for all $t>0$ and all $r \in \mathbb{Z}/t\mathbb{Z}$ the function

$$\sum_{n\geq 0}\Bigl(N(r,t;n) \;-\; \frac{1}{t}\,p(n)\Bigr)\, q^{n-1/24}$$

is a mock modular form of weight $\frac{1}{2}$, with shadow proportional to

$$\Biggl(\sum_{n\equiv 2r+1 \pmod{2t}} \;-\; \sum_{n\equiv 2r-1 \pmod{2t}}\Biggr)\left(\frac{12}{n}\right) n\, q^{n^2/24}\,.$$

Applying the general principle formulated at the end of §5, one deduces that the sum

$$\sum_{n\in\mathcal{A},\, n\geq 0}\Bigl(N(r,t;n) \;-\; \frac{1}{t}\,p(n)\Bigr)\, q^{n-1/24}$$

is a (weakly holomorphic) modular form for any arithmetic progression $\mathcal{A} \subset \mathbb{Z}$ not containing any number of the form $(1-h^2)/24$ with $h \equiv 2r \pm 1 \pmod{2t}$. In particular, this holds if $\mathcal{A}$ is the set of n with $\left(\frac{1-24n}{p}\right) = -1$ for some prime $p > 3$, and using this and methods from classical modular form theory the authors deduce the following nice result (stated there only for t odd and Q prime to t) about divisibility of the Dyson counting function $N(r,t;n)$:

THEOREM 7.2. — *Let $t > 0$ and Q a prime power prime to* 6. *Then there exist $A > 0$ and $B \in \mathbb{Z}/A\mathbb{Z}$ such that $N(r,t;n) \equiv 0 \pmod{Q}$ for all $n \equiv B \pmod{A}$ and $r \in \mathbb{Z}/t\mathbb{Z}$.*

In a different direction, knowing the modularity properties of mock theta functions permits one to obtain asymptotic results, as well as congruences, for their coefficients. We give two examples. In §2 we described the weak Maass form $\widehat{h}_3(\tau)$ associated to Ramanujan's order 3 mock theta function $f(q)$. In [5], Bringmann and Ono construct a weak Maass–Poincaré series that they can identify (essentially by comparing the modular transformation properties and the asymptotics at cusps) with $\widehat{h}_3(\tau)$, and from this they deduce a Rademacher-type closed formula for the coefficient $\alpha(n)$ of q^n in $f(q)$ of the form

$$\alpha(n) \;=\; \frac{1}{\sqrt{n-1/24}} \sum_{k=1}^{\infty} c_k(n)\, \sinh\Bigl(\frac{\pi}{12k}\sqrt{24n-1}\Bigr)\,,$$

where $c_k(n)$ is an explicit finite exponential sum depending only on n modulo $2k$, e.g., $c_1(n) = (-1)^{n-1}$. This formula had been conjectured by Andrews and Dragonette in 1966 (after Ramanujan had stated, and Dragonette and Andrews had proven, weaker asymptotic statements corresponding to keeping only the first term of this series), but had resisted previous attempts at proof because the circle method, which is the natural tool to use, requires having a very precise description of the behavior of $f(q)$ as q approaches roots of unity, and this in turn requires knowing the modular transformation properties of $h_3(\tau) = q^{-1/24} f(q)$. As a second example, Bringmann [4] was able to use this type of explicit formulas for the coefficients of mock theta functions, combined with Theorem 7.1, to prove an inequality that had been conjectured earlier by Andrews and Lewis, saying that $N(0,3;n)$ is larger than $N(1,3;n)$ for all $n \equiv 1 \pmod 3$ and smaller for all other values of n (except $n = 3$, 9 or 21, where they are equal).

We close by mentioning that mock theta functions (both in the guises of Appell–Lerch sums and of indefinite theta series) also arise in connection with characters of infinite-dimensional Lie superalgebras and conformal field theory [20], and that they also occur in connection with certain quantum invariants of special 3-dimensional manifolds [13]. This suggests that mock modular forms may have interesting applications even outside the domain of pure combinatorics and number theory.

tends to 0 as $\mathrm{Im}(z) \to \infty$, so $f \equiv 0$ by Liouville's theorem.
(4) Replace x by $-x$ in the integral.
(5) Let $g(x) = \frac{1}{\cosh \pi x}$. We first compute the Fourier transform $\mathcal{F}g$ of g: Using Cauchy's formula we get

$$\left(\int_{\mathbf{R}} - \int_{\mathbf{R}+i}\right) \frac{e^{2\pi izx}}{\cosh \pi x}\, dx = 2\pi i \operatorname*{Res}_{x=i/2} \frac{e^{2\pi izx}}{\cosh \pi x} = 2e^{-\pi z},$$

but

$$\int_{\mathbf{R}+i} \frac{e^{2\pi izx}}{\cosh \pi x}\, dx = \int_{\mathbf{R}} \frac{e^{2\pi iz(x+i)}}{\cosh \pi(x+i)}\, dx = -e^{-2\pi z} \int_{\mathbf{R}} \frac{e^{2\pi izx}}{\cosh \pi x}\, dx,$$

so we find

$$(\mathcal{F}g)(z) := \int_{\mathbf{R}} \frac{e^{2\pi izx}}{\cosh \pi x}\, dx = \frac{2e^{-\pi z}}{1+e^{-2\pi z}} = g(z).$$

We see that g is its own Fourier transform! (Note the unusual plus sign in the definition of the Fourier transform).

