6.5.30

$$\gamma(a,x+y)-\gamma(a,x)$$
$$=e^{-x}x^{a-1}\sum_{n=0}^{\infty}\frac{(a-1)(a-2)\ldots(a-n)}{x^n}[1-e^{-y}e_n(y)]$$
$$(|y|<|x|)$$

Continued Fraction

6.5.31

$$\Gamma(a,x)=e^{-x}x^a\left(\frac{1}{x+}\,\frac{1-a}{1+}\,\frac{1}{x+}\,\frac{2-a}{1+}\,\frac{2}{x+}\cdots\right)$$
$$(x>0,|a|<\infty)$$

Asymptotic Expansions

6.5.32

$$\Gamma(a,z)\sim z^{a-1}e^{-z}\left[1+\frac{a-1}{z}+\frac{(a-1)(a-2)}{z^2}+\ldots\right]$$
$$\left(z\to\infty \text{ in } |\arg z|<\frac{3\pi}{2}\right)$$

Suppose $R_n(a,z)=u_{n+1}(a,z)+\ \ldots$ is the remainder after n terms in this series. Then if a,z are real, we have for $n>a-2$

$$|R_n(a,z)|\leq|u_{n+1}(a,z)|$$

and sign $R_n(a,z)=\text{sign } u_{n+1}(a,z)$.

6.5.33 $$\gamma(a,z)\sim\sum_{n=0}^{\infty}\frac{(-1)^n z^{a+n}}{(a+n)n!} \qquad (a\to+\infty)$$

6.5.34 $$\lim_{n\to\infty}\frac{e_n(\alpha n)}{e^{\alpha n}}=\begin{cases}0 \text{ for } \alpha>1\\ \frac{1}{2} \text{ for } \alpha=1\\ 1 \text{ for } 0\leq\alpha<1\end{cases}$$

6.5.35

$$\Gamma(z+1,z)\sim e^{-z}z^z\left(\sqrt{\frac{\pi}{2}}z^{\frac{1}{2}}+\frac{2}{3}+\frac{\sqrt{2\pi}}{24}\frac{1}{z^{\frac{1}{2}}}+\cdots\right)$$
$$(z\to\infty \text{ in } |\arg z|<\tfrac{1}{2}\pi)$$

Definite Integrals

6.5.36

$$\int_0^\infty e^{-at}\Gamma(b,ct)\,dt=\frac{\Gamma(b)}{a}\left[1-\frac{c^b}{(a+c)^b}\right] \quad *$$
$$(\mathscr{R}(a+c)>0, \mathscr{R}b>-1)$$

6.5.37

$$\int_0^\infty t^{a-1}\Gamma(b,t)\,dt=\frac{\Gamma(a+b)}{a}$$
$$(\mathscr{R}(a+b)>0,\quad \mathscr{R}a>0)$$

6.6. Incomplete Beta Function

6.6.1 $$B_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}dt$$

6.6.2 $$I_x(a,b)=B_x(a,b)/B(a,b)$$

For statistical applications, see **26.5**.

Symmetry

6.6.3 $$I_x(a,b)=1-I_{1-x}(b,a)$$

Relation to Binomial Expansion

6.6.4 $$I_p(a,n-a+1)=\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}$$

For binomial distribution, see **26.1**.

Recurrence Formulas

6.6.5 $$I_x(a,b)=xI_x(a-1,b)+(1-x)I_x(a,b-1)$$

6.6.6 $$(a+b-ax)I_x(a,b)=a(1-x)I_x(a+1,b-1)+bI_x(a,b+1) \quad *$$

6.6.7 $$(a+b)I_x(a,b)=aI_x(a+1,b)+bI_x(a,b+1)$$

Relation to Hypergeometric Function

6.6.8 $$B_x(a,b)=a^{-1}x^aF(a,1-b;a+1;x)$$

Numerical Methods

6.7. Use and Extension of the Tables

Example 1. Compute $\Gamma(6.38)$ to 8S. Using the recurrence relation **6.1.16** and **Table 6.1** we have,

$$\Gamma(6.38)=[(5.38)(4.38)(3.38)(2.38)(1.38)]\Gamma(1.38)$$
$$=232.43671.$$

Example 2. Compute $\ln\Gamma(56.38)$, using **Table 6.4** and linear interpolation in f_2. We have

$$\ln\Gamma(56.38)=(56.38-\tfrac{1}{2})\ln(56.38)-(56.38)+f_2(56.38)$$

The error of linear interpolation in the table of the function f_2 is smaller than 10^{-7} in this region. Hence, $f_2(56.38)=.92041\ 67$ and $\ln\Gamma(56.38)=169.85497\ 42$.

Direct interpolation in **Table 6.4** of $\log_{10}\Gamma(n)$ eliminates the necessity of employing logarithms. However, the error of linear interpolation is .002 so that $\log_{10}\Gamma(n)$ is obtained with a relative error of 10^{-5}.

*See page II.

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Inventiones mathematicae

The $f(q)$ mock theta function conjecture and partition ranks

Kathrin Bringmann, Ken Ono[⋆]

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706, USA
(e-mail: bringman@math.wisc.edu/ono@math.wisc.edu)

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Abstract. In 1944, Freeman Dyson initiated the study of ranks of integer partitions. Here we solve the classical problem of obtaining formulas for $N_e(n)$ (resp. $N_o(n)$), the number of partitions of n with even (resp. odd) rank. Thanks to Rademacher's celebrated formula for the partition function, this problem is equivalent to that of obtaining a formula for the coefficients of the mock theta function $f(q)$, a problem with its own long history dating to Ramanujan's last letter to Hardy. Little was known about this problem until Dragonette in 1952 obtained asymptotic results. In 1966, G.E. Andrews refined Dragonette's results, and conjectured an exact formula for the coefficients of $f(q)$. By constructing a weak Maass-Poincaré series whose "holomorphic part" is $q^{-1}f(q^{24})$, we prove the Andrews-Dragonette conjecture, and as a consequence obtain the desired formulas for $N_e(n)$ and $N_o(n)$.

1. Introduction and statement of results

A *partition* of a positive integer n is any non-increasing sequence of positive integers whose sum is n. As usual, let $p(n)$ denote the number of partitions of n. The partition function $p(n)$ has the well known generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

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Mathematics Subject Classification (2000): 11P82, 05A17 which is easily seen to coincide with $q^{\frac{1}{24}}/\eta(z)$, where

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad \left(q := e^{2\pi i z}\right)$$

is Dedekind's eta-function, a weight $1/2$ modular form. Rademacher famously employed this modularity to perfect the Hardy-Ramanujan asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}} \tag{1.1}$$

to obtain his exact formula for $p(n)$ (for example, see Chap. 14 of [22]).

To state his formula, let $I_s(x)$ be the usual I-Bessel function of order s, and let $e(x) := e^{2\pi i x}$. Furthermore, if $k \geq 1$ and n are integers, then let

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \chi_{12}(x) \cdot e\left(\frac{x}{12k}\right), \tag{1.2}$$

where the sum runs over the residue classes modulo $24k$, and where

$$\chi_{12}(x) := \left(\frac{12}{x}\right). \tag{1.3}$$

If n is a positive integer, then one version of Rademacher's formula reads

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \tag{1.4}$$

In an effort to provide a combinatorial explanation of Ramanujan's congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

Dyson introduced [17] the so-called "rank" of a partition, a delightfully simple statistic. The rank of a partition is defined to be its largest part minus the number of its parts. In this famous paper [17], Dyson conjectured that ranks could be used to "explain" the congruences above with modulus 5 and 7. More precisely, he conjectured that the partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7)[1]. He further postulated the existence of another statistic,

[1] A short calculation reveals that this phenomenon cannot hold modulo 11.

the so-called "crank"[2], which allegedly would explain all three congruences. In 1954, Atkin and Swinnerton-Dyer proved [9] Dyson's rank conjectures, consequently cementing the central role that ranks play in the theory of partitions.

To study ranks, it is natural to investigate a generating function. If $N(m,n)$ denotes the number of partitions of n with rank m, then it is well known that

$$1+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n},$$

where

$$(a;q)_n := (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right).$$

Therefore, if $N_e(n)$ (resp. $N_o(n)$) denotes the number of partitions of n with even (resp. odd) rank, then by letting $z=-1$ we obtain

(1.5)
$$1+\sum_{n=1}^{\infty}(N_e(n)-N_o(n))q^n = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

We address the following classical problem: Determine exact formulas for $N_e(n)$ and $N_o(n)$. In view of (1.4) and (1.5), since

$$p(n) = N_e(n)+N_o(n),$$

this question is equivalent to the problem of deriving exact formulas for the coefficients $\alpha(n)$ of the series

$$\begin{aligned} f(q) = 1+\sum_{n=1}^{\infty}\alpha(n)q^n &:= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} \\ &= 1+q-2q^2+3q^3-3q^4+3q^5-5q^6+\cdots. \end{aligned} \tag{1.6}$$

The series $f(q)$ is one of the third order mock theta functions Ramanujan defined in his last letter to Hardy dated January 1920 (see pp. 127–131 of [23]). Surprisingly, very little is known about mock theta functions in general. For example, Ramanujan's claims about their analytic properties remain open. There is even debate concerning the rigorous definition of a mock theta function, which, of course, precedes the formulation of one's order. Despite these seemingly problematic issues, Ramanujan's mock theta functions possess many striking properties, and they have been the subject of an astonishing number of important works, (for example, see [2–5,7,

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[2] In 1988, Andrews and Garvan [8] found the crank, and they indeed confirmed Dyson's speculation that it "explains" the three Ramanujan congruences above. Recent work of Mahlburg [21] establishes that the Andrews-Dyson-Garvan crank plays an even more central role in the theory partition congruences. His work concerns partition congruences modulo arbitrary powers of all primes ≥ 5.

12–14, 16, 19, 26, 27] to name a few). This activity realizes G.N. Watson's[3] prophetic words:

"*Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine (a.k.a. Persephone)...*"

Returning to $f(q)$, the problem of estimating its coefficients $\alpha(n)$ has a long history, one which even precedes Dyson's definition of partition ranks. Indeed, Ramanujan's last letter to Hardy already includes the claim that

$$\alpha(n) = (-1)^{n-1}\frac{\exp\left(\pi\sqrt{\frac{n}{6}-\frac{1}{144}}\right)}{2\sqrt{n-\frac{1}{24}}} + O\left(\frac{\exp\left(\frac{1}{2}\pi\sqrt{\frac{n}{6}-\frac{1}{144}}\right)}{\sqrt{n-\frac{1}{24}}}\right).$$

Typical of his writings, Ramanujan offered no proof of this claim. Dragonette finally proved this claim in her 1951 Ph.D. thesis [16] written under the direction of Rademacher. In his 1964 Ph.D. thesis, also written under Rademacher, Andrews improved upon Dragonette's work, and he proved[4] that

$$\begin{aligned}\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{[\sqrt{n}]} \frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor} A_{2k}\left(n-\frac{k(1+(-1)^k)}{4}\right)}{k}\\ \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) + O(n^{\epsilon}).\end{aligned} \tag{1.7}$$

This result falls short of the problem of obtaining an exact formula for $\alpha(n)$, and as a consequence represents the obstruction to obtaining formulas for $N_e(n)$ and $N_o(n)$. In his plenary address "Partitions: At the interface of q-series and modular forms", delivered at the Millenial Number Theory Conference at the University of Illinois in 2000, Andrews highlighted this classical problem by promoting his conjecture[5] of 1966 (see p. 456 of [2], and Sect. 5 of [4]) for the coefficients $\alpha(n)$.

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[3] This quote is taken from Watson's 1936 Presidential Address to the London Mathematical Society entitled "The final problem: An account of the mock theta functions" (see p. 80 of [26]).

[4] This is a reformulation of Theorem 5.1 of [2] using the identity $I_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cdot \sinh(z)$.

[5] This conjecture is suggested as a speculation by Dragonette in [16].

Conjecture (Andrews-Dragonette). *If n is a positive integer, then*

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right). \tag{1.8}$$

The following theorem gives the first exact formulas for the coefficients of a mock theta function.

Theorem 1.1. *The Andrews-Dragonette Conjecture is true.*

Remark. Since $N_e(n) = (p(n) + \alpha(n))/2$ and $N_o(n) = (p(n) - \alpha(n))/2$, Theorem 1.1, combined with (1.4), provides the desired formulas for $N_e(n)$ and $N_o(n)$.

To prove Theorem 1.1, we use recent work of Zwegers [27] which nicely packages Watson's transformation properties of $f(q)$ in terms of real analytic vector valued modular forms. Loosely speaking, Zwegers "completes" $q^{-1/24}f(q)$ to obtain a three dimensional real analytic vector valued modular form of weight $1/2$. We recall his results in Sect. 2. To prove Theorem 1.1, we realize the q-series, whose coefficients are given by the infinite series expansions in (1.8), as the "holomorphic part" of a weak Maass form. This form is defined in Sect. 3.1 as a specialization of a Poincaré series, and in Sect. 3.2 we confirm that the coefficients of its holomorphic part are indeed in agreement with the expansions in (1.8). To complete the proof of Theorem 1.1, it then suffices to establish a suitable identity relating this weak Maass form to Zwegers' form. We achieve this in Sect. 5 by analyzing the image of these forms under the differential operator $\xi_{\frac{1}{2}}$ (defined in Sect. 5). This task requires the Serre-Stark Basis Theorem for weight $1/2$ holomorphic modular forms, and estimates on sums of the A_{2k}-sums derived in Sect. 4.

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2. Modular transformation properties of $q^{-\frac{1}{24}}f(q)$

Here we recall what is known about $q^{-1/24}f(q)$ and its modular transformation properties. An important first step was already achieved by G.N. Watson in [26]. Although $f(q)$ is not the Fourier expansion of a usual meromorphic modular form, in this classic paper Watson determined modular transformation properties which strongly suggested that $f(q)$ is a "piece" of a real analytic modular form, as opposed to a classical meromorphic modular form.

Watson's modular transformation formulas are very complicated, and are difficult to grasp at first glance. In particular, the collection of these formulas involve another third order mock theta function, as well as terms arising from Mordell integrals. Recent work of Zwegers [27] nicely packages Watson's results in the modern language of real analytic vector valued modular forms. We recall some of his results as they pertain to $f(q)$.

We begin by fixing notation. Let $\omega(q)$ be the third order mock theta function

$$\begin{aligned}\omega(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} \\ &= \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} \\ &\quad + \frac{q^{12}}{(1-q)^2(1-q^3)^2(1-q^5)^2} + \cdots .\end{aligned} \tag{2.1}$$

If $q := e^{2\pi i z}$, where $z \in \mathbb{H}$, then define the vector valued function $F(z)$ by

$$\begin{aligned}F(z) &= (F_0(z),\ F_1(z),\ F_2(z))^T \\ &:= (q^{-\frac{1}{24}} f(q),\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T.\end{aligned} \tag{2.2}$$

Similarly, let $G(z)$ be the vector valued non-holomorphic function defined by

$$\begin{aligned}G(z) &= (G_0(z),\ G_1(z),\ G_2(z))^T \\ &:= 2i\sqrt{3}\int_{-\bar{z}}^{i\infty} \frac{(g_1(\tau),\ g_0(\tau),\ -g_2(\tau))^T}{\sqrt{-i(\tau+z)}}\, d\tau,\end{aligned} \tag{2.3}$$

where the $g_i(\tau)$ are the cuspidal weight $3/2$ theta functions

$$\begin{aligned}g_0(\tau) &:= \sum_{n=-\infty}^{\infty} (-1)^n \left(n+\frac{1}{3}\right) e^{3\pi i \left(n+\frac{1}{3}\right)^2 \tau}, \\ g_1(\tau) &:= -\sum_{n=-\infty}^{\infty} \left(n+\frac{1}{6}\right) e^{3\pi i \left(n+\frac{1}{6}\right)^2 \tau}, \\ g_2(\tau) &:= \sum_{n=-\infty}^{\infty} \left(n+\frac{1}{3}\right) e^{3\pi i \left(n+\frac{1}{3}\right)^2 \tau}.\end{aligned} \tag{2.4}$$

Using these vector valued functions, Zwegers defines $H(z)$ by

$$H(z) := F(z) - G(z). \tag{2.5}$$

The following description of $H(z)$ is the main result of [27].

Theorem 2.1 (Zwegers). *The function $H(z)$ is a vector valued real analytic modular form of weight $1/2$ satisfying*

$$H(z+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(z),$$

$$H(-1/z) = \sqrt{-iz} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(z),$$

where $\zeta_n := e^{2\pi i/n}$. Furthermore, $H(z)$ is an eigenfunction of the Casimir operator $\Omega_{\frac{1}{2}} := -4y^2 \frac{\partial^2}{\partial z \partial \overline{z}} + iy\frac{\partial}{\partial \overline{z}} + \frac{3}{16}$ with eigenvalue $\frac{3}{16}$, where $z = x+iy$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

We give a consequence of Zwegers' result in terms of weak Maass forms of half-integral weight. To make this precise, suppose that $k \in \frac{1}{2} + \mathbb{Z}$. If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{2.6}$$

The weight k Casimir operator is defined by

$$\Omega_k := -4y^2 \frac{\partial^2}{\partial z \partial \overline{z}} + 2iky\frac{\partial}{\partial \overline{z}} + \frac{2k-k^2}{4}. \tag{2.7}$$

Notice that the weight k hyperbolic Laplacian

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{2.8}$$

is related to the Casimir operator Ω_k by the simple identity

$$\Omega_k = \Delta_k + \frac{2k-k^2}{4},$$

where $z = x + iy$ with $x, y \in \mathbb{R}$.

Following Bruinier and Funke, we now recall the notion [11] of a weak Maass form of half-integral weight.

Definition 2.2. *Suppose that $k \in \frac{1}{2} + \mathbb{Z}$, N is a positive integer, and that ψ is a Dirichlet character with modulus $4N$. A weak Maass form of weight k on $\Gamma_0(4N)$ with Nebentypus character ψ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following:*

(1) *For all* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$ *and all* $z \in \mathbb{H}$, *we have*[6]

$$f(Az) = \psi(d) \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \, (cz+d)^k \, f(z).$$

(2) *We have that* $\Delta_k f = 0$.
(3) *The function* $f(z)$ *has at most linear exponential growth at all the cusps of* $\Gamma_0(4N)$.

Before we state a useful corollary to Theorem 2.1, we recall certain facts about Dedekind sums and their role in describing the modular transformation properties of Dedekind's eta-function. If $x \in \mathbb{R}$, then let

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{for } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

For coprime integers c and d, let $s(d,c)$ be the usual Dedekind sum

$$s(d,c) := \sum_{\mu \pmod{c}} \left(\left(\frac{\mu}{c}\right)\right)\left(\left(\frac{d\mu}{c}\right)\right).$$

In terms of these sums, we define $\omega_{d,c}$ by

$$\omega_{d,c} := e^{\pi i s(d,c)}. \tag{2.9}$$

Using this notation, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, with $c > 0$, then we have[7]

$$\eta\left(\frac{az+b}{cz+d}\right) = i^{-\frac{1}{2}} \cdot \omega_{-d,c} \cdot \exp\left(\frac{\pi i(a+d)}{12c}\right) \cdot (cz+d)^{\frac{1}{2}} \cdot \eta(z). \tag{2.10}$$

Remark. The exponential sums defined by (1.2) may also be described in terms of Dedekind sums. In particular, if $k \geq 1$ and n are integers, then $A_k(n)$ is also given by (see (120.5) on p. 272 of [22])

$$A_k(n) = \sum_{x \pmod{k}^*} \omega_{-x,k} \cdot e\left(\frac{nx}{k}\right), \tag{2.11}$$

where the sum runs over the primitive residue classes x modulo k.

Theorem 2.1 implies the following convenient corollary.

Corollary 2.3. *The function* $M(z) := F_0(24z) - G_0(24z)$ *is a weak Maass form of weight* $1/2$ *on* $\Gamma_0(144)$ *with Nebentypus character* χ_{12}.

[6] This transformation law agrees with Shimura's notion of half-integral weight modular forms [25].
[7] This formula is easily derived from the formulas appearing in Chap. 9 of [22].

Sketch of the proof. It is well known that $\eta(24z)$ is a cusp form of weight $1/2$ for the group $\Gamma_0(576)$ with Nebentypus χ_{12}. For integers n, we obviously have

$$\widetilde{M}(z+n) = e(-n/24) \cdot \widetilde{M}(z),$$

where $\widetilde{M}(z) = F_0(z) - G_0(z)$. Therefore, to prove the claim it suffices to compare the automorphy factors of $M(z)$ with those appearing in (2.10) when interpreted for $\eta(24z)$. By Theorem 2.1 (see also Theorem[8] 2.2 of [2]), if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, with $c > 0$, then

$$\widetilde{M}\left(\frac{az+b}{cz+d}\right) = i^{-\frac{1}{2}} \cdot \omega_{-d,c}^{-1} \cdot (-1)^{\frac{c+1+ad}{2}} \cdot e\left(-\frac{a+d}{24c} - \frac{a}{4} + \frac{3dc}{8}\right)(cz+d)^{\frac{1}{2}} \cdot \widetilde{M}(z).$$

In view of these formulas, it is then straightforward to verify that the automorphy factors above agree with those for $\eta(24z)$ when restricted to $\Gamma_0(576)$. Consequently, it then follows that $M(z)$ is also a weak Maass form of weight $1/2$ on $\Gamma_0(576)$ with Nebentypus χ_{12}.

In order to verify that $M(z)$ satisfies the desired transformation law under $\Gamma_0(144)$, it suffices to check that its images under the representatives for the non-trivial classes in $\Gamma_0(144)/\Gamma_0(576)$ behave properly. For example, if $H(z) = (H_0(z), H_1(z), H_2(z))^T$, then Theorem 2.1 gives

$$\begin{aligned} M\left(\frac{z}{288z+1}\right) &= H_0\left(\frac{24z}{12(24z)+1}\right) \\ &= \left(-i\left(\frac{12(24z)+1}{-24z}\right)\right)^{\frac{1}{2}} \cdot H_1\left(\frac{12(24z)+1}{-24z}\right) \\ &= \left(-i\left(\frac{12(24z)+1}{-24z}\right)\right)^{\frac{1}{2}} \cdot H_1\left(-\frac{1}{24z}\right) \\ &= (288z+1)^{\frac{1}{2}} \cdot H_0(24z) = (288z+1)^{\frac{1}{2}} \cdot M(z). \end{aligned}$$

This is the desired transformation law under $z \to \frac{1}{288z+1}$. The analogous computation for the remaining representatives completes the proof. $\square$

Remark. Let $\psi \pmod 6$ be the Dirichlet character

$$\psi(n) := \begin{cases} 1 & \text{if } n \equiv 1 \pmod 6, \\ -1 & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

The theta-function

$$\vartheta(\psi; z) := \sum_{n=1}^{\infty} \psi(n)\, n\, q^{n^2}$$

[8] There is a minor typo in the displayed formula which is easily found when reading the proof.

is well known to be a cusp form of weight 3/2 on $\Gamma_0(144)$ with Nebentypus χ_{12} (for example, see [25]). One easily sees that

$$g_1(24z) = -\frac{1}{6} \cdot \vartheta(\psi; z).$$

In fact, one could also use this fact to deduce Corollary 2.3.

3. The Poincaré series $P_k(s; z)$

Here we construct a Poincaré series which has the property that the Fourier coefficients of its "holomorphic part", when $s = 3/4$ and $k = 1/2$, are given by the infinite series expansions appearing in (1.8). In Sect. 3.1, we begin by defining this series as a trace over Möbius transformations, and in Sect. 3.2 we compute its Fourier expansion. The main result of this calculation is the reproduction of the infinite series formulas in (1.8) as coefficients of the holomorphic part of a weak Maass form of weight 1/2 on $\Gamma_0(144)$ with Nebentypus character χ_{12}.

3.1. The construction. Suppose that $k \in \frac{1}{2} + \mathbb{Z}$. We now define an important class of Poincaré series $P_k(s; z)$. For matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, with $c \geq 0$, define the character $\chi(\cdot)$ by

(3.1)
$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{cases} e\left(-\frac{b}{24}\right) & \text{if } c = 0, \\ i^{-1/2}(-1)^{\frac{1}{2}(c+ad+1)} e\left(-\frac{a+d}{24c} - \frac{a}{4} + \frac{3dc}{8}\right) \cdot \omega_{-d,c}^{-1} & \text{if } c > 0. \end{cases}$$

Remark. The character χ is defined to coincide with the automorphy factor for the real analytic form $F_0(z) - G_0(z)$ when restricted to $\Gamma_0(2)$.

Throughout, let $z = x+iy$, and for $s \in \mathbb{C}$, $k \in \frac{1}{2}+\mathbb{Z}$, and $y \in \mathbb{R} \setminus \{0\}$, let

(3.2)
$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|),$$

where $M_{\nu,\mu}(z)$ is the standard M-Whittaker function which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) u = 0.$$

Furthermore, let

$$\varphi_{s,k}(z) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right) e\left(-\frac{x}{24}\right).$$

Suppose that c is a positive odd integer. For integers $0 \leq a < c$ and $0 < b < c$, define the functions

$$T_1\left(\frac{a}{c}; z\right) := -\frac{i}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c}; \tau\right)}{\sqrt{-i(\tau + z)}}\, d\tau,$$

$$T_2\left(\frac{a}{c}; z\right) := -\frac{i}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c}; -\frac{1}{\tau}\right)}{\sqrt{-i(\tau + z)}}\, d\tau,$$

$$T_1(a,b,c;z) := -\frac{\zeta_{2c}^{-5b}}{2\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta(a,b,c;\tau)}{\sqrt{-i(\tau + z)}}\, d\tau,$$

$$T_2(a,b,c;z) := -\frac{\zeta_{2c}^{-5b}}{2\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}}\Theta\left(a,b,c; -\frac{1}{\tau}\right)}{\sqrt{-i(\tau + z)}}\, d\tau.$$

If we let $t_c := \operatorname{lcm}(c, 6)$, then define $\Theta(a,b,c;\tau)$ by

$$\Theta(a,b,c;\tau) := \sum_{m \pmod{t_c}} (-1)^m \sin\left(\frac{\pi}{3}(2m+1)\right) e^{2\pi i m \frac{a}{c}} \theta\left(2cm+6b+c, 2ct_c; \frac{\tau}{24c^2}\right).$$

Recall that the theta functions $\theta(\alpha, \beta; \tau)$ are defined by (1.7). Using this notation, define the following functions

$$\mathcal{G}_1\left(\frac{a}{c}; z\right) := \mathcal{N}\left(\frac{a}{c}; z\right) - T_1\left(\frac{a}{c}; z\right), \tag{3.3}$$

$$\mathcal{G}_2\left(\frac{a}{c}; z\right) := \mathcal{M}\left(\frac{a}{c}; z\right) - T_2\left(\frac{a}{c}; z\right), \tag{3.4}$$

$$\mathcal{G}_1(a,b,c;z) := \mathcal{N}(a,b,c;z) - T_1(a,b,c;z), \tag{3.5}$$

$$\mathcal{G}_2(a,b,c;z) := \mathcal{M}(a,b,c;z) - T_2(a,b,c;z). \tag{3.6}$$

These functions constitute a vector valued weak Maass form of weight $1/2$. Here we recall this notion more precisely. A *vector valued weak Maass form of weight k for* $\mathrm{SL}_2(\mathbb{Z})$ is any finite set of smooth functions, say $v_1(z), \ldots, v_m(z) : \mathbb{H} \to \mathbb{C}$, which satisfy the following:

(1) If $1 \leq n_1 \leq m$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, then there is a root of unity $\epsilon(A, n_1)$ and an index $1 \leq n_2 \leq m$ for which

$$v_{n_1}(Az) = \epsilon(A, n_1)(cz+d)^k v_{n_2}(z)$$

for all $z \in \mathbb{H}$.

(2) For each $1 \leq n \leq m$ we have that $\Delta_k v_n = 0$.

If c is a positive odd integer, let V_c be the "vector" of functions defined by

$$V_c := \left\{\mathcal{G}_1\left(\frac{a}{c}; z\right), \mathcal{G}_2\left(\frac{a}{c}; z\right) \ : \ \text{with } 0 < a < c\right\}$$
$$\bigcup \left\{\mathcal{G}_1(a, b, c; z), \mathcal{G}_2(a, b, c; z) \ : \ (a, b) \text{ with } 0 \leq a < c \text{ and } 0 < b < c\right\}.$$

THEOREM 3.4. *Assume the notation above. If c is a positive odd integer, then V_c is a vector valued weak Maass form of weight $1/2$ for the full modular group* $\mathrm{SL}_2(\mathbb{Z})$.

Sketch of the proof. The proof of Theorem 3.4 follows along the lines of the proof of Theorem 1.1. Therefore, for brevity here we simply provide a sketch of the proof and make key observations.

As in the proof of Lemma 3.2, one first shows that

$$\frac{2\sqrt{3}}{iz} \cdot J\left(\frac{a}{c}; \frac{2\pi i}{z}\right) = \frac{i}{\sqrt{3}} \int_0^{i\infty} \frac{\Theta\left(\frac{a}{c}; \tau\right)}{\sqrt{-i(\tau + z)}}\, d\tau,$$

$$2\sqrt{3}\sqrt{-iz} \cdot J\left(\frac{a}{c}; -2\pi i z\right) = \frac{i}{\sqrt{3}} \int_0^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c}; -\frac{1}{\tau}\right)}{\sqrt{-i(\tau + z)}}\, d\tau,$$

$$\frac{\zeta_{2c}^{-5b}\sqrt{3}}{iz} \cdot J\left(a, b, c; \frac{2\pi i}{z}\right) = \frac{\zeta_{2c}^{-5b}}{6c} \int_0^{i\infty} \frac{\Theta(a, b, c; \tau)}{\sqrt{-i(\tau + z)}}\, d\tau,$$

$$\zeta_{2c}^{-5b}\sqrt{3}\sqrt{-iz} \cdot J(a, b, c; -2\pi i z) = \frac{\zeta_{2c}^{-5b}}{6c} \int_0^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}}\Theta\left(a, b, c; -\frac{1}{\tau}\right)}{\sqrt{-i(\tau + z)}}\, d\tau.$$

Arguing as in the proof of Lemma 3.3, one then establishes that the functions T_i satisfy the same transformation laws under the generators of $\mathrm{SL}_2(\mathbb{Z})$ as the corresponding functions $\mathcal{N}$ and $\mathcal{M}$ appearing in (3.3)–(3.6). That the functions $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy suitable transformation laws under $\mathrm{SL}_2(\mathbb{Z})$ follows easily from the "closure" of the formulas in Theorem 2.3.