Let $f_\tau(x) = e^{\pi i\tau x^2}$, $\tau \in \mathcal{H}$. The Fourier transform of f_τ is given by

$$\mathcal{F}f_\tau = \frac{1}{\sqrt{-i\tau}} f_{-\frac{1}{\tau}}.$$

We now see

$$\begin{aligned}\int_{\mathbf{R}} \frac{e^{\pi i\tau x^2+2\pi izx}}{\cosh \pi x}\, dx &= \mathcal{F}(f_\tau \cdot g)(z) = (\mathcal{F}f_\tau) * (\mathcal{F}g)(z) \\ &= \frac{1}{\sqrt{-i\tau}} f_{-\frac{1}{\tau}} * g(z) = \frac{1}{\sqrt{-i\tau}} \int_{\mathbf{R}} \frac{e^{\pi i\frac{-1}{\tau}(z-x)^2}}{\cosh \pi x}\, dx.\end{aligned}$$

This identity holds for $z \in \mathbf{R}$. Since both sides are analytic functions of z, the identity holds for all $z \in \mathbf{C}$. If we replace z by iz we get the desired result.

We may also prove the identity of part (5) by using (1) and (2) to show that $z \mapsto \frac{1}{\sqrt{-i\tau}} e^{\pi iz^2/\tau} h(\frac{z}{\tau}; -\frac{1}{\tau})$ also satisfies the two equations (1) and (2). By uniqueness we get the equation.
(6) Using (1) and (2) we can show that the right hand side, considered as a function of z, also satisfies (1) and (2). The equation now follows from (3). $\square$

1.3 Lerch sums

In this section we will study the function

$$\sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi inv}}{1-e^{2\pi in\tau+2\pi iu}} \qquad (\tau \in \mathcal{H}, v \in \mathbf{C}, u \in \mathbf{C} \setminus (\mathbf{Z}\tau + \mathbf{Z})).$$

This function was also studied by Lerch. The original paper [15] is in Czech and is not very easy to obtain. See [14] for an abstract in German. We will prove elliptic and modular transformation properties of this function in Proposition 1.4 and 1.5 respectively. These results are equivalent to the results found by Lerch.

It is more convenient to normalize the above sum by dividing by the classical Jacobi theta function ϑ. (Lerch did this too.) We will first give, without proof, some standard properties of ϑ. For the theory of ϑ-functions see [19].

Proposition 1.3 *For $z \in \mathbf{C}$ and $\tau \in \mathcal{H}$ define*

$$\vartheta(z) = \vartheta(z;\tau) := \sum_{\nu \in \frac{1}{2}+\mathbf{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu (z+\frac{1}{2})}.$$

Then ϑ satisfies:

(1) $\vartheta(z+1) = -\vartheta(z)$.

(2) $\vartheta(z+\tau) = -e^{-\pi i \tau - 2\pi i z}\vartheta(z)$.

(3) *Up to a multiplicative constant, $z \mapsto \vartheta(z)$ is the unique holomorphic function satisfying (1) and (2).*

(4) $\vartheta(-z) = -\vartheta(z)$.

(5) *The zeros of ϑ are the points $z = n\tau + m$, with $n, m \in \mathbf{Z}$. These are simple zeros.*

(6) $\vartheta(z;\tau+1) = e^{\frac{\pi i}{4}}\vartheta(z;\tau)$.

(7) $\vartheta(\frac{z}{\tau}; -\frac{1}{\tau}) = -i\sqrt{-i\tau}\, e^{\pi i z^2/\tau}\vartheta(z;\tau)$.

(8) $\vartheta(z;\tau) = -iq^{\frac{1}{8}}\zeta^{-\frac{1}{2}} \prod_{n=1}^{\infty}(1-q^n)(1-\zeta q^{n-1})(1-\zeta^{-1}q^n)$, *with $q = e^{2\pi i \tau}$, $\zeta = e^{2\pi i z}$. This is the Jacobi triple product identity.*

(9) $\vartheta'(0;\tau+1) = e^{\frac{\pi i}{4}}\vartheta'(0;\tau)$ *and* $\vartheta'(0;-\frac{1}{\tau}) = (-i\tau)^{3/2}\ \vartheta'(0;\tau)$.