To complete the proof, it suffices to show that each component is annihilated by the weight $1/2$ hyperbolic Laplacian $\Delta_{\frac{1}{2}}$, and satisfies the required growth conditions at the cusps. These facts follow *mutatis mutandis* as in the proof of Theorem 1.1. □

Sketch of the proof of Theorem 1.2. By Theorem 3.4, the transformation laws of the components of the given vector valued weak Maass forms are completely determined under all of $\mathrm{SL}_2(\mathbb{Z})$. Observe that $D\left(\frac{a}{c}; z\right)$ is the image of $\mathcal{G}_1\left(\frac{a}{c}; z\right)$ by letting $z \to \ell_c z$. Therefore, the modular transformation properties of $D\left(\frac{a}{c}; z\right)$ are inherited by the modularity properties of $\Theta\left(\frac{a}{c}; \ell_c\tau\right)$ when applied to the definition of $\widetilde{S}_1\left(\frac{a}{c}; z\right)$. By Proposition 2.1 of [29], it is known that $\Theta\left(\frac{a}{c}; \ell_c\tau\right)$ is on $\Gamma_1(144 f_c^2 \ell_c)$, and the result follows. □

on $\mathfrak{H}$. For a holomorphic modular form f of weight k we get

$$\begin{aligned} &\omega f_k \left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) \\ &= \left(-y^2\partial_x^2 - y^2\partial_y^2 + y\partial_x\partial_\theta \right) e^{ik\theta} y^{k/2} f(x+iy) \\ &= \frac{k}{2}\left(1-\frac{k}{2}\right) f_k \left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right). \end{aligned}$$

Thus we see that all modular forms considered up to now correspond to eigenfunctions of ω on G that are Γ_{mod}-invariant on the left, and transform on the right according to a character of K.

1.3.6 *Representations.* If the function f_0 or f_k is square integrable on $\Gamma_{\text{mod}}\backslash G$, then it generates an irreducible subspace of $L^2(\Gamma_{\text{mod}}\backslash G)$ for the action of G by right translation. If f is a holomorphic modular form, then this irreducible representation belongs to the *discrete series* of representations of G. If f is a square integrable real analytic modular form with positive eigenvalue, the function f_0 is a weight zero vector in an irreducible representation of the *principal series*. For more information on the representational point of view one may consult §2 of [15].

Hecke operators are not discussed in thus book. But they reveal very interesting properties of modular forms. See, e.g., Chapter II of [29] for the holomorphic case, and Chapter V of [35] for Maass forms. The representational point of view incorporates the Hecke operators by working with functions on the adele group of GL_2, see [15].

1.4 Fourier expansion of modular forms

Up till now we have motivated the study of modular forms from harmonic analysis: spectral decomposition of the Laplace operator, and irreducible subspaces for the right representation of G in $L^2(\Gamma_{\text{mod}}\backslash G)$. Number theoretically interesting formulas arise as soon as one writes down the Fourier expansion of modular forms.

1.4.1 *Fourier expansion.* For both types of modular forms discussed thus far, the transformation behavior implies periodicity in $x = \text{Re}(z)$: take $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in condition i) to conclude that $f(z+1) = f(z)$. Hence there is a Fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(y) e^{2\pi inx}.$$

Condition ii) in the definitions above implies that the a_n satisfy ordinary differential equations. All Fourier terms $a_n(y)e^{2\pi inx}$ inherit the growth condition iii).

We see in the holomorphic case that $a_n(y)$ is a multiple of $e^{-2\pi ny}$, and has to vanish for $n < 0$. Thus we get

$$f(z) = \sum_{n\geq 0} c_n(f) e^{2\pi inz}.$$

The $c_n(f)$ are called the *Fourier coefficients* of f.

In the real analytic case the differential equation is

$$-y^2 a_n''(y) = (\lambda - 4\pi^2 n^2 y^2) a_n(y).$$

We write $\lambda = \frac{1}{4} - s^2$, $s \in \mathbb{C}$. For $n = 0$ there is a two dimensional space of solutions, with basis $y^{s+1/2}$, $y^{-s+1/2}$ (if $s \neq 0$). For $n \neq 0$ we get a variant of the Whittaker differential equation. The growth condition restricts the possibilities to a one dimensional space, spanned by $W_{0,s}(4\pi|n|y)$; see, e.g., [56], 1.7. (The Whittaker function $W_{0,s}$ decreases exponentially: $W_{0,s}(t) \sim e^{-t/2}$ $(t \to \infty)$.) This leads to

$$f(z) = b_0(f) y^{s+1/2} + c_0(f) y^{-s+1/2} + \sum_{n \neq 0} c_n(f) W_{0,s}(4\pi|n|y) e^{2\pi i n x}.$$

To distinguish between the Fourier coefficients b_0 and c_0, a choice of s such that $\lambda = \frac{1}{4} - s^2$ is necessary.

Often one uses a modified Bessel function in the terms with $n \neq 0$; the function $y \mapsto \sqrt{y} K_s(2\pi|n|y)$ spans the space of possible $a_n(y)$. I prefer Whittaker functions, as they can be used in weights other than 0 as well. When comparing the Fourier expansions below with those at other places, one should keep in mind that $s_{\text{here}} = s_{\text{usual}} - \frac{1}{2}$, and that $W_{0,s}(y) = \sqrt{y/\pi} K_s(y/2)$.

1.4.2 *Holomorphic modular forms.* [53], (2.2.1), on p. 32, gives the Fourier expansion of holomorphic Eisenstein series ($k \geq 4$ even):

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) e^{2\pi i n z},$$

with the *divisor function* $\sigma_u(n) = \sum_{d|n} d^u$.

All coefficients in the expansion of $\frac{1}{2\zeta(k)} G_k(z)$ are rational numbers, even integers if $k = 4$ or 6. This implies (after some computations) that $\Delta(z) = \sum_{n \geq 1} \tau(n) e^{2\pi i n z}$, with all $\tau(n) \in \mathbb{Z}$, $\tau(1) = 1$. In particular, $c_0(\Delta) = 0$.

1.4.3 *Definition.* A *cusp form* is a modular form for which the Fourier term of order zero vanishes. This means that $c_0 = 0$ in the holomorphic case, and $b_0 = c_0 = 0$ in the real analytic case. The holomorphic modular form Δ is a cusp form.

1.4.4 *Real analytic Eisenstein series.* If the integers c and d are relatively prime, then there are a and b such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\text{mod}}$, and the coset $\Gamma_{\text{mod}}^{\infty} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ depends only on $\pm(c, d)$. In this way we get for $\operatorname{Re} s > \frac{1}{2}$:

$$\begin{aligned} e(s; z) &= \frac{1}{2} \sum_{c,d \in \mathbb{Z},\, (c,d)=1} y^{s+1/2} |cz + d|^{-2s-1} \\ &= \frac{y^{s+1/2}}{2\zeta(2s+1)} \sum_{n,m \in \mathbb{Z}}{}' |mz + n|^{-2s-1}. \end{aligned}$$

Hence $e(s;z) = \frac{1}{2}y^{s+1/2}\zeta(2s+1)^{-1}G(z,\bar{z};s+1/2,s+1/2)$, with Maass's Eisenstein series $G(\cdot,\cdot;\alpha,\beta)$ as on p. 207 of [35]. The Fourier expansion is given on p. 210 of *loc. cit.*, and leads to

$$\begin{aligned} e(s;z) &= y^{s+1/2} + \sqrt{\pi}\frac{\Gamma(s)\zeta(2s)}{\Gamma(s+1/2)\zeta(2s+1)}y^{-s+1/2} \\ &\quad + \frac{\pi^{s+1/2}}{\Gamma(s+1/2)\zeta(2s+1)}\sum_{n\neq 0}\frac{\sigma_{2s}(|n|)}{|n|^{s+1/2}}W_{0,s}(4\pi|n|y)e^{2\pi i n x}. \end{aligned}$$

Again we see that the divisor function appears in the Fourier coefficients of Eisenstein series.

This Fourier expansion defines the function $e(s)$ on $\mathfrak{H}$ for all $s \in \mathbb{C}$ that satisfy $s \neq 0$, 1 and $\Gamma(s+1/2)\zeta(2s+1) \neq 0$. The Γ_{mod}-invariance is preserved, as are the other conditions in Definition 1.2.2. In this way we get e as a meromorphic family on $\mathbb{C}$ of real analytic modular forms. The singularities have order one, and the residues are again modular forms. Moreover, the functional equation of the zeta function of Riemann implies the *functional equation* of e:

$$e(-s;z) = c_0(e(-s)) \cdot e(s;z).$$

1.4.5 *Cuspidal Maass forms.* In 1.2.6 we mentioned that there is a countable set $\psi_0, \psi_1, \ldots$ of square integrable real analytic modular forms that constitute an orthonormal basis of the part of $L^2(\Gamma_{\text{mod}}\backslash\mathfrak{H})$ in which the selfadjoint extension A_0 of the Laplacian has a discrete spectrum. We may arrange the ψ_j such that their eigenvalues λ_j increase. Take $\psi_0 = \sqrt{3/\pi}$. For $j \geq 1$ one knows that $\lambda_j > \frac{1}{4}$, hence $\lambda_j = \frac{1}{4} - s_j^2$ with $s_j \in i\mathbb{R}$. The square integrability is inherited by the Fourier coefficients: $\int_1^\infty |a_n(y)|^2\, y^{-2}\, dy < \infty$. Hence $b_0(\psi_j) = c_0(\psi_j) = 0$ for $j \geq 1$. So $\psi_1, \psi_2, \ldots$ are *cusp forms.*

We can choose all ψ_j to be real-valued. (Use that $f \mapsto \bar{f}$ preserves the space of real analytic cusp forms for a real eigenvalue.) We may even arrange that each ψ_j is an eigenfunction of all Hecke operators. Much more information can be found in §3.5 of [57]. We only mention the *Ramanujan-Petersson conjecture for real analytic modular forms* (not proved up till now):

$$c_n(\psi_j) = \mathcal{O}\left(|n|^{-1/2+\varepsilon}\right) \quad (|n| \to \infty) \qquad \text{for each } j \geq 1, \text{ for each } \varepsilon > 0.$$

1.5 More modular forms

There are more general types of modular forms. First we consider real analytic modular forms of even weight. Next we introduce a multiplier system to be able to define modular forms of arbitrary complex weight. This opens the possibility to consider families of modular forms for which the weight varies continuously.

1.5.1 *Real analytic modular forms of even weight.* We have seen in 1.3.5 that the modular forms considered up till now correspond to functions on $\Gamma_{\text{mod}}\backslash G$ that are eigenfunctions of the Casimir operator. These forms transform according to a character of K. Usually, one calls all such functions modular forms, provided a growth condition at the cusp is satisfied.

All characters of K are of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto e^{ik\theta}$ with $k \in \mathbb{Z}$. But as $-\text{Id} \in \Gamma_{\text{mod}} \cap K$ is central in G, only characters with even k admit non-zero functions with the prescribed transformation properties.

The correspondence in 1.3.4 between holomorphic modular forms and functions on G is not the most convenient one if one wants to study real analytic modular forms. This is caused by the fact that the factor $(cz+d)^k$ in the transformation behavior of holomorphic modular forms does not have absolute value 1. We follow the convention to relate functions f on $\mathfrak{H}$ and functions F on G by

$$F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = e^{-ik\arg(ci+d)} f\left(\frac{ai+b}{ci+d}\right).$$

As $k \in 2\mathbb{Z}$ the choice of the argument does not matter. This leads to the following definition.

1.5.2 *Definition.* A *real analytic modular form* of *even weight* $k \in 2\mathbb{Z}$ with *eigenvalue* $\lambda \in \mathbb{C}$ is a function $f : \mathfrak{H} \to \mathbb{C}$ that satisfies the conditions

i) $f(\gamma \cdot z) = e^{ik\arg(cz+d)} f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\text{mod}}$.

ii) $L_k f = \lambda f$, with $L_k = -y^2\partial_x^2 - y^2\partial_y^2 + iky\partial_x$.

iii) There is a real number a such that $f(z) = \mathcal{O}(y^a)$ $(y \to \infty)$, uniformly in $x \in \mathbb{R}$.

1.5.3 *Examples* The real analytic modular forms defined in 1.2.2 have weight 0. If h is a holomorphic modular form of weight $k \in \mathbb{Z}$, then $z \mapsto y^{k/2}h(z)$ is a real analytic modular form in the sense just defined, of weight k, with eigenvalue $\frac{k}{2}(1-\frac{k}{2})$.

For each $k \in 2\mathbb{Z}$ there are Eisenstein series of weight k with eigenvalue $\frac{1}{4} - s^2$. They have a meromorphic extension, and satisfy a functional equation. If $k \geq 4$, then the value at $s = \frac{1}{2}(k-1)$ corresponds to a multiple of G_k.

There are countably many cuspidal real analytic modular forms of weight k with eigenvalues $\lambda_1, \lambda_2, \ldots$, obtained from the ψ_j by differential operators (see Proposition 4.5.3). Those differential operators are multiples of the operators described on p. 177 of [35], often called *Maass operators.*

1.5.4 *The eta function of Dedekind.* If $k \notin 2\mathbb{Z}$ the definition above admits only the zero function as a modular form. But there are modular forms with other weight, Theorem 6.3 shows that $\widehat{\mu}(u,v;\tau)$ is essentially a weight $1/2$ non-holomorphic Jacobi form. In analogy with the classical theory of Jacobi forms, one may then obtain harmonic Maass forms by making suitable specializations for u and v by elements in $\mathbb{Q}\tau+\mathbb{Q}$, and by multiplying by appropriate powers of q. Without this result, it would be very difficult to explicitly construct examples of weight $1/2$ harmonic Maass forms.

Harmonic Maass forms of weight k are mapped to classical modular forms (see Lemma 7.4), their so-called *shadows*, by the differential operator

$$\xi_k := 2iy^k \cdot \overline{\frac{\partial}{\partial \overline{\tau}}}.$$

The following lemma makes it clear that the shadows of the real analytic forms arising from $\widehat{\mu}$ can be described in terms of weight $3/2$ theta functions.

Lemma 6.4. [Lemma 1.8 of [219]] *The function R is real analytic and satisfies*

$$\frac{\partial R}{\partial \overline{u}}(u;\tau) = \sqrt{2}y^{-\frac{1}{2}}e^{-2\pi c^2 y}\vartheta(\overline{u};-\overline{\tau}),$$

where $c := \mathrm{Im}(u)/\mathrm{Im}(\tau)$. Moreover, we have that

$$\frac{\partial}{\partial \overline{\tau}}R(a\tau-b;\tau) = -\frac{i}{\sqrt{2y}}e^{-2\pi a^2 y}\sum_{\nu\in\mathbb{Z}+\frac{1}{2}}(-1)^{\nu-\frac{1}{2}}(\nu+a)e^{-\pi i\nu^2\overline{\tau}-2\pi i\nu(a\overline{\tau}-b)}.$$

7. Harmonic Maass forms

For the remainder of the paper, we shall assume that the reader is familiar with the classical theory of elliptic modular forms (for example, see [71, 84, 125, 134, 143, 155, 164, 177, 185, 193, 196]).