(10) $\vartheta'(0;\tau) = -2\pi\eta(\tau)^3$, *with η as in the introduction.*

We now turn to the normalized version of Lerch's function:

Proposition 1.4 *For $u, v \in \mathbf{C} \setminus (\mathbf{Z}\tau + \mathbf{Z})$ and $\tau \in \mathcal{H}$, define*

$$\mu(u,v) = \mu(u,v;\tau) := \frac{e^{\pi i u}}{\vartheta(v;\tau)} \sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i (n^2+n)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}}.$$

Then μ satisfies:

(1) $\mu(u+1,v) = -\mu(u,v)$,

(2) $\mu(u, v+1) = -\mu(u,v)$,

(3) $\mu(u,v) + e^{-2\pi i(u-v)-\pi i\tau}\mu(u+\tau, v) = -ie^{-\pi i(u-v)-\pi i\tau/4}$,

(4) $\mu(u+\tau, v+\tau) = \mu(u,v)$,

(5) $\mu(-u,-v) = \mu(u,v)$,

(6) *$u \mapsto \mu(u,v)$ is a meromorphic function, with simple poles in the points $u = n\tau + m$ $(n, m \in \mathbf{Z})$, and residue $\frac{-1}{2\pi i}\frac{1}{\vartheta(v)}$ in $u = 0$,*

(7) $\mu(u+z, v+z) - \mu(u,v) = \dfrac{1}{2\pi i}\dfrac{\vartheta'(0)\vartheta(u+v+z)\vartheta(z)}{\vartheta(u)\vartheta(v)\vartheta(u+z)\vartheta(v+z)}$,
for $u, v, u+z, v+z \notin \mathbf{Z}\tau + \mathbf{Z}$,

(8) $\mu(v,u) = \mu(u,v)$.

Proof: (1) is trivial and (2) follows from (1) of Proposition 1.3.
(3) The definition of ϑ gives the following:

$$ie^{-\pi i\tau/4+\pi iv}\vartheta(v) = \sum_{n\in\mathbf{Z}}(-1)^n e^{\pi i(n^2-n)\tau+2\pi inv}$$
$$= \sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i(n^2-n)\tau+2\pi inv}}{1-e^{2\pi in\tau+2\pi iu}}\left(1-e^{2\pi in\tau+2\pi iu}\right)$$
$$= -e^{2\pi iv}\sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi inv}}{1-e^{2\pi in\tau+2\pi i(u+\tau)}} - e^{2\pi iu}\sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi inv}}{1-e^{2\pi in\tau+2\pi iu}}.$$

Dividing both sides by $-e^{\pi iu}\vartheta(v)$, we get the desired result.
(4) Part (2) of Proposition 1.3 gives

$$\mu(u+\tau, v+\tau) = \frac{e^{\pi i(u+\tau)}}{\vartheta(v+\tau)}\sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi in(v+\tau)}}{1-e^{2\pi in\tau+2\pi i(u+\tau)}}$$
$$= -\frac{e^{\pi i(u+\tau)+\pi i\tau+2\pi iv}}{\vartheta(v)}\sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i(n^2+3n)\tau+2\pi inv}}{1-e^{2\pi i(n+1)\tau+2\pi iu}}.$$

Replace n by $n-1$ in the last sum to get the desired result.
(5) If we replace n by $-n$ in the definition of μ we see

$$\mu(u,v) = \frac{e^{\pi iu}}{\vartheta(v)}\sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i(n^2-n)\tau-2\pi inv}}{1-e^{-2\pi in\tau+2\pi iu}}.$$

We multiply by $\frac{-e^{2\pi in\tau-2\pi iu}}{-e^{2\pi in\tau-2\pi iu}}$ to find

$$\mu(u,v) = -\frac{e^{-\pi iu}}{\vartheta(v)}\sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau-2\pi inv}}{1-e^{2\pi in\tau-2\pi iu}}.$$

Now using (4) of Proposition 1.3 we find

$$\mu(u,v) = \mu(-u,-v).$$

(6) From the definition we see that $u \mapsto \mu(u,v)$ has a simple pole if $1-e^{2\pi i n\tau+2\pi i u}=0$, for some $n \in \mathbf{Z}$. So $u \mapsto \mu(u,v)$ has simple poles in the points $u = -n\tau+m$ $(n,m \in \mathbf{Z})$.

The pole in $u=0$ comes from the term $n=0$. We see

$$\lim_{u\to 0} u\,\mu(u,v) = \frac{1}{\vartheta(v)} \lim_{u\to 0} \frac{u}{1-e^{2\pi i u}} = \frac{-1}{2\pi i}\frac{1}{\vartheta(v)}.$$

(7) Consider $f(z) = \vartheta(u+z)\vartheta(v+z)\left(\mu(u+z,v+z)-\mu(u,v)\right)$. Using (1), (2) and (5) of Proposition 1.3, and (1), (2), (4) and (6) of this proposition, we see that f has no poles, a zero for $z=0$, and satisfies

$$\begin{cases} f(z+1) = f(z) \\ f(z+\tau) = e^{-2\pi i\tau - 2\pi i(u+v+2z)} f(z). \end{cases}$$

It follows that the quotient $f(z)/\vartheta(z)\vartheta(u+v+z)$ is a double periodic function with at most one simple pole in each fundamental parallelogram, and hence constant:

$$f(z) = C(u,v)\vartheta(z)\vartheta(u+v+z). \tag{1.1}$$

To compute C we consider $z=-u$. If we take $z=-u$ in (1.1) we find

$$f(-u) = C(u,v)\vartheta(-u)\vartheta(v) = -C(u,v)\vartheta(u)\vartheta(v) \tag{1.2}$$

by (4) of Proposition 1.3.

By definition we have

$$\begin{aligned} f(-u) &= \lim_{z\to -u} \vartheta(u+z)\vartheta(v+z)\left(\mu(u+z,v+z)-\mu(u,v)\right) \\ &= \vartheta(v-u)\cdot \lim_{z\to 0} \vartheta(z)\mu(z,v-u) \\ &= \vartheta(v-u)\cdot \lim_{z\to 0}\frac{\vartheta(z)}{z} \cdot \lim_{z\to 0} z\mu(z,v-u) = -\frac{1}{2\pi i}\vartheta'(0), \end{aligned} \tag{1.3}$$

where we have used (6).

Combining (1.2) and (1.3) gives the desired result.

(8) Take $z=-u-v$ in (7) and use (5) of Proposition 1.3 to find

$$\mu(-v,-u) = \mu(u,v).$$

If we now use (5), we get the desired result. □

Proposition 1.5 *Let μ be as in Proposition 1.4. Then μ satisfies the following modular transformation properties:*

(1) $\mu(u,v;\tau+1)=e^{-\frac{\pi i}{4}}\mu(u,v;\tau)$,

(2) $\displaystyle\frac{1}{\sqrt{-i\tau}}\,e^{\pi i(u-v)^2/\tau}\mu\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right)+\mu(u,v;\tau)=\frac{1}{2i}h(u-v;\tau)$,
with h as in Definition 1.1.

Proof: (1) Use (6) of Proposition 1.3.
(2) Replacing (u,v,z,τ) by $(\frac{u}{\tau},\frac{v}{\tau},\frac{z}{\tau},-\frac{1}{\tau})$ in (7) of Proposition 1.4 and using (7) and (9) of Proposition 1.3 we see that the left hand side depends only on $u-v$, not on u and v separately. Call it $\frac{1}{2i}\tilde{h}(u-v;\tau)$. Using (1) and (3) of Proposition 1.4 we see that $\tilde{h}$ satisfies the two identities (1) and (2) of Proposition 1.2, so if we can prove that $\tilde{h}$ is a holomorphic function, then we may conclude that $\tilde{h}=h$, as desired.

The poles of both $u\mapsto\mu(u,v)$ and $u\mapsto\mu(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau})$ are simple, and occur at $u\in\mathbf{Z}\tau+\mathbf{Z}$, so the only poles of $u\mapsto\tilde{h}(u-v)$ could be simple poles for $u\in\mathbf{Z}\tau+\mathbf{Z}$. Since this is a function of $u-v$ it has no poles at all, and hence is holomorphic.

Alternatively, we can check, using (6) of Proposition 1.4 and (7) of Proposition 1.3, that the residue at $u=0$ vanishes. By (1) and (3) of Proposition 1.4 the residues vanish for all $u\in\mathbf{Z}\tau+\mathbf{Z}$, hence $\tilde{h}$ is holomorphic. □

1.4 A real-analytic Jacobi form?

Definition 1.6 For $z\in\mathbf{C}$ we define

$$E(z)=2\int_0^z e^{-\pi u^2}du=\sum_{n=0}^{\infty}\frac{(-\pi)^n}{n!}\frac{z^{2n+1}}{n+1/2}.$$

This is an odd entire function of z.

Lemma 1.7 *For $z\in\mathbf{R}$ we have*

$$E(z)=\operatorname{sgn}(z)\left(1-\beta(z^2)\right),$$

where

$$\beta(x)=\int_x^{\infty}u^{-\frac{1}{2}}e^{-\pi u}du\qquad(x\in\mathbf{R}_{\geq 0}).$$

Proof: Write $\int_0^z e^{-\pi u^2}du$ as $\operatorname{sgn}(z)\int_0^{|z|}e^{-\pi u^2}du$ and substitute $u=\sqrt{v}$. □

We consider for $u\in\mathbf{C}$ and $\tau\in\mathcal{H}$ the series

$$R(u;\tau)=\sum_{\nu\in\frac{1}{2}+\mathbf{Z}}\left\{\operatorname{sgn}(\nu)-E\Big((\nu+a)\sqrt{2y}\Big)\right\}(-1)^{\nu-\frac{1}{2}}e^{-\pi i\nu^2\tau-2\pi i\nu u},$$

with $y=\operatorname{Im}(\tau)$ and $a=\frac{\operatorname{Im}(u)}{\operatorname{Im}(\tau)}$.