D. Niebur [160, 161] and D. Hejhal [117] constructed certain non-holomorphic Poincaré series which turn out to be examples of *harmonic Maass forms.* Bruinier [61] made great use of these Poincaré series in his early work on Borcherds lifts and Green's functions. He then realized the importance of developing a "theory of harmonic Maass forms" in its own right. Later in joint work with Funke [63], he developed the fundamental results of this theory, some of which we describe here. After making the necessary definitions, we shall discuss Hecke operators and various differential operators. The interplay between harmonic Maass forms and classical modular forms shall play an important role throughout this paper.

7.1. **Definitions.** In 1949, H. Maass introduced the notion of a *Maass form*[13] (see [149, 150]). He constructed these non-holomorphic automorphic forms using Hecke characters of real quadratic fields, in analogy with Hecke's theory [115] of *modular forms with complex multiplication* (see [180] for a modern treatment).

To define these functions, let $\Delta=\Delta_0$ be the *hyperbolic Laplacian*

$$\Delta := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$

[13]In analogy with the eigenvalue problem for the vibrating membrane, Maass referred to these automorphic forms as *Wellenformen*, or *waveforms*.

where $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. It is a second-order differential operator which acts on functions on $\mathbb{H}$, and it is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$.

A *Maass form* for a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

(1) For every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have
$$f\left(\frac{az+b}{cz+d}\right) = f(z).$$
(2) We have that f is an eigenfunction of Δ.
(3) There is some $N > 0$ such that
$$f(x+iy) = O(y^N)$$
as $y \to +\infty$.

Furthermore, we call f a *Maass cusp form* if
$$\int_0^1 f(z+x)dx = 0.$$
There is now a vast literature on Maass forms thanks to the works of many authors such as Hejhal, Iwaniec, Maass, Roelcke, Selberg, Terras, Venkov, among many others (for example, see [116, 117, 124, 126, 149, 150, 181, 191, 199, 200, 203]).

This paper concerns a generalization of this notion of Maass form. Following Bruinier and Funke [63], we define the notion of a harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ as follows. As before, we let $z = x+iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. We define the weight k hyperbolic Laplacian Δ_k by
$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{7.1}$$
For odd integers d, define ϵ_d by
$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases} \tag{7.2}$$

Definition 7.1. If N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), then a *weight k harmonic Maass form on* $\Gamma \in \{\Gamma_1(N), \Gamma_0(N)\}$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}$, we have
$$M\left(\frac{az+b}{cz+d}\right) = \begin{cases} (cz+d)^k M(z) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz+d)^k \ M(z) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$
Here $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol, and $\sqrt{z}$ is the principal branch of the holomorphic square root.
(2) We have that $\Delta_k M = 0$.
(3) There is a polynomial $P_M = \sum_{n \leq 0} c^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that
$$M(z) - P_M(z) = O(e^{-\epsilon y})$$
as $y \to +\infty$ for some $\epsilon > 0$. Analogous conditions are required at all cusps.

Remark 9. Maass forms and classical modular forms are required to satisfy moderate growth conditions at cusps, and it is for this reason that harmonic Maass forms are often referred to as "harmonic weak Maass forms". The term "weak" refers to the relaxed condition Definition 7.1 (3) which gives rise to a rich theory. For convenience, we use the terminology "harmonic Maass form" instead of "harmonic weak Maass form".

Remark 10. We refer to the polynomial P_M as the *principal part* of $M(z)$ at ∞. Obviously, if P_M is non-constant, then $M(z)$ has exponential growth at ∞. Similar remarks apply at all cusps.

Remark 11. Bruinier and Funke [63] define two types of harmonic Maass forms based on varying the growth conditions at cusps. For a group Γ, they refer to these spaces as $H_k(\Gamma)$ and $H_k^+(\Gamma)$. Definition 7.1 (3) corresponds to their $H_k^+(\Gamma)$ definition.

Remark 12. Since holomorphic functions on $\mathbb{H}$ are harmonic, it follows that weakly holomorphic modular forms are harmonic Maass forms.

Remark 13. Here we recall the congruence subgroups. If N is a positive integer, then define the level N *congruence subgroups* $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$ by

$$\begin{aligned}
\Gamma_0(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ c \equiv 0 \pmod N \right\}, \\
\Gamma_1(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ a \equiv d \equiv 1 \pmod N, \text{ and } c \equiv 0 \pmod N \right\}, \\
\Gamma(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ a \equiv d \equiv 1 \pmod N, \text{ and } b \equiv c \equiv 0 \pmod N \right\}.
\end{aligned}$$

Remark 14. For $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, the transformation law in Definition 7.1 (1) coincides with those in Shimura's theory of half-integral weight modular forms [192].

Remark 15. Later we shall require the classical "slash" operator. For convenience, we recall its definition here. Suppose that $k \in \frac{1}{2}\mathbb{Z}$. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ ($\Gamma_0(4)$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), define $j(A, z)$ by

$$j(A, z) := \begin{cases} \sqrt{cz + d} & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz + d} & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases} \tag{7.3}$$

where ε_d is defined by (7.2), and where $\sqrt{z}$ is the principal branch of the holomorphic square root as before. For functions $f : \mathbb{H} \to \mathbb{C}$, we define the action of the "slash" operator by

$$(f \mid_k A)(z) := j(A, z)^{-2k} f(Az) = j(A, z)^{-2k} f\left(\frac{az + b}{cz + d}\right). \tag{7.4}$$

Notice that Definition 7.1 (1) may be rephrased as

$$(M \mid_k A)(z) = M(z).$$

Remark 16. We shall also consider *level N weight $k \in \frac{1}{2}\mathbb{Z}$ forms with Nebentypus χ*. To define such forms, suppose that N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), and let χ be a Dirichlet character modulo N. To define these forms, one merely requires[14], for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, that

$$M\left(\frac{az+b}{cz+d}\right) = \begin{cases} \chi(d)(cz+d)^k M(z) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \chi(d)(cz+d)^k \; M(z) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

Throughout, we shall adopt the following notation. If $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup, then we let

$$\begin{aligned} S_k(\Gamma) &:= \text{weight } k \text{ cusp forms on } \Gamma, \\ M_k(\Gamma) &:= \text{weight } k \text{ holomorphic modular forms on } \Gamma, \\ M_k^!(\Gamma) &:= \text{weight } k \text{ weakly holomorphic modular forms on } \Gamma, \\ H_k(\Gamma) &:= \text{weight } k \text{ harmonic Maass forms on } \Gamma. \end{aligned}$$

Furthermore, if χ is a Dirichlet character modulo N, then we let

$$\begin{aligned} S_k(N,\chi) &:= \text{level } N \text{ weight } k \text{ cusp forms with Nebentypus } \chi, \\ M_k(N,\chi) &:= \text{level } N \text{ weight } k \text{ holomorphic modular forms with Nebentypus } \chi, \\ M_k^!(N,\chi) &:= \text{level } N \text{ weight } k \text{ weakly holomorphic modular forms with Nebentypus } \chi, \\ H_k(N,\chi) &:= \text{level } N \text{ weight } k \text{ harmonic Maass forms with Nebentypus } \chi. \end{aligned}$$

When the Nebentypus character is trivial, we shall suppress χ from the notation.

The real analytic forms in Theorem 6.1 provide non-trivial examples of weight 1/2 harmonic Maass forms. More generally, the work of Zwegers [218, 219], shows how to complete all of Ramanujan's mock theta functions to obtain weight 1/2 harmonic Maass forms. In Section 8, we shall present further examples of harmonic Maass forms.

7.2. **Fourier expansions.** In this paper we consider harmonic Maass forms with weight $2-k \in \frac{1}{2}\mathbb{Z}$ with $k > 1$. Therefore, throughout we assume that $1 < k \in \frac{1}{2}\mathbb{Z}$.

Harmonic Maass forms have distinguished Fourier expansions which are described in terms of the incomplete Gamma-function $\Gamma(\alpha; x)$

$$\Gamma(\alpha;x) := \int_x^\infty e^{-t} t^{\alpha-1} \, dt, \tag{7.5}$$

and the usual parameter $q := e^{2\pi i z}$. The following characterization is straightforward (for example, see Section 3 of [63]).

Lemma 7.2. *Assume the notation and hypotheses above, and suppose that N is a positive integer. If $f(z) \in H_{2-k}(\Gamma_1(N))$, then its Fourier expansion is of the form*

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n, \tag{7.6}$$

where $z = x+iy \in \mathbb{H}$, with $x, y \in \mathbb{R}$.

As Lemma 7.2 reveals, $f(z)$ naturally decomposes into two summands. In view of this fact, we make the following definition.

[14]This replaces (1) in Definition 7.1.

Definition 7.3. Assuming the notation and hypotheses in Lemma 7.2, we refer to

$$f^+(z) := \sum_{n \gg -\infty} c_f^+(n) q^n$$

as the *holomorphic part of* $f(z)$, and we refer to

$$f^-(z) := \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n$$

as the *non-holomorphic part of* $f(z)$.

Remark 17. A harmonic Maass form with trivial non-holomorphic part is a weakly holomorphic modular form. We shall make use of this fact as follows. If $f_1, f_2 \in H_{2-k}(\Gamma)$ are two harmonic Maass forms with equal non-holomorphic parts, then $f_1 - f_2 \in M_{2-k}^!(\Gamma)$.

7.3. **The ξ-operator and period integrals of cusp forms.** Harmonic Maass forms are related to classical modular forms thanks to the properties of differential operators. The first nontrivial relationship depends on the differential operator

$$\xi_w := 2iy^w \cdot \overline{\frac{\partial}{\partial \overline{z}}}. \tag{7.7}$$

The following lemma[15], which is a straightforward refinement of a proposition of Bruinier and Funke (see Proposition 3.2 of [63]), shall play a central role throughout this paper.

Lemma 7.4. *If* $f \in H_{2-k}(N, \chi)$, *then*

$$\xi_{2-k} : H_{2-k}(N, \chi) \longrightarrow S_k(N, \overline{\chi})$$

is a surjective map. Moreover, assuming the notation in Definition 7.3, we have that

$$\xi_{2-k}(f) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} \overline{c_f^-(-n)} n^{k-1} q^n.$$

Thanks to Lemma 7.4, we are in a position to relate the non-holomorphic parts of harmonic Maass forms, the expansions

$$f^-(z) := \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n,$$

with "period integrals" of modular forms. This observation was critical in Zwegers's work on Ramanujan's mock theta functions.

To make this connection, we must relate the Fourier expansion of the cusp form $\xi_{2-k}(f)$ with $f^-(z)$. This connection is made by applying the simple integral identity

$$\int_{-\overline{z}}^{i\infty} \frac{e^{2\pi i n \tau}}{\left(-i(\tau+z)\right)^{2-k}}\, d\tau = i(2\pi n)^{1-k} \cdot \Gamma(k-1, 4\pi n y) q^{-n}. \tag{7.8}$$

This identity follows by the direct calculation

$$\int_{-\overline{z}}^{i\infty} \frac{e^{2\pi i n \tau}}{\left(-i(\tau+z)\right)^{2-k}}\, d\tau = \int_{2iy}^{i\infty} \frac{e^{2\pi i n(\tau - z)}}{(-i\tau)^{2-k}}\, d\tau = i(2\pi n)^{1-k} \cdot \Gamma(k-1, 4\pi n y)\, q^{-n}.$$

[15]The formula for $\xi_{2-k}(f)$ corrects a typographical error in [63].

Hence the number of covariants of degree s and order q of the binary quantic of order n is the coefficient of $\rho^{q+1}x^s$ in (1), or the coefficient of ρ^{q+1} in (2).

We may obtain by this method a proof of the formula for the canonical form of an invariant matrix of a matrix with repeated characteristic roots*, without the use of Aitken's method of chains†.

Let A be the matrix $\begin{bmatrix} \mu, & 1 \\ 0, & \mu \end{bmatrix}$. Denote by $[n]_\mu$ the matrix of order n^2 with μ in each position in the leading diagonal, unity in each position in the diagonal next above, and zero elsewhere, so that $A = [2]_\mu$.

Clearly, confining our attention to the canonical form, we have

$$A^{[n]} = [n+1]_{\mu^n}.$$

Then, if
$$[A^{[n]}]^{[s]} = \Sigma\, k_{pq} A^{[p,\, q]},$$

we have
$$[n+1]^{[s]}_{\mu^n} = \Sigma\, k_{pq}[p-q+1]_{\mu^{ns}},$$

and
$$[n+1]^{[s]}_{\mu} = \Sigma\, k_{pq}[p-q+1]_{\mu^s},$$

and the above generating function is applicable. The proof now follows the lines of the preceding paper*.

University College,
Swansea.

THE FINAL PROBLEM: AN ACCOUNT OF THE MOCK THETA FUNCTIONS

G. N. WATSON‡.

It is not unnatural in one who has held office in this Society for sixteen years that his mode of approach to the preparation of his valedictory Address should have taken the form of an investigation into the procedure of his similarly situated predecessors.

Of the thirty-five previous Presidents, all but three have delivered Addresses on resigning office. Two of the exceptions were the first two Presidents, de Morgan and Sylvester; de Morgan, however, had had his

* D. E. Littlewood, *Proc. London Math. Soc.* (2), 40 (1936), 370–381.

† A. C. Aitken, *Proc. London Math. Soc.* (2), 38 (1935), 354–376.

‡ Presidential Address delivered at the meeting of 14 November, 1935.

say in an inaugural speech at the first meeting of the Society. The third exception was Henrici, who confined himself to thanking the members for "the kind indulgence they had shown him", a sentiment which I would wish to echo to-day.

It had occurred to me that the survival of the Society for seventy years might make an historical topic appropriate for my Address; and, in fact, that a President at a loss for a subject might do far worse than give some account of the Addresses of his remoter predecessors. I was, however, deterred from this course by the examples of those two Presidents, Prof. Love and Prof. Hardy, who, like me, had held the office of Secretary for a lengthy period*; after making two statements on the progress of the Society, I shall follow them by confining myself to a mathematical topic.

In 1928 Prof. Hardy was able to report that the membership had grown to 410 and the annual output to 1,280 pages; the number of members is now 440 and the annual output is 1,440 pages. Each increase in our rate of publication in the last sixteen years has been followed by an increase in the number of papers received, until at last the inelasticity of our financial resources has led the Council reluctantly to announce the adoption of measures tending to limit the amount which we accept for publication.

The topic which I have selected, though unfortunately not too well adapted for oral exposition, will, I hope, be considered to be as characteristic of its author as the choices of most of my predecessors have been. I make no apologies for my subject being what is now regarded as old-fashioned, because, as a friend remarked to me a few months ago, I am an old-fashioned mathematician. Practically everything that I have to say to-night would be immediately comprehensible to Gauss or Jacobi; on the other hand, Euler, though he might enjoy listening, would probably encounter difficulties both of form and substance.