Lemma 1.8 *For all $c, \epsilon > 0$, this series converges absolutely and uniformly on the set $\{u \in \mathbf{C}, \tau \in \mathcal{H} \mid |a| < c, y > \epsilon\}$. The function R it defines is real-analytic and satisfies*

$$\frac{\partial R}{\partial \overline{u}}(u;\tau) = \sqrt{2}y^{-1/2}e^{-2\pi a^2 y}\vartheta(\overline{u}; -\overline{\tau}) \tag{1.4}$$

and

$$\frac{\partial}{\partial \overline{\tau}} R(a\tau - b;\tau) = -\frac{i}{\sqrt{2y}}e^{-2\pi a^2 y} \sum_{\nu \in \frac{1}{2}+\mathbf{Z}} (-1)^{\nu-\frac{1}{2}}(\nu + a)e^{-\pi i \nu^2 \overline{\tau} - 2\pi i \nu(a\overline{\tau} - b)}. \tag{1.5}$$

Proof: We split $\mathrm{sgn}(\nu) - E((\nu + a)\sqrt{2y})$ into the sum of $\mathrm{sgn}(\nu) - \mathrm{sgn}((\nu + a)\sqrt{2y})$ and $\mathrm{sgn}((\nu + a)\sqrt{2y})\beta(2(\nu + a)^2 y)$. We see that $\mathrm{sgn}(\nu) - \mathrm{sgn}((\nu + a)\sqrt{2y})$ is nonzero for only a finite number of values $\nu \in \frac{1}{2} + \mathbf{Z}$ (this number depends on a, but since a is bounded, so is this number). Hence the series

$$\sum_{\nu \in \frac{1}{2}+\mathbf{Z}} \left\{\mathrm{sgn}(\nu) - \mathrm{sgn}\Big((\nu + a)\sqrt{2y}\Big)\right\}(-1)^{\nu-\frac{1}{2}}e^{-\pi i \nu^2 \tau - 2\pi i \nu u}$$

converges absolutely and uniformly.

We can easily see that $0 \leq \beta(x) \leq e^{-\pi x}$ for all $x \in \mathbf{R}_{\geq 0}$, hence

$$\begin{aligned}
&\left|\left\{\mathrm{sgn}\Big((\nu + a)\sqrt{2y}\Big)\beta\Big(2(\nu + a)^2 y\Big)\right\}(-1)^{\nu-\frac{1}{2}}e^{-\pi i \nu^2 \tau - 2\pi i \nu u}\right| \\
&\leq e^{-2\pi(\nu+a)^2 y}\left|e^{-\pi i \nu^2 \tau - 2\pi i \nu u}\right| \\
&= e^{-\pi(\nu+a)^2 y - \pi a^2 y} \leq e^{-\pi(\nu+a)^2 \epsilon}.
\end{aligned}$$

We have the inequality

$$(\nu + a)^2 \geq \frac{1}{2}\nu^2,$$

for $|\nu| \geq \nu_0$, for some $\nu_0 \in \mathbf{R}$ which depends only on c (a is bounded by c). Hence we see that the series

$$\sum_{\nu \in \frac{1}{2}+\mathbf{Z}} \left\{\mathrm{sgn}\Big((\nu + a)\sqrt{2y}\Big)\beta\Big(2(\nu + a)^2 y\Big)\right\}(-1)^{\nu-\frac{1}{2}}e^{-\pi i \nu^2 \tau - 2\pi i \nu u}$$

converges absolutely and uniformly on the given set.

Since R is the (infinite) sum of real-analytic functions, and the series converges absolutely and uniformly, it is real-analytic.

We fix $\tau \in \mathcal{H}$, and determine $u = a\tau - b$ by the coordinates $a, b \in \mathbf{R}$. We see

$$\begin{aligned}
&\left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right) R(a\tau - b;\tau) \\
&= \left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right) \sum_{\nu \in \frac{1}{2}+\mathbf{Z}} \left\{\mathrm{sgn}(\nu) - E\Big((\nu + a)\sqrt{2y}\Big)\right\}(-1)^{\nu-\frac{1}{2}}e^{-\pi i \nu^2 \tau - 2\pi i \nu(a\tau - b)}
\end{aligned}$$

$$
\begin{aligned}
&= -\sqrt{2y}\sum_{\nu\in\frac12+\mathbf{Z}} E'\Big((\nu+a)\sqrt{2y}\Big)(-1)^{\nu-\frac12}e^{-\pi i\nu^2\tau-2\pi i\nu(a\tau-b)}\\
&= -2\sqrt{2y}\sum_{\nu\in\frac12+\mathbf{Z}} e^{-2\pi(\nu+a)^2y}(-1)^{\nu-\frac12}e^{-\pi i\nu^2\tau-2\pi i\nu(a\tau-b)}\\
&= -2\sqrt{2y}e^{-2\pi a^2 y}\sum_{\nu\in\frac12+\mathbf{Z}}(-1)^{\nu-\frac12}e^{-\pi i\nu^2\overline{\tau}-2\pi i\nu(a\overline{\tau}-b)}\\
&= -2i\sqrt{2y}e^{-2\pi a^2 y}\vartheta(a\overline{\tau}-b;-\overline{\tau}),
\end{aligned}
$$