Early in 1920, three months before his death, Ramanujan wrote his last letter to Hardy. In the course of it he said: "I discovered very interesting functions recently which I call 'Mock' ϑ-functions. Unlike the 'False' ϑ-functions (studied partially by Prof. Rogers in his interesting paper†) they enter into mathematics as beautifully as the ordinary ϑ-functions. I am sending you with this letter some examples".

The study of some of the five foolscap pages of notes which accompanied the letter is the subject which I have chosen for my Address; I doubt

* With the exception of Prof. Burnside (Secretary 1902, 1903; President 1906, 1907) no other person has held both offices.

† L. J. Rogers, *Proc. London Math. Soc.* (2), 16 (1917), 315–336. A "false ϑ-function" is a function such as $1-q+q^2-q^5+q^7-\ldots$, which differs from an ordinary theta function in the signs of alternate terms.

whether a more suitable title could be found for it than the title used by John H. Watson, M.D., for what he imagined to be his final memoir on Sherlock Holmes.

The first three pages in which Ramanujan explained what he meant by a "mock ϑ-function" are very obscure. They will be made clearer if I preface them by Hardy's comment that a mock ϑ-function is a function defined by a q-series convergent when $|q| < 1$, for which we can calculate asymptotic formulae, when q tends to a "rational point" $e^{2r\pi i/s}$ of the unit circle, of the same degree of precision as those furnished for the ordinary ϑ-functions by the theory of linear transformation.

The three pages of explanation (with a few modifications of the formulae to simplify the type-setting) are as follows:

"If we consider a ϑ-function in the transformed Eulerian form, *e.g.*

$$\text{(A)} \qquad 1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\dots,$$

$$\text{(B)} \qquad 1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\dots,$$

and determine the nature of the singularities at the points

$$q=1, \quad q^2=1, \quad q^3=1, \quad q^4=1, \quad q^5=1, \quad \dots,$$

we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance, when $q=e^{-t}$ and $t \to 0$,

$$\text{(A)} = \sqrt{\left(\frac{t}{2\pi}\right)} \exp\left(\frac{\pi^2}{6t}-\frac{t}{24}{}^{*}\right)+o(1)\dagger,$$

$$\text{(B)} = \sqrt{\left(\frac{2}{5-\sqrt{5}}\right)} \exp\left(\frac{\pi^2}{15t}-\frac{t}{60}{}^{*}\right)+o(1),$$

and similar results at other singularities.

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* It is not necessary that there should be only one term like this. There may be many terms but *the number of terms must be finite.*

† Also $o(1)$ may turn out to be $O(1)$. That is all. For instance, when $q \to 1$, the function $\{(1-q)(1-q^2)(1-q^3)\dots\}^{-120}$ is equivalent to the sum of five terms like (*) together with $O(1)$ instead of $o(1)$.

If we take a number of functions like (A) and (B), it is only in a limited number of cases the terms close as above; but in the majority of cases they never close as above. For instance, when $q = e^{-t}$ and $t \to 0$,

$$\text{(C)} \quad 1+\frac{q}{(1-q)^2}+\frac{q^3}{(1-q)^2(1-q^2)^2}+\frac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\dots$$

$$= \sqrt{\left(\frac{t}{2\pi\sqrt{5}}\right)} \exp\left[\frac{\pi^2}{5t}{}^{*} + a_1{}^{\dagger}t + a_2 t^2 + \dots + O(a_k t^k)\right],$$

where $a_1 = 1/8\sqrt{5}$, and so on. The function (C) is a simple example of a function behaving in an unclosed form at the singularities".

At this point I interpose a few explanatory remarks.

The "Eulerian form" of a function apparently refers to the character of the denominators of the terms in the series. The phrase is probably suggested by Euler's formulae

$$\prod_{n=0}^{\infty} (1+q^n z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{(1-q)(1-q^2)\dots(1-q^n)},$$

$$\prod_{n=0}^{\infty} (1-q^n z)^{-1} = \sum_{n=0}^{\infty} \frac{z^n}{(1-q)(1-q^2)\dots(1-q^n)}.$$

As regards the illustrative functions, (A) is immediately derivable from Heine's formula for basic hypergeometric series,

$${}_2\Phi_1\left[\begin{matrix} a, b; & \dfrac{c}{ab} \\ c & \end{matrix}\right] = \prod_{n=0}^{\infty} \left[\frac{(1-cq^n/a)(1-cq^n/b)}{(1-cq^n)\{1-cq^n/(ab)\}}\right],$$

by making $a \to \infty$, $b \to \infty$, and $c \to q$. It is thus seen that (A) is the partition function

$$\prod_{n=0}^{\infty} (1-q^{n+1})^{-1}.$$

The function (B) is $G(q)$, one of the two functions, $G(q)$ and $H(q)$, which occur in the Rogers-Ramanujan identities. These identities and the

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* The coefficient $1/t$ (*sic*) in the index of e happens to be $\pi^2/5$ in this particular case. It may be some other transcendental numbers in other cases.

† The coefficients of t, t^2, ... happen to be $1/8\sqrt{5}$, ... in this case. In other cases they may turn out to be some other algebraic numbers.

definitions of the functions are contained in the assertions that

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)}$$

$$= [(1-q)(1-q^4)(1-q^6)(1-q^9)\ldots]^{-1},$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\ldots(1-q^n)}$$

$$= [(1-q^2)(1-q^3)(1-q^7)(1-q^8)\ldots]^{-1}.$$

The asymptotic formulae for the functions are derivable from known properties of ϑ-functions, the formula for $H(q)$, which corresponds to the formula already quoted for $G(q)$, being

$$H(q) = \sqrt{\left(\frac{2}{5+\sqrt{5}}\right)} \exp\left(\frac{\pi^2}{15t} + \frac{11t}{60}\right) + o(1).$$

The function (C) is more remarkable. It is easy to obtain a first approximation to the value of the function when t is small by the method of estimating the sum of the terms in the neighbourhood of the greatest term of the series*; but the term $t/8\sqrt{5}$, still more the following terms, cannot be determined very satisfactorily in this manner. The only simple procedure which I have devised depends upon making a preliminary transformation of the function.

It is easy to see that

$$\sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m+1)}}{(1-q)^2(1-q^2)^2\ldots(1-q^m)^2}$$

$$= \prod_{r=0}^{\infty} (1-q^{r+1})^{-1} \,.\, \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m+1)}}{(1-q)(1-q^2)\ldots(1-q^m)} (1-q^{m+1})(1-q^{m+2})\ldots$$

$$= \prod_{r=0}^{\infty} (1-q^{r+1})^{-1} \,.\, \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}m(m+1)} \,.\, (-)^n q^{\frac{1}{2}n(n+1)+mn}}{(1-q)(1-q^2)\ldots(1-q^m) \,.\, (1-q)(1-q^2)\ldots(1-q^n)}.$$

We sum this double series by diagonals. The sum of the terms for which $m+n$ has a given odd value, say $2p+1$, is zero; the sum of the terms for which $m+n$ has a given even value, say $2p$, is

$$\frac{q^{p(2p+1)}}{(1-q^2)(1-q^4)\ldots(1-q^{2p})}.$$

* Perhaps this method played a larger part in Ramanujan's work on mock ϑ-functions than my own studies of the subject would suggest.

We thus find that the function (C) is equal to

$$\prod_{r=0}^{\infty} (1-q^{r+1})^{-1} G_{\frac{1}{2}}(q^2),$$

where $G_\mu(q)$ denotes the function

$$\sum_{n=0}^{\infty} \frac{q^{n(n+\mu)}}{(1-q)(1-q^2)\dots(1-q^n)},$$

which occurs in Ramanujan's proof of his identities. Now, for integral values of μ, it can easily be shown that

$$G_\mu(q) = \frac{1}{\sqrt[4]{5}}\left(\frac{\sqrt{5}-1}{2}\right)^{\mu-\frac{1}{2}} \exp\left[\frac{\pi^2}{15t}+\frac{(5\mu^2-\mu)-(\mu^2-\mu)\sqrt{5}-\frac{1}{3}}{20}t+O(t^2)\right],$$

from a consideration of the difference equation

$$G_\mu(q) = G_{\mu+1}(q)+q^{\mu+1}G_{\mu+2}(q),$$

combined with the asymptotic values of $G_0(q) \equiv G(q)$ and $G_1(q) \equiv H(q)$. If we make the plausible assumption that the formula is valid for any fixed μ, whether an integer or not, we can obtain Ramanujan's asymptotic formula for his function (C) by taking $\mu = \frac{1}{2}$.

I now revert to Ramanujan's notes; they continue thus:

"Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say: Suppose there is a function in the Eulerian form and suppose that all or an infinity of points are exponential singularities, and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: Is the function taken the sum of two functions one of which is an ordinary ϑ-function and the other a (trivial) function which is $O(1)$ at *all* the points $e^{2m\pi i/n}$? The answer is *it is not necessarily so.* When it is not so, I call the function a Mock ϑ-function. I have not proved rigorously that *it is not necessarily so.* But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function. Also I have shown that if *it is necessarily so* then it leads to the following assertion:— viz. it is possible to construct two power series in x, namely $\Sigma a_n x^n$ and $\Sigma b_n x^n$, both of which have *essential singularities* on the unit circle, are convergent when $|x| < 1$, and tend to *finite limits at every point* $x = e^{2r\pi i/s}$, and that at the same time the limit of $\Sigma a_n x^n$ at the point $x = e^{2r\pi i/s}$ is equal to the limit of $\Sigma b_n x^n$ at the point $x = e^{-2r\pi i/s}$.

This assertion seems to be untrue. Anyhow, we shall go to the examples and see how far our assertions are true.

I have proved that, if

$$f(q) = 1+\frac{q}{(1+q)^2}+\frac{q^4}{(1+q)^2(1+q^2)^2}+\ldots,$$

then

$$f(q)+(1-q)(1-q^3)(1-q^5)\ldots(1-2q+2q^4-2q^9+\ldots) = O(1)$$

at all the points $q=-1$, $q^3=-1$, $q^5=-1$, $q^7=-1$, ...; and at the same time

$$f(q)-(1-q)(1-q^3)(1-q^5)\ldots(1-2q+2q^4-2q^9+\ldots) = O(1)$$

at all the points $q^2=-1$, $q^4=-1$, $q^6=-1$, ... Also, obviously, $f(q)=O(1)$ at all the points $q=1$, $q^3=1$, $q^5=1$, ... And so $f(q)$ is a Mock ϑ-function.

When $q=-e^{-t}$ and $t\to 0$,

$$f(q)+\sqrt{\left(\frac{\pi}{t}\right)}\exp\left(\frac{\pi^2}{24t}-\frac{t}{24}\right)\to 4.$$

The coefficient of q^n in $f(q)$ is

$$(-1)^{n-1}\frac{\exp\{\pi\sqrt{(\frac{1}{6}n-\frac{1}{144})}\}}{2\sqrt{(n-\frac{1}{24})}}+O\left(\frac{\exp\{\frac{1}{2}\pi\sqrt{(\frac{1}{6}n-\frac{1}{144})}\}}{\sqrt{(n-\frac{1}{24})}}\right).$$

It is inconceivable that a single ϑ-function could be found to cut out the singularities of $f(q)$".

This completes Ramanujan's general description of mock ϑ-functions. His remarks about lack of rigorous proof indicate that he was not completely convinced that the functions which he had constructed actually cannot be expressed in terms of ϑ-functions and "trivial" functions. It would therefore seem that his work on the transformation theory of mock ϑ-functions did not lead him to the precise formulae (such as I shall describe presently) for transformations of mock ϑ-functions of the third order. The precise forms of the transformation formulae make it clear that the behaviour of mock ϑ-functions near the unit circle is of a more complex character than that of ordinary ϑ-functions.

The subsequent results about $f(q)$ which I have quoted are all immediate consequences of my transformation formulae, except for the approximation for the general coefficient of the expansion of $f(q)$ in power series. I have not troubled to verify this approximation; it is presumably derivable from the transformation formulae in the manner in which Hardy and Ramanujan* obtained the corresponding formula for $p(n)$, the number of partitions of n.

The last two pages of Ramanujan's notes consist of lists of definitions of four sets of mock ϑ-functions with statements of relations connecting members of each of the first three sets; for fairly obvious reasons the functions in the various sets are described as being of orders 3, 5, 5, and 7 respectively. These lists and statements have already been published†.

On this occasion I propose to restrict myself to the consideration of functions of order 3. In addition to the function $f(q)$ defined above, Ramanujan has discovered three such functions. Rather strangely‡ he seems to have overlooked the existence of the set of functions which I call $\omega(q)$, $\nu(q)$, $\rho(q)$.

The definitions of the complete set of functions are as follows:

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \dots (1+q^n)^2},$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4) \dots (1+q^{2n})},$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^3) \dots (1-q^{2n-1})},$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4) \dots (1-q^n+q^{2n})},$$

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2(1-q^5)^2 \dots (1-q^{2n+1})^2},$$

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1+q)(1+q^3) \dots (1+q^{2n+1})},$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6) \dots (1+q^{2n+1}+q^{4n+2})}.$$

* *Proc. London Math. Soc.* (2), 17 (1918), 75–115.

† *Collected papers of S. Ramanujan* (1927), 354–355.

‡ Particularly in view of his having discovered both sets of functions of order 5.

These several functions are connected by the following relations:

$$2\phi(-q)-f(q)=f(q)+4\psi(-q)=\vartheta_4(0,\,q)\prod_{r=1}^{\infty}(1+q^r)^{-1},$$

$$4\chi(q)-f(q)=3\vartheta_4^{\,2}(0,\,q^3)\prod_{r=1}^{\infty}(1-q^r)^{-1},$$

$$2\rho(q)+\omega(q)=3[\tfrac{1}{2}q^{-\frac{3}{8}}\vartheta_2(0,\,q^{\frac{3}{2}})]^2\prod_{r=1}^{\infty}(1-q^{2r})^{-1},$$

$$v(\pm q)\pm q\omega(q^2)=\tfrac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,\,q)\prod_{r=1}^{\infty}(1+q^{2r}),$$

$$f(q^8)\pm 2q\omega(\pm q)\pm 2q^3\,\omega(-q^4)=\vartheta_3(0,\,\pm q)\,\vartheta_3^{\,2}(0,\,q^2)\prod_{r=1}^{\infty}(1-q^{4r})^{-2}.$$

Whether Ramanujan's proofs of the relations involving $f(q)$, $\phi(q)$, and $\psi(q)$ are the same as mine must remain unknown.

The first stage in my discussion of the functions consists in obtaining new definitions of the functions by transforming the series by which they are defined into series more amenable to manipulation. For this purpose I use a limiting case of a general formula connecting basic hypergeometric series which I discovered some years ago* in the course of the construction of the seventh proof of the Rogers-Ramanujan identities.

If we write

$${}_r\Phi_s\begin{bmatrix}\alpha,\,\beta,\,\gamma,\,\ldots;\; x\\ \delta,\,\epsilon,\,\ldots;\end{bmatrix}=1+\sum_{n=1}^{\infty}\prod_{m=0}^{n-1}\left\{\frac{(1-\alpha q^m)(1-\beta q^m)(1-\gamma q^m)\ldots}{(1-q^{m+1})(1-\delta q^m)(1-\epsilon q^m)\ldots}\right\}x^n,$$

where r is the number of the symbols $\alpha,\beta,\gamma,\ldots$, and s is the number of the symbols $\delta,\epsilon,\ldots$, the general formula is

$${}_8\Phi_7\begin{bmatrix}a,\; q\sqrt{a},\; -q\sqrt{a},\; c,\; d,\; e,\; f,\; g;\; \dfrac{a^2q^2}{cdefg}\\ \sqrt{a},\; -\sqrt{a},\; aq/c,\; aq/d,\; aq/e,\; aq/f,\; aq/g;\end{bmatrix}$$

$$=\prod_{n=1}^{\infty}\left[\frac{\{1-aq^n\}\{1-aq^n/(fg)\}\{1-aq^n/(ge)\}\{1-aq^n/(ef)\}}{\{1-aq^n/e\}\{1-aq^n/f\}\{1-aq^n/g\}\{1-aq^n/(efg)\}}\right]$$

$$\times{}_4\Phi_3\begin{bmatrix}aq/(cd),\; e,\; f,\; g;\; q\\ efg/a,\; aq/c,\; aq/d;\end{bmatrix},$$

provided that e, f, or g is of the form q^{-N}, where N is a positive integer.

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* G. N. Watson, *Journal London Math. Soc.*, 4 (1929), 4–9.

Make $a \to 1,\ e \to \infty,\ f \to \infty,\ g \to \infty$; and let

$$c = \exp i\theta, \quad d = \exp(-i\theta);$$

we find that*

$$1 + \sum_{n=1}^{\infty} \frac{(-)^n (1+q^n)(2-2\cos\theta)\, q^{\frac{1}{2}n(3n+1)}}{1-2q^n \cos\theta + q^{2n}}$$

$$= \prod_{r=1}^{\infty} (1-q^r) \left[1 + \sum_{n=1}^{n} \frac{q^{n^2}}{\prod_{m=1}^{n} (1-2q^m \cos\theta + q^{2m})} \right].$$

This is the general relation which is fundamental in the construction of the new definitions of the mock ϑ-functions. In this relation take successively

$$\theta = \pi, \quad \theta = \tfrac{1}{2}\pi, \quad \theta = \tfrac{1}{3}\pi,$$

and we get immediately

$$f(q) \prod_{r=1}^{\infty} (1-q^r) = 1 + 4 \sum_{n=1}^{\infty} \frac{(-)^n q^{\frac{1}{2}n(3n+1)}}{1+q^n},$$

$$\phi(q) \prod_{r=1}^{\infty} (1-q^r) = 1 + 2 \sum_{n=1}^{\infty} \frac{(-)^n (1+q^n)\, q^{\frac{1}{2}n(3n+1)}}{1+q^{2n}},$$

$$\chi(q) \prod_{r=1}^{\infty} (1-q^r) = 1 + \sum_{n=1}^{\infty} \frac{(-)^n (1+q^n)\, q^{\frac{1}{2}n(3n+1)}}{1-q^n+q^{2n}}.$$

These are the formulae which will henceforth be adopted as the definitions of $f(q)$, $\phi(q)$, and $\chi(q)$. They render obvious the connection between $\chi(q)$ and $f(q)$; for, by combining like terms of the series on the right, we have

$$[4\chi(q) - f(q)] \prod_{r=1}^{\infty} (1-q^r) = 3 \left[1 + 4 \sum_{n=1}^{\infty} \frac{(-)^n q^{\frac{3}{2}n(n+1)}}{1+q^{3n}} \right],$$

and the required relation follows from the formula†

$$1 + 4 \sum_{n=1}^{\infty} \frac{(-)^n q^{\frac{1}{2}n(n+1)}}{1+q^n} = \vartheta_4^2(0, q).$$

* This formula may also be obtained by expressing the series on the right (*qua* function of $\cos\theta$) as a sum of partial fractions.

† This simple formula (well known to Ramanujan) apparently is not given explicitly by Tannery and Molk; to obtain it, take $z = \frac{1}{2}\pi$ in the expression for $1/\vartheta_1(z, \sqrt{q})$ as a sum of partial fractions. *Cf.* J. Tannery et J. Molk, *Théorie des fonctions elliptiques*, 3 (1898), 136.

The new definition of $\psi(q)$ is not such a direct consequence of my formula. To transform $\psi(q)$, take $\exp i\theta = q^{\frac{1}{2}}$ and then replace q by q^4; we thus have

$$1+\sum_{n=1}^{\infty}\frac{(-)^n(1+q^{4n})(2-q-q^{-1})q^{2n(3n+1)}}{(1-q^{4n-1})(1-q^{4n+1})}$$

$$=\prod_{r=1}^{\infty}(1-q^{4r})\left[1+\sum_{n=1}^{\infty}\frac{q^{4n^2}}{(1-q^3)(1-q^5)(1-q^7)\dots(1-q^{4n+1})}\right],$$

that is to say

$$1+\sum_{n=1}^{\infty}(-)^n q^{2n(3n+1)}\left\{\frac{1-q^{-1}}{1-q^{4n-1}}+\frac{1-q}{1-q^{4n+1}}\right\}$$

$$=\prod_{r=1}^{\infty}(1-q^{4r})\left[1+\sum_{n=1}^{\infty}\frac{q^{4n^2}(1-q^{4n+1})+q^{(2n+1)^2}}{(1-q^3)(1-q^5)\dots(1-q^{4n+1})}\right]$$

$$=\prod_{r=1}^{\infty}(1-q^{4r})\left[1+\sum_{m=2}^{\infty}\frac{q^{m^2}}{(1-q^3)(1-q^5)\dots(1-q^{2m-1})}\right]$$

$$(m=2n \text{ or } 2n+1)$$

$$=\prod_{r=1}^{\infty}(1-q^{4r})\,[1-q+(1-q)\,\psi(q)].$$

Hence we have

$$\psi(q)\prod_{r=1}^{\infty}(1-q^{4r})$$

$$=\frac{1}{1-q}+\sum_{n=1}^{\infty}(-)^n q^{2n(3n+1)}\left\{\frac{1}{1-q^{4n+1}}-\frac{q^{-1}}{1-q^{4n-1}}\right\}-\prod_{r=1}^{\infty}(1-q^{4r})$$

$$=\frac{1}{1-q}-1+\sum_{n=1}^{\infty}(-)^n q^{2n(3n+1)}\left\{\frac{1}{1-q^{4n+1}}-\frac{q^{-1}}{1-q^{4n-1}}-(1+q^{-4n})\right\}$$

$$=\frac{q}{1-q}+\sum_{n=1}^{\infty}(-)^n q^{2n(3n+1)}\left\{\frac{q^{4n+1}}{1-q^{4n+1}}+\frac{q^{1-8n}}{1-q^{-4n+1}}\right\}$$

$$=\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{6n(n+1)+1}}{1-q^{4n+1}}.$$

We accordingly adopt the formula

$$\psi(q)\prod_{r=1}^{\infty}(1-q^{4r})=\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{6n(n+1)+1}}{1-q^{4n+1}}$$

as the modified definition of $\psi(q)$.

We next take another limiting case of the formula connecting basic hypergeometric series. Make $e\to\infty$, $f\to\infty$, $g\to\infty$, and let $a=q$ and $cd=q$, writing

$$c=q^{\frac{1}{2}}\exp i\theta,\quad d=q^{\frac{1}{2}}\exp(-i\theta).$$

On reduction we find that

$$\sum_{n=0}^{\infty}\frac{(-)^n(1-q^{2n+1})\,q^{\frac{3}{2}n(n+1)}}{1-2q^{n+\frac{1}{2}}\cos\theta+q^{2n+1}}=\prod_{r=1}^{\infty}(1-q^r)\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{\prod\limits_{m=0}^{n}(1-2q^{m+\frac{1}{2}}\cos\theta+q^{2m+1})}.$$

In this relation take successively

$$\theta=0,\quad \theta=\tfrac{1}{2}\pi,\quad \theta=\tfrac{2}{3}\pi;$$

and at the same time replace q by q^2 in the first and third of the results which are obtained.

We get immediately

$$\omega(q)\prod_{r=1}^{\infty}(1-q^{2r})=\sum_{n=0}^{\infty}(-)^n q^{3n(n+1)}\frac{1+q^{2n+1}}{1-q^{2n+1}},$$

$$\upsilon(q)\prod_{r=1}^{\infty}(1-q^{r})=\sum_{n=0}^{\infty}(-)^n q^{\frac{3}{2}n(n+1)}\frac{1-q^{2n+1}}{1+q^{2n+1}},$$

$$\rho(q)\prod_{r=1}^{\infty}(1-q^{2r})=\sum_{n=0}^{\infty}(-)^n q^{3n(n+1)}\frac{1-q^{4n+2}}{1+q^{2n+1}+q^{4n+2}}.$$

These are the formulae which will be adopted as the definitions of $\omega(q)$, $\upsilon(q)$, and $\rho(q)$. They render obvious the connection between $\rho(q)$ and $\omega(q)$; for, by combining like terms of the series on the right, we have

$$[2\rho(q)-\omega(q)]\prod_{r=1}^{\infty}(1-q^{2r})=3\sum_{n=0}^{\infty}(-)^n q^{3n(n+1)}\frac{1+q^{6n+3}}{1-q^{6n+3}};$$

and the required relation follows from the formula*

$$\sum_{n=0}^{\infty}(-)^n q^{n(n+1)}\frac{1+q^{2n+1}}{1-q^{2n+1}}=\frac{\vartheta_2(0,q)\,\vartheta_3(0,q)}{2q^{\frac{1}{4}}}$$

$$=\left[\frac{\vartheta_2(0,\sqrt{q})}{2q^{\frac{1}{8}}}\right]^2.$$

* This formula is immediately derivable from the expression for $1/\vartheta_4(z)$ as a sum of partial fractions. *Cf.* J. Tannery et J. Molk, *Théorie des fonctions elliptiques*, 3 (1898), 136.

Now that the new definitions of the mock ϑ-functions have been constructed, it is a fairly easy matter to establish the relations which connect the functions.

Apart from the relations connecting $\chi(q)$ with $f(q)$ and $\rho(q)$ with $\omega(q)$ which have been obtained already, these relations are special cases of an expansion of the reciprocal of the product of three ϑ-functions. Easy though this expansion is to establish, I do not remember having encountered it previously. It may be stated as follows:

Let r be any integer (positive, zero, or negative) and let α, β, γ be any constants such that

$$\vartheta_1(\beta-\gamma)\,\vartheta_1(\gamma-\alpha)\,\vartheta_1(\alpha-\beta) \neq 0.$$

Then the function

$$\frac{e^{(2r-1)iz}}{\vartheta_2(z-\alpha)\,\vartheta_2(z-\beta)\,\vartheta_2(z-\gamma)}$$

is expressible as the sum of partial fractions

$$\sum_{m=-\infty}^{\infty} \frac{A_m}{e^{2iz}+q^{2m}\,e^{2i\alpha}} + \sum_{m=-\infty}^{\infty} \frac{B_m}{e^{2iz}+q^{2m}\,e^{2i\beta}} + \sum_{m=-\infty}^{\infty} \frac{C_m}{e^{2iz}+q^{2m}\,e^{2i\gamma}},$$

where
$$A_m = \frac{2(-)^{m+r}\,q^{m(3m+1)+2mr}\,e^{2mi(2\alpha-\beta-\gamma)+(2r+1)i\alpha}}{\vartheta_1'(0)\,\vartheta_1(\alpha-\beta)\,\vartheta_1(\alpha-\gamma)},$$

with corresponding values for B_m and C_m. The expansion is valid for all values of z except the poles of the function under consideration.

First observe that, if $q = e^{\pi i\tau}$, it is easy enough to verify that

$$\lim_{z\to\alpha+\frac{1}{2}\pi+m\pi\tau} \frac{(e^{2iz}+q^{2m}\,e^{2i\alpha})\,e^{(2r-1)iz}}{\vartheta_2(z-\alpha)\,\vartheta_2(z-\beta)\,\vartheta_2(z-\gamma)}$$

is equal to the value of A_m given above.

We now proceed to establish the expansion by obtaining it as a limiting case of a similar expansion in which the functions concerned are algebraic. Write

$$\vartheta_{2;\,N}(z) = 2q^{\frac{1}{4}}\cos z \prod_{n=1}^{\infty}(1-q^{2n}) \prod_{n=1}^{N}(1+2q^{2n}\cos 2z+q^{4n})$$

so that
$$\frac{e^{(2r-1)iz}}{\vartheta_{2;\,N}(z-\alpha)\,\vartheta_{2;\,N}(z-\beta)\,\vartheta_{2;\,N}(z-\gamma)}$$

is the quotient of two polynomials in e^{2iz}. The elementary theory of partial fractions then shows that, when $3N+r+1 \geqslant 0$, we have

$$\frac{e^{(2r-1)iz}}{\vartheta_{2;N}(z-\alpha)\,\vartheta_{2;N}(z-\beta)\,\vartheta_{2;N}(z-\gamma)}$$

$$= \sum_{m=-N}^{N} \frac{A_{m;N}}{e^{2iz}+q^{2m}e^{2i\alpha}} + \sum_{m=-N}^{N} \frac{B_{m;N}}{e^{2iz}+q^{2m}e^{2i\beta}} + \sum_{m=-N}^{N} \frac{C_{m;N}}{e^{2iz}+q^{2m}e^{2i\gamma}},$$

where

$$A_{m;N} = A_m \prod_{n=N-m+1}^{\infty} [(1-q^{2n})(1-q^{2n}e^{-2i(\alpha-\beta)})(1-q^{2n}e^{-2i(\alpha-\gamma)})]$$

$$\times \prod_{n=N+m+1}^{\infty} [(1-q^{2n})(1-q^{2n}e^{2i(\alpha-\beta)})(1-q^{2n}e^{2i(\alpha-\gamma)})]$$

with corresponding values for $B_{m;N}$ and $C_{m;N}$.