with ϑ as in Proposition 1.3 and the term-by-term differentiation being easily justified. Since $\frac{\partial}{\partial\overline{u}}=\frac{i}{2y}\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)$, this gives the differential equation (1.4). Similarly

$$
\begin{aligned}
&\frac{\partial}{\partial\overline{\tau}}R(a\tau-b;\tau)\\
&=\frac12\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\sum_{\nu\in\frac12+\mathbf{Z}}\Big\{\mathrm{sgn}(\nu)-E\Big((\nu+a)\sqrt{2y}\Big)\Big\}(-1)^{\nu-\frac12}e^{-\pi i\nu^2\tau-2\pi i\nu(a\tau-b)}\\
&=-\frac{i}{2}\frac{1}{\sqrt{2y}}\sum_{\nu\in\frac12+\mathbf{Z}}(\nu+a)\;E'\Big((\nu+a)\sqrt{2y}\Big)\;(-1)^{\nu-\frac12}e^{-\pi i\nu^2\tau-2\pi i\nu(a\tau-b)}\\
&=-\frac{i}{\sqrt{2y}}e^{-2\pi a^2y}\sum_{\nu\in\frac12+\mathbf{Z}}(-1)^{\nu-\frac12}(\nu+a)\;e^{-\pi i\nu^2\overline{\tau}-2\pi i\nu(a\overline{\tau}-b)},
\end{aligned}
$$

proving equation (1.5). □

Proposition 1.9 *The function R has the following elliptic transformation properties:*

(1) $R(u+1)=-R(u)$,

(2) $R(u)+e^{-2\pi iu-\pi i\tau}R(u+\tau)=2e^{-\pi iu-\pi i\tau/4}$,

(3) $R(-u)=R(u)$.

Proof: Part (1) is trivial, and for (3) we replace ν by $-\nu$ in the sum and use the fact that E is an odd function. To prove (2), we start with

$$
\begin{aligned}
&e^{-2\pi iu-\pi i\tau}R(u+\tau)\\
&=e^{-2\pi iu-\pi i\tau}\sum_{\nu\in\frac12+\mathbf{Z}}\Big\{\mathrm{sgn}(\nu)-E\Big((\nu+a+1)\sqrt{2y}\Big)\Big\}(-1)^{\nu-\frac12}e^{-\pi i\nu^2\tau-2\pi i\nu(u+\tau)}\\
&=-\sum_{\nu\in\frac12+\mathbf{Z}}\Big\{\mathrm{sgn}(\nu-1)-E\Big((\nu+a)\sqrt{2y}\Big)\Big\}(-1)^{\nu-\frac12}e^{-\pi i\nu^2\tau-2\pi i\nu u},
\end{aligned}
$$

where we have replaced ν by $\nu - 1$. We now find

$$R(u)+e^{-2\pi iu-\pi i\tau}R(u+\tau) = \sum_{\nu\in\frac{1}{2}+\mathbf{Z}}\Big\{\mathrm{sgn}(\nu)-\mathrm{sgn}(\nu-1)\Big\}(-1)^{\nu-\frac{1}{2}}e^{-\pi i\nu^2\tau-2\pi i\nu u} = 2e^{-\pi iu-\pi i\tau/4},$$

since $\mathrm{sgn}(\nu) - \mathrm{sgn}(\nu - 1)$ is zero for all $\nu \in \frac{1}{2} + \mathbf{Z}$ except for $\nu = \frac{1}{2}$. $\square$

Proposition 1.10 *R has the following modular transformation properties:*

(1) $R(u;\tau+1) = e^{-\frac{\pi i}{4}}R(u;\tau)$,

(2) $\dfrac{1}{\sqrt{-i\tau}}\,e^{\pi iu^2/\tau}R\Big(\dfrac{u}{\tau};-\dfrac{1}{\tau}\Big) + R(u;\tau) = h(u;\tau)$.

Proof: Part (1) is trivial. The left hand side of (2) we call $\tilde{h}(u;\tau)$. Using (1) and (2) of Proposition 1.9 we can see that $\tilde{h}$ satisfies:

$$\begin{cases}\tilde{h}(u)+\tilde{h}(u+1) = \frac{2}{\sqrt{-i\tau}}\,e^{\pi i(u+\frac{1}{2})^2/\tau},\\ \tilde{h}(u)+e^{-2\pi iu-\pi i\tau}\tilde{h}(u+\tau) = 2e^{-\pi iu-\pi i\tau/4}.\end{cases}$$

Part (3) of Proposition 1.2 determines h as the unique holomorphic function with these properties. This reduces the proof to showing that $\tilde{h}$ is a holomorphic function of u.

We fix $\tau \in \mathcal{H}$, and determine $u = a\tau - b$ by the coordinates $a, b \in \mathbf{R}$ (this implies $a = \frac{\mathrm{Im}(u)}{\mathrm{Im}(\tau)}$ as in Lemma 1.8). Since $\frac{\partial}{\partial\overline{u}} = \frac{i}{2y}\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)$, we have to show that

$$\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)\tilde{h}(a\tau-b;\tau) = 0.$$

According to Lemma 1.8 we have

$$\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R(a\tau-b;\tau) = -2i\sqrt{2y}e^{-2\pi a^2y}\vartheta(a\overline{\tau}-b;-\overline{\tau}) \tag{1.6}$$

We have

$$\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R\left(\frac{a\tau-b}{\tau};-\frac{1}{\tau}\right) = \tau\left(\frac{\partial}{\partial b}+\frac{1}{\tau}\frac{\partial}{\partial a}\right)R\left(a-\frac{b}{\tau};-\frac{1}{\tau}\right).$$

Up to a factor τ this is the same as $\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R(a\tau-b;\tau)$, with (a, b, τ) replaced by $(b, -a, -\frac{1}{\tau})$. Hence by (1.6) we find

$$\begin{aligned}\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R\left(\frac{a\tau-b}{\tau};-\frac{1}{\tau}\right) &= -2i\tau\sqrt{2y'}e^{-2\pi b^2y'}\vartheta\left(-\frac{b}{\overline{\tau}}+a;\frac{1}{\overline{\tau}}\right)\\ &= 2i\tau\sqrt{2y'}e^{-2\pi b^2y'}\vartheta\left(-\frac{a\overline{\tau}-b}{\overline{\tau}};\frac{1}{\overline{\tau}}\right),\end{aligned}$$

with $y' = \mathrm{Im}(-\frac{1}{\tau}) = \frac{y}{\tau\overline{\tau}}$. In the last step we have used (4) of Proposition 1.3.
If we now use (7) of Proposition 1.3, with $z = a\overline{\tau} - b$ and τ replaced by $-\overline{\tau}$, we see that this equals

$$\begin{aligned} &2i\tau\sqrt{2y'}e^{-2\pi b^2 y'} \cdot -i\sqrt{i\overline{\tau}}e^{-\pi i(a\overline{\tau}-b)^2/\overline{\tau}}\vartheta(a\overline{\tau}-b;-\overline{\tau}) \\ &\quad = 2i\sqrt{2y}\sqrt{-i\tau}e^{-\pi i(a\tau-b)^2/\tau}e^{-2\pi a^2 y}\vartheta(a\overline{\tau}-b;-\overline{\tau}). \end{aligned} \tag{1.7}$$

Using (1.6) and (1.7) we find

$$\begin{aligned} &\left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right)\tilde{h}(a\tau - b;\tau) \\ &\qquad = \frac{1}{\sqrt{-i\tau}}e^{\pi i(a\tau-b)^2/\tau}\left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right)R\left(\frac{a\tau-b}{\tau};-\frac{1}{\tau}\right) \\ &\qquad\quad + \left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right)R(a\tau-b;\tau) = 0. \end{aligned}$$

We have established the fact that $\tilde{h}$ is holomorphic, and hence equals h. □

In the next theorem we combine the properties of μ and R to find a function $\tilde{\mu}$ which is no longer meromorphic, but has better elliptic and modular transformation properties than μ.

Theorem 1.11 *We set*

$$\tilde{\mu}(u,v;\tau) = \mu(u,v;\tau) + \frac{i}{2}R(u-v;\tau), \tag{1.8}$$

then

(1) $\tilde{\mu}(u + k\tau + l, v + m\tau + n) = (-1)^{k+l+m+n}e^{\pi i(k-m)^2\tau + 2\pi i(k-m)(u-v)}\tilde{\mu}(u,v)$, *for* $k,l,m,n \in \mathbf{Z}$,

(2) $\tilde{\mu}\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = v(\gamma)^{-3}(c\tau+d)^{\frac{1}{2}}e^{-\pi ic(u-v)^2/(c\tau+d)}\tilde{\mu}(u,v;\tau)$, *for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$, *with* $v(\gamma) = \eta(\frac{a\tau+b}{c\tau+d})/\left((c\tau+d)^{\frac{1}{2}}\eta(\tau)\right)$

(3) $\tilde{\mu}(-u,-v) = \tilde{\mu}(v,u) = \tilde{\mu}(u,v)$,

(4) $\tilde{\mu}(u+z,v+z) - \tilde{\mu}(u,v) = \dfrac{1}{2\pi i}\dfrac{\vartheta'(0)\vartheta(u+v+z)\vartheta(z)}{\vartheta(u)\vartheta(v)\vartheta(u+z)\vartheta(v+z)}$, *for* $u, v, u+z, v+z \notin \mathbf{Z}\tau + \mathbf{Z}$,

(5) $u \mapsto \tilde{\mu}(u,v)$ *has singularities in the points* $u = n\tau + m$ $(n,m \in \mathbf{Z})$*. Furthermore we have* $\lim_{u\to 0} u\tilde{\mu}(u,v) = \frac{-1}{2\pi i}\frac{1}{\vartheta(v)}$.