We now make $N\to\infty$. The observation that

$$A_{m;N}/A_m$$

is a bounded function of m and N which, for any fixed m, tends to unity as $N\to\infty$, combined with the obvious remark that the series

$$\sum_{m=-\infty}^{\infty} \frac{A_m}{e^{2iz}+q^{2m}e^{2i\alpha}}$$

is absolutely convergent, justifies an appeal to Tannery's theorem; we thus have

$$\lim_{N\to\infty} \sum_{m=-N}^{N} \frac{A_{m;N}}{e^{2iz}+q^{2m}e^{2i\alpha}} = \sum_{m=-\infty}^{\infty} \frac{A_m}{e^{2iz}+q^{2m}e^{2i\alpha}},$$

and the other details of the passage to the limit present no difficulties. We have therefore established the expansion

$$\frac{e^{(2r-1)iz}}{\vartheta_2(z-\alpha)\,\vartheta_2(z-\beta)\,\vartheta_2(z-\gamma)}$$

$$= \sum_{\alpha,\beta,\gamma} \sum_{m=-\infty}^{\infty} \frac{2(-)^{m+r}\,q^{m(3m+1)+2mr}\,e^{2mi(2\alpha-\beta-\gamma)+(2r+1)i\alpha}}{\vartheta_1'(0)\,\vartheta_1(\alpha-\beta)\,\vartheta_1(\alpha-\gamma)(e^{2iz}+q^{2m}e^{2i\alpha})}.$$

In this expansion take

$$r=0, \quad z=0, \quad \alpha=\tfrac{1}{4}\pi, \quad \beta=-\tfrac{1}{4}\pi, \quad \gamma=0;$$

we get

$$\frac{\vartheta_1'(0, q)}{\vartheta_2(0, q)} = \frac{2\vartheta_2(\frac{1}{4}\pi, q)}{\vartheta_2(0, q)} \sum_{m=-\infty}^{\infty} (-)^m q^{m(3m+1)} \left\{ \frac{e^{(\frac{3}{2}m+\frac{1}{4})\pi i}}{1+iq^{2m}} + \frac{e^{-(\frac{3}{2}m+\frac{1}{4})\pi i}}{1-iq^{2m}} \right\}$$

$$-2 \sum_{m=-\infty}^{\infty} \frac{(-)^m q^{m(3m+1)}}{1+q^{2m}}.$$

Now it is easy to verify that

$$e^{(\frac{3}{2}m+\frac{1}{4})\pi i}(1-iq^{2m}) + e^{-(\frac{3}{2}m+\frac{1}{4})\pi i}(1+iq^{2m})$$

$$= 2\cos(\tfrac{1}{2}m - \tfrac{1}{4})\pi - 2q^{2m}\sin(\tfrac{1}{2}m - \tfrac{1}{4})\pi$$

$$= (-)^{\frac{1}{2}m(3m+1)}\{1+(-q^2)^m\}\sqrt{2},$$

and hence we have

$$\frac{\vartheta_1'(0, q)}{\vartheta_2(0, q)} = \frac{2\vartheta_2(\frac{1}{4}\pi, q)\sqrt{2}}{\vartheta_2(0, q)} \phi(-q^2) \prod_{r=1}^{\infty} \{1-(-q^2)^r\} - f(q^2) \prod_{r=1}^{\infty} (1-q^{2r}).$$

Further,

$$\frac{\vartheta_2(\frac{1}{4}\pi, q)\sqrt{2}}{\vartheta_2(0, q)} \prod_{r=1}^{\infty} \{1-(-q^2)^r\} = \prod_{n=1}^{\infty} \left[\frac{1+q^{4n}}{(1+q^{2n})^2}\right] \prod_{n=1}^{\infty} [(1+q^{4n-2})(1-q^{4n})]$$

$$= \prod_{n=1}^{\infty} \left[\frac{(1+q^{2n})(1-q^{4n})}{(1+q^{2n})^2}\right]$$

$$= \prod_{n=1}^{\infty} (1-q^{2n}),$$

so that we have

$$2\phi(-q^2) - f(q^2) = \frac{\vartheta_1'(0, q)}{\vartheta_2(0, q)} \prod_{n=1}^{\infty} (1-q^{2n})^{-1}$$

$$= \prod_{n=1}^{\infty} \left[\frac{1-q^{2n}}{(1+q^{2n})^2}\right] = \vartheta_4(0, q^2) \prod_{n=1}^{\infty} (1+q^{2n})^{-1},$$

whence Ramanujan's relation connecting $\phi(-q)$ with $f(q)$ follows immediately.

Again, in the partial fraction formula take

$$r = 1, \quad z = 0, \quad \alpha = 0, \quad \beta = \tfrac{1}{2}\pi\tau, \quad \gamma = \tfrac{1}{4}\pi\tau;$$

we get

$$\frac{\vartheta_1'(0,q)\,\vartheta_1^2(\tfrac{1}{4}\pi\tau,q)}{\vartheta_2(0,q)\,\vartheta_2(\tfrac{1}{4}\pi\tau,q)\,\vartheta_2(\tfrac{1}{2}\pi\tau,q)}$$

$$= -\frac{2\vartheta_1(\tfrac{1}{4}\pi\tau,q)}{\vartheta_1(\tfrac{1}{2}\pi\tau,q)}\left[\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+\frac{1}{2})}}{1+q^{2m}}+\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3(m+\frac{1}{2})(m+1)}}{1+q^{2m+1}}\right]$$

$$+2\sum_{m=-\infty}^{\infty}\frac{q^{3m(m+1)+\frac{3}{4}}}{1+q^{2m+\frac{1}{2}}}$$

$$= -\frac{2\vartheta_1(\tfrac{1}{4}\pi\tau,q)}{\vartheta_1(\tfrac{1}{2}\pi\tau,q)}\sum_{n=-\infty}^{\infty}\frac{(-)^{\frac{1}{2}n(3n+1)}q^{\frac{3}{4}n(n+1)}}{1+q^n}+2q^{\frac{1}{4}}\sum_{m=-\infty}^{\infty}\frac{q^{3m(m+1)+\frac{1}{2}}}{1+q^{2m+\frac{1}{2}}}$$

$$(n = 2m \text{ or } 2m+1)$$

$$= -\frac{\vartheta_1(\tfrac{1}{4}\pi\tau,q)}{\vartheta_1(\tfrac{1}{2}\pi\tau,q)}\,\phi(-\sqrt{q})\prod_{r=1}^{\infty}\{1-(-\sqrt{q})^r\}-2q^{\frac{1}{4}}\psi(-\sqrt{q})\prod_{r=1}^{\infty}(1-q^{2r}).$$

Now evidently

$$\frac{q^{-\frac{1}{4}}\vartheta_1(\tfrac{1}{4}\pi\tau,q)}{\vartheta_1(\tfrac{1}{2}\pi\tau,q)}\prod_{r=1}^{\infty}\{1-(-\sqrt{q})^r\} = \prod_{n=1}^{\infty}\left[\frac{1-q^{n-\frac{1}{2}}}{(1-q^{2n-1})^2}\right]\prod_{n=1}^{\infty}[(1+q^{n-\frac{1}{2}})(1-q^n)]$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-q^{2n-1})(1-q^n)}{(1-q^{2n-1})^2}\right]$$

$$= \prod_{n=1}^{\infty}(1-q^{2n}).$$

Hence we have

$$\phi(-\sqrt{q})+2\psi(-\sqrt{q}) = -\frac{q^{-\frac{1}{4}}\vartheta_1'(0,q)\,\vartheta_1^2(\tfrac{1}{4}\pi\tau,q)}{\vartheta_2(0,q)\,\vartheta_2(\tfrac{1}{4}\pi\tau,q)\,\vartheta_2(\tfrac{1}{2}\pi\tau,q)}\prod_{n=1}^{\infty}(1-q^{2n})^{-1}$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})(1-q^{n-\frac{1}{2}})^2}{(1+q^{2n})^2(1+q^{n-\frac{1}{2}})(1+q^{2n-1})^2}\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})(1-q^{n-\frac{1}{2}})^2}{(1+q^n)^2(1+q^{n-\frac{1}{2}})}\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-q^n)(1-q^{n-\frac{1}{2}})^2}{(1+q^n)(1+q^{n-\frac{1}{2}})}\right]$$

$$= \prod_{n=1}^{\infty}\left(\frac{1-q^{\frac{1}{2}n}}{1+q^{\frac{1}{2}n}}\right)\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})$$

$$= \vartheta_4(0,\sqrt{q})\prod_{n=1}^{\infty}(1+q^{\frac{1}{2}n})^{-1}.$$

We have therefore obtained the two results

$$\vartheta_4(0,q)\prod_{n=1}^{\infty}(1+q^n)^{-1} = 2\phi(-q)-f(q) = \phi(-q)+2\psi(-q),$$

and from them the formula

$$f(q)+4\psi(-q)=\vartheta_4(0,\,q)\prod_{n=1}^{\infty}(1+q^n)^{-1}$$

follows at once.

Next, in the partial fraction formula take

$$r=1,\quad z=0,\quad \alpha=\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\tau,\quad \beta=\tfrac{1}{4}\pi\tau,\quad \gamma=\pi+\tfrac{3}{4}\pi\tau;$$

we get

$$\frac{\vartheta_1'(0,\,q)}{2q^{\frac{3}{8}}\vartheta_2(\frac{3}{4}\pi\tau,\,q)}=-\frac{iq^{\frac{3}{8}}\vartheta_1(\frac{1}{2}\pi\tau,\,q)}{\vartheta_2(\frac{1}{4}\pi\tau,\,q)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}$$

$$+\sum_{m=-\infty}^{\infty}\frac{q^{3m(m+\frac{1}{2})}}{1+q^{2m+\frac{1}{2}}}-\sum_{m=-\infty}^{\infty}\frac{q^{3(m+\frac{1}{2})(m+1)}}{1+q^{2m+\frac{3}{2}}}$$

$$=\frac{q^{\frac{1}{4}}\vartheta_4(0,\,q)}{\vartheta_2(\frac{1}{4}\pi\tau,\,q)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}+\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{\frac{3}{4}n(n+1)}}{1+q^{n+\frac{1}{2}}}$$

$$(n=2m\ \text{ or }\ 2m+1)$$

$$=\frac{q^{\frac{1}{4}}\vartheta_4(0,\,q)}{\vartheta_2(\frac{1}{4}\pi\tau,\,q)}\,\omega(q)\prod_{r=1}^{\infty}(1-q^{2r})+v(\sqrt{q})\prod_{r=1}^{\infty}(1-q^{\frac{1}{2}r}).$$

Now evidently

$$\frac{\vartheta_4(0,\,q)}{\vartheta_2(\frac{1}{4}\pi\tau,\,q)}\prod_{r=1}^{\infty}\left(\frac{1-q^{2r}}{1-q^{\frac{1}{2}r}}\right)=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n-1})^2}{1+q^{n-\frac{1}{2}}}\right]\prod_{n=1}^{\infty}\left[\frac{1-q^{2n}}{(1-q^{n-\frac{1}{2}})(1-q^n)}\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n-1})(1-q^{2n})}{1-q^n}\right]=1.$$

Hence we have

$$v(\sqrt{q})+q^{\frac{1}{4}}\,\omega(q)=\frac{\vartheta_1'(0,\,q)}{2q^{\frac{3}{8}}\vartheta_2(\frac{3}{4}\pi\tau,\,q)}\prod_{r=1}^{\infty}(1-q^{\frac{1}{2}r})^{-1}$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})^2}{(1+q^{n-\frac{1}{2}})(1-q^{n-\frac{1}{2}})(1-q^n)}\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-q^{2n})^2}{(1-q^{2n-1})(1-q^n)}\right]$$

$$=\prod_{n=1}^{\infty}\left[(1+q^n)^3(1-q^n)\right]$$

$$=\tfrac{1}{2}q^{-\frac{1}{8}}\vartheta_2(0,\,\sqrt{q})\prod_{n=1}^{\infty}(1+q^n).$$

Since the expression on the right is a one-valued function of q, we immediately deduce the pair of formulae

$$v(q)+q\omega(q^2)=v(-q)-q\omega(q)^2=\tfrac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,q)\prod_{n=1}^{\infty}(1+q^{2n}).$$

Lastly, in the partial fraction formula take

$$r=1,\quad z=0,\quad \alpha=\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\tau,\quad \beta=\tfrac{1}{4}\pi,\quad \gamma=\tfrac{3}{4}\pi+\pi\tau;$$

we get

$$\frac{e^{-\frac{1}{4}\pi i}\vartheta_1'(0)\vartheta_1(\frac{1}{4}\pi+\frac{1}{2}\pi\tau)\vartheta_2(\pi\tau)}{2\vartheta_1(\frac{1}{2}\pi\tau)\vartheta_2(\frac{1}{4}\pi)\vartheta_2(\frac{3}{4}\pi+\pi\tau)}=\frac{e^{\frac{1}{4}\pi i}q^{\frac{3}{4}}\vartheta_1(\frac{1}{2}\pi+\pi\tau)}{\vartheta_1(\frac{1}{4}\pi+\frac{1}{2}\pi\tau)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}$$
$$+i\sum_{m=-\infty}^{\infty}\frac{q^{3m^2}e^{-\frac{1}{2}m\pi i}}{1+iq^{2m}}+\sum_{m=-\infty}^{\infty}\frac{q^{3(m+1)^2}e^{\frac{1}{2}m\pi i}}{1-iq^{2m+2}},$$

so that, replacing m by n or $n-1$, we have

$$\frac{\vartheta_1'(0)\vartheta_4(\frac{1}{4}\pi)\vartheta_2(0)}{2\vartheta_4(0)\vartheta_2^2(\frac{1}{4}\pi)}$$
$$=\frac{q^{\frac{3}{4}}\vartheta_2(0)}{\vartheta_4(\frac{1}{4}\pi)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}+i\sum_{n=-\infty}^{\infty}q^{3n^2}\left\{\frac{e^{-\frac{1}{2}n\pi i}}{1+iq^{2n}}-\frac{e^{\frac{1}{2}n\pi i}}{1-iq^{2n}}\right\}$$
$$=\frac{q^{\frac{3}{4}}\vartheta_2(0)}{\vartheta_4(\frac{1}{4}\pi)}\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{3m(m+1)}}{1-q^{2m+1}}+2\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{4m(3m+1)}}{1+q^{8m}}$$
$$+2q^3\sum_{m=-\infty}^{\infty}\frac{(-)^m q^{12m(m+1)}}{1+q^{8m+4}}$$
$$(n=2m \text{ or } 2m+1)$$
$$=\frac{q^{\frac{3}{4}}\vartheta_2(0)}{\vartheta_4(\frac{1}{4}\pi)}\omega(q)\prod_{r=1}^{\infty}(1-q^{2r})+[f(q^8)+2q^3\omega(-q^4)]\prod_{r=1}^{\infty}(1-q^{8r}).$$

When we reduce this in the usual manner, we find that

$$f(q^8)+2q\,\omega(q)+2q^3\omega(-q^4)=\vartheta_3(0,q)\vartheta_3^2(0,q^2)\prod_{n=1}^{\infty}(1-q^{4n})^{-2},$$

and hence, by changing the sign of q,

$$f(q^8)-2q\,\omega(-q)-2q^3\omega(-q^4)=\vartheta_4(0,q)\vartheta_3^2(0,q^2)\prod_{n=1}^{\infty}(1-q^{4n})^{-2}.$$

Any other relations of this kind connecting mock ϑ-functions of order 3 would appear to be derivable from the relations now obtained.

It is now feasible to construct the linear transformations of the mock ϑ-functions. Since any substitution of the modular group can be resolved into a number of substitutions of the forms

$$\tau' = \tau + 1, \quad \tau' = -1/\tau,$$

it is sufficient to construct the transformations which express the fourteen functions* $f(\pm q)$, ... in terms of similar functions of q_1 (or powers of q_1), where q and q_1 are connected by the relations†

$$q = e^{-\alpha}, \quad \alpha\beta = \pi^2, \quad q_1 = e^{-\beta}.$$

The general similarity between the series involved in the new definitions of the mock ϑ-functions and the series which are generating functions of class-numbers of binary quadratic forms suggests that it may be possible to construct the required transformations by means of functional equations such as have been used by Mordell‡ in connection with class-numbers. Since, however, I lacked the ingenuity necessary for the construction of the functional equations (if indeed they exist), I decided to use the more prosaic methods of contour integration by which a writer subsequent to Mordell has treated the generating functions of class-numbers§.