Remark 1.12 Parts (1) and (2) of the theorem say that the function $\tilde{\mu}$ transforms like a two-variable Jacobi form of weight $\frac{1}{2}$ and index $\left(\begin{smallmatrix}-1 & 1\\ 1 & -1\end{smallmatrix}\right)$ (for the theory of Jacobi forms, see [9], where, however, only Jacobi forms of one variable are considered). Furthermore we can find several differential equations satisfied by $\tilde{\mu}$. Therefore we would like to call this function a real-analytic Jacobi form. However, in the literature I haven't been able to find a satisfying definition of a real-analytic Jacobi form. I intend to return to this problem in the future.

Remark 1.13 All three function in (1.8) have a property that the other two do not have: $\tilde{\mu}$ transforms well (like a Jacobi form), μ is meromorphic and $u, v \mapsto R(u-v)$ depends only on $u-v$.

Proof: (1) Using the first four parts of Proposition 1.4 and the first two of Proposition 1.9 we find

$$\begin{aligned}
\tilde{\mu}(u+1,v) &= -\tilde{\mu}(u,v),\\
\tilde{\mu}(u,v+1) &= -\tilde{\mu}(u,v),\\
\tilde{\mu}(u+\tau,v) &= -e^{2\pi i(u-v)+\pi i\tau}\tilde{\mu}(u,v),\\
\tilde{\mu}(u,v+\tau) &= -e^{2\pi i(v-u)+\pi i\tau}\tilde{\mu}(u,v).
\end{aligned}$$

Combining these equations we get the desired result.
(2) Using Proposition 1.5 and Proposition 1.10 we find

$$\begin{aligned}
\tilde{\mu}(u,v;\tau+1) &= e^{-\frac{\pi i}{4}}\tilde{\mu}(u,v;\tau)\\
\tilde{\mu}\Big(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\Big) &= -\sqrt{-i\tau}e^{-\pi i(u-v)^2/\tau}\tilde{\mu}(u,v;\tau)
\end{aligned}$$

Set $m(u,v;\tau) := \vartheta(u-v;\tau)\tilde{\mu}(u,v;\tau)$. Using (6) and (7) of Proposition 1.3 we see

$$\begin{aligned}
m(u,v;\tau+1) &= m(u,v;\tau)\\
m\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right) &= \tau\ m(u,v;\tau)
\end{aligned}$$

and so

$$m\left(\frac{u}{c\tau+d},\frac{v}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)\ m(u,v;\tau),$$

for all $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$. Hence

$$\tilde{\mu}\left(\frac{u}{c\tau+d},\frac{v}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)\frac{\vartheta(u-v;\tau)}{\vartheta\left(\frac{u-v}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)}\tilde{\mu}(u,v;\tau). \tag{1.9}$$

From (6) and (7) of Proposition 1.3 we find

$$\vartheta\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = \chi(\gamma)\sqrt{c\tau+d}\ e^{\pi icz^2/(c\tau+d)}\vartheta(z;\tau), \tag{1.10}$$

with $\chi(\gamma)$ some eighth root of unity. Applying $\frac{d}{dz}\big|_{z=0}$ on both sides gives

$$\vartheta'\left(0; \frac{a\tau+b}{c\tau+d}\right) = \chi(\gamma)(c\tau+d)^{\frac{3}{2}}\vartheta'(0;\tau).$$

Using (10) of Proposition 1.3 we find

$$\chi(\gamma) = v(\gamma)^3$$

If we combine this with (1.9) and (1.10) we get the desired result.
(3) Using (5) of Proposition 1.4 and (3) of Proposition 1.9 we find

$$\tilde{\mu}(-u,-v) = \tilde{\mu}(u,v)$$

Using (8) of Proposition 1.4 and (3) of Proposition 1.9 we find

$$\tilde{\mu}(v,u) = \tilde{\mu}(u,v)$$

(4) This follows directly from (7) of Proposition 1.4.
(5) R has no singularities, so the singularities come from μ. The location and nature of these singularities is already given in (6) of Proposition 1.4. □

1.5 Period integrals of weight 3/2 unary theta functions

In this section we will rewrite h in terms of the period integral of a unary theta function of weight 3/2.

To state the main result we need the following definition:

Definition 1.14 Let $a, b \in \mathbf{R}$ and $\tau \in \mathcal{H}$ then

$$g_{a,b}(\tau) := \sum_{\nu \in a+\mathbf{Z}} \nu e^{\pi i \nu^2 \tau + 2\pi i \nu b}.$$

The function $g_{a,b}$ is a unary theta function.

Proposition 1.15 *$g_{a,b}$ satisfies:*

(1) $g_{a+1,b}(\tau) = g_{a,b}(\tau)$

(2) $g_{a,b+1}(\tau) = e^{2\pi i a} g_{a,b}(\tau)$

(3) $g_{-a,-b}(\tau) = -g_{a,b}(\tau)$

(4) $g_{a,b}(\tau+1) = e^{-\pi i a(a+1)} g_{a,a+b+\frac{1}{2}}(\tau)$