It is unnecessary to work out all the fourteen transformation formulae by contour integration; when the transformation formulae for $f(q)$ and $\phi(q)$ have been constructed, the remainder can be deduced immediately from the relations connecting the various mock ϑ-functions.

First consider $f(q)$. We have, by Cauchy's theorem,

$$f(q) \prod_{r=1}^{\infty} (1-q^r) = \frac{1}{2\pi i} \left\{ \int_{-\infty-ic}^{\infty-ic} + \int_{\infty+ic}^{-\infty+ic} \right\} \frac{\pi}{\sin \pi z} \frac{\exp(-\frac{3}{2}\alpha z^2)}{\cosh \frac{1}{2}\alpha z}\, dz,$$

where c is a positive number so small that the zeros of $\sin \pi z$ are the only poles of the integrand between the lines forming the contour. On the higher of these two lines we write

$$\frac{1}{\sin \pi z} = -2i \sum_{n=0}^{\infty} e^{(2n+1)\pi i z},$$

* Actually I do not trouble to deal with the functions $\chi(\pm q)$ and $\rho(\pm q)$ which are less interesting than the rest.

† The numbers α and β, which are positive when q is positive, are slightly easier to work with than the complex τ.

‡ L. J. Mordell, *Quart. J. of Math.*, 48 (1920), 329–342.

§ G. N. Watson, *Compositio Math.*, 1 (1934), 39–68.

so that

$$\frac{1}{2\pi i}\int_{\infty+ic}^{-\infty+ic} \frac{\pi}{\sin \pi z} \frac{\exp(-\frac{3}{2}az^2)}{\cosh\frac{1}{2}az}\,dz$$

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic} 4\pi i \exp\{(2n+1)\pi i z-\tfrac{3}{2}az^2\}\frac{e^{az}+e^{-az}-1}{e^{\frac{3}{2}az}+e^{-\frac{3}{2}az}}\,dz$$

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic} F_n(z)\,dz,$$

say. We calculate these integrals in the following manner. The poles of $F_n(z)$ are (at most) simple poles at the points

$$z_m=\frac{(2m+1)\pi i}{3a} \quad (m=-\infty, \ldots, -1, 0, 1, \ldots, +\infty),$$

and the residue at z_m is

$$\frac{4\pi}{3a}(-)^m \exp\{(2n+1)\pi i z_m-\tfrac{3}{2}a{z_m}^2\}.(2\cosh az_m-1)=\lambda_{n,m},$$

say. Now, by Cauchy's theorem,

$$\frac{1}{2\pi i}\left\{\int_{-\infty+ic}^{\infty+ic}-P\int_{-\infty+z_n}^{\infty+z_n}\right\}F_n(z)\,dz=\lambda_{n,0}+\lambda_{n,1}+\ldots+\lambda_{n,n-1}+\tfrac{1}{2}\lambda_{n,n},$$

where P denotes the "principal value" of the integral. Next, by rearrangement of repeated series,

$$\tfrac{1}{2}\lambda_{0,0}+\sum_{n=1}^{\infty}(\lambda_{n,0}+\lambda_{n,1}+\ldots+\lambda_{n,n-1}+\tfrac{1}{2}\lambda_{n,n})$$

$$=\sum_{m=0}^{\infty}(\tfrac{1}{2}\lambda_{m,m}+\lambda_{m+1,m}+\lambda_{m+2,m}+\ldots)$$

$$=\tfrac{1}{2}\sum_{m=0}^{\infty}\lambda_{m,m}\frac{1+\exp 2\pi i z_m}{1-\exp 2\pi i z_m}$$

$$=\frac{2\pi}{3a}\sum_{m=0}^{\infty}(-)^m\{2\cos\tfrac{1}{3}(2m+1)\pi-1\}\,q_1^{\frac{1}{3}(2m+1)^2}\frac{1+q_1^{\frac{2}{3}(2m+1)}}{1-q_1^{\frac{2}{3}(2m+1)}}$$

$$=\frac{2\pi}{a}\sum_{p=0}^{\infty}(-)^p q_1^{3(2p+1)^2}\frac{1+q_1^{4p+2}}{1-q_1^{4p+2}},$$

where $m=3p+1$, the terms for which $m\neq 3p+1$ vanishing.

Further, we have

$$P\int_{-\infty+z_n}^{\infty+z_n} F_n(z)\,dz$$

$$= P\int_{-\infty}^{\infty} F_n(z_n+x)\,dx$$

$$= P\int_{-\infty}^{\infty} 4\pi i \exp\left\{-\frac{(2n+1)^2\pi^2}{6a}-\frac{3ax^2}{2}\right\}\frac{\cosh\{ax+\frac{1}{3}(2n+1)\pi i\}-\frac{1}{2}}{(-)^n i\sinh\frac{3}{2}ax}\,dx$$

$$= 4\pi i(-)^n \sin\tfrac{1}{3}(2n+1)\pi\,.\,q_1^{\frac{1}{6}(2n+1)^2}\int_{-\infty}^{\infty} e^{-\frac{3}{2}ax^2}\frac{\sinh ax}{\sinh\frac{3}{2}ax}\,dx.$$

This simplification in the integral under consideration is due to the modified contour having been chosen to pass through the stationary point of the function

$$\exp\{(2n+1)\pi iz-\tfrac{3}{2}az^2\},$$

which occurs in the integrand, in the manner of the "method of steepest descents".

The integral along the lower line can be evaluated at once by changing the sign of i throughout the previous work. On combining the results we get

$$f(q)\prod_{r=1}^{\infty}(1-q^r)=\frac{4\pi}{a}q_1^{\frac{4}{3}}\omega(q_1^2)\prod_{r=1}^{\infty}(1-q_1^{4r})+4\vartheta_1(\tfrac{1}{3}\pi,\,q_1^{\frac{3}{2}})\int_0^{\infty} e^{-\frac{3}{2}ax^2}\frac{\sinh ax}{\sinh\frac{3}{2}ax}\,dx.$$

By Jacobi's imaginary transformation this reduces to

$$q^{-\frac{1}{24}}f(q)=2\sqrt{\left(\frac{2\pi}{a}\right)}\,q_1^{\frac{4}{3}}\omega(q_1^2)+4\sqrt{\left(\frac{3a}{2\pi}\right)}\int_0^{\infty} e^{-\frac{3}{2}ax^2}\frac{\sinh ax}{\sinh\frac{3}{2}ax}\,dx,$$

which is the transformation for $f(q)$. We shall consider the integral on the right presently.

We now turn to $\phi(q)$. We have, by Cauchy's theorem,

$$\phi(q)\prod_{r=1}^{\infty}(1-q^r)=\frac{1}{2\pi i}\left\{\int_{-\infty-ic}^{\infty-ic}+\int_{\infty+ic}^{-\infty+ic}\right\}\frac{\pi}{\sin\pi z}\frac{\cosh\frac{1}{2}az}{\cosh az}\exp(-\tfrac{3}{2}az^2)\,dz,$$

and, as before, we get

$$\frac{1}{2\pi i}\int_{\infty+ic}^{-\infty+ic}\frac{\pi}{\sin\pi z}\frac{\cosh\frac{1}{2}az}{\cosh az}\exp(-\tfrac{3}{2}az^2)\,dz=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic}\Phi_n(z)\,dz,$$

where

$$\Phi_n(z)=2\pi i\exp\{(2n+1)\pi iz-\tfrac{3}{2}az^2\}\frac{\cosh\frac{1}{2}az\{2\cosh 2az-1\}}{\cosh 3az}.$$

The poles of $\Phi_n(z)$ are (at most) simple poles at the points

$$\zeta_m = \frac{(4m+1)\pi i}{6a}, \quad \eta_m = \frac{(4m-1)\pi i}{6a} \quad (m = -\infty, \ldots, -1, 0, 1, \ldots, \infty),$$

and the residues at ζ_m and η_m are

$$\frac{2\pi}{3a} \exp\{(2n+1)\pi i\,\zeta_m - \tfrac{3}{2}a\zeta_m{}^2\} . \cosh \tfrac{1}{2}a\zeta_m \{2\cosh 2a\zeta_m - 1\} = \mu_{n,m},$$

$$-\frac{2\pi}{3a} \exp\{(2n+1)\pi i\,\eta_m - \tfrac{3}{2}a\eta_m{}^2\} . \cosh \tfrac{1}{2}a\eta_m \{2\cosh 2a\eta_m - 1\} = \nu_{n,m},$$

say. Now, by Cauchy's theorem,

$$\frac{1}{2\pi i}\left\{\int_{-\infty+ic}^{\infty+ic} - \int_{-\infty+z_n}^{\infty+z_n}\right\} \Phi_n(z)\,dz$$

$$= \mu_{n,0} + \mu_{n,1} + \mu_{n,2} + \cdots + \mu_{n,n} + \nu_{n,1} + \nu_{n,2} + \cdots + \nu_{n,n},$$

and, as before, by rearrangement of repeated series,

$$\sum_{n=0}^{\infty} (\mu_{n,0} + \mu_{n,1} + \ldots + \mu_{n,n}) + \sum_{n=1}^{\infty} (\nu_{n,1} + \nu_{n,2} + \ldots + \nu_{n,n})$$

$$= \sum_{m=0}^{\infty} (\mu_{m,m} + \mu_{m+1,m} + \mu_{m+2,m} + \ldots) + \sum_{m=1}^{\infty} (\nu_{m,m} + \nu_{m+1,m} + \nu_{m+2,m} + \ldots)$$

$$= \sum_{m=0}^{\infty} \frac{\mu_{m,m}}{1 - \exp 2\pi i\zeta_m} + \sum_{m=1}^{\infty} \frac{\nu_{m,m}}{1 - \exp 2\pi i\eta_m}$$

$$= \frac{2\pi}{3a} \sum_{m=0}^{\infty} \frac{q_1^{(4m+1)(4m+3)/24}}{1 + q_1^{(4m+1)/3}} \cos\frac{(4m+1)\pi}{12} \left\{2\cos\frac{(4m+1)\pi}{3} - 1\right\}$$

$$- \frac{2\pi}{3a} \sum_{m=1}^{\infty} \frac{q_1^{(4m-1)(4m+5)/24}}{1 - q_1^{(4m-1)/3}} \cos\frac{(4m-1)\pi}{12} \left\{2\cos\frac{4m-1)\pi}{3} - 1\right\}$$

$$= \frac{\pi\sqrt{2}}{a} q_1^{\frac{3}{8}} \left[\sum_{n=0}^{\infty} \frac{(-)^n q_1^{2n(3n+5)+1}}{1 - q_1^{4n+3}} + \sum_{p=0}^{\infty} \frac{(-)^p q_1^{6p(p+1)}}{1 - q_1^{4p+1}}\right]$$

$$(m = 3n+2;\ m = 3p+1).$$

Further, we have

$$\int_{-\infty+z_n}^{\infty+z_n} \Phi_n(z)\,dz = \int_{-\infty}^{\infty} \Phi_n(z_n + x)\,dx$$

$$= 2\pi i \int_{-\infty}^{\infty} \exp\left\{-\frac{(2n+1)^2\pi^2}{6a} - \frac{3ax^2}{2}\right\}$$

$$\times \frac{\cosh\frac{5}{2}az + \cosh\frac{3}{2}az - \cosh\frac{1}{2}az}{-\cosh 3ax}\,dx$$

$$= 2\pi i q_1^{(2n+1)^2/6} \cos\frac{(2n+1)\pi}{6} \int_{-\infty}^{\infty} e^{-\frac{3}{2}ax^2} \frac{\cosh\frac{5}{2}ax + \cosh\frac{1}{2}ax}{\cosh 3ax}\,dx.$$

The integral along the lower line can be evaluated by changing the sign of i throughout the previous work. On combining the results we get.

$$\phi(q) \prod_{r=1}^{\infty} (1-q^r)$$

$$= \frac{2\pi\sqrt{2}}{\alpha} q_1^{\frac{1}{8}} \psi(q_1) \prod_{r=1}^{\infty} (1-q_1^{4r}) + \vartheta_2(\tfrac{1}{6}\pi, q_1^{\frac{8}{3}}) \int_{-\infty}^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x}\, dx.$$

This reduces to

$$q^{-\frac{1}{24}} \phi(q) = 2\sqrt{\left(\frac{\pi}{\alpha}\right)}\, q_1^{-\frac{1}{24}} \psi(q_1) + \sqrt{\left(\frac{6\alpha}{\pi}\right)} \int_0^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x}\, dx,$$

which is the transformation for $\phi(q)$.

We next consider the integrals on the right of the transformation formulae. Let

$$\int_0^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x}\, dx = J(\alpha);$$

then it is easy to see that

$$J(\alpha) = \sqrt{\left(\frac{6\beta}{\pi}\right)} \int_0^{\infty}\int_0^{\infty} e^{-\frac{3}{2}\beta y^2} \cos 3\pi xy\, \frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x}\, dy\, dx$$

$$= \frac{\pi}{3\alpha}\sqrt{\left(\frac{6\beta}{\pi}\right)} \int_0^{\infty} e^{-\frac{3}{2}\beta y^2} \left\{\frac{\cosh\frac{1}{2}\beta y \cos\frac{5}{12}\pi}{\cosh\beta y + \cos\frac{5}{6}\pi} + \frac{\cosh\frac{1}{2}\beta y \cos\frac{1}{12}\pi}{\cosh\beta y + \cos\frac{1}{6}\pi}\right\} dy$$

$$= \sqrt{\left(\frac{\beta\pi}{\alpha^2}\right)} \int_0^{\infty} e^{-\frac{3}{2}\beta y^2} \frac{2\cosh\frac{3}{2}\beta y \cosh\beta y}{\cosh 3\beta y}\, dy,$$

so that
$$J(\alpha) = \sqrt{\left(\frac{\pi^3}{\alpha^3}\right)}\, J(\beta).$$

Next let
$$\int_0^{\infty} e^{-\frac{3}{2}\alpha x^2} \frac{\sinh \alpha x}{\sinh\frac{3}{2}\alpha x}\, dx = J_1(\alpha),$$

and it is found in a similar manner that

$$J_1(\alpha) = \sqrt{\left(\frac{2\pi^3}{\beta^3}\right)}\, J_2(\beta),$$

where
$$J_2(\beta) = \int_0^{\infty} e^{-\frac{3}{2}\beta x^2} \frac{\cosh\beta x}{\cosh 3\beta x}\, dx.$$

It is easy to obtain asymptotic expansions for $J(\alpha)$, $J_1(\alpha)$, and $J_2(\alpha)$ proceeding in ascending powers of α and valid when $|\alpha|$ is small and the real part of α is positive; the first few terms of the expansions are

$$J(\alpha)=\sqrt{\left(\frac{2\pi}{3\alpha}\right)}\,[1-\tfrac{23}{24}\alpha+\tfrac{3985}{1152}\alpha^2-\dots],$$

$$J_1(\alpha)=\sqrt{\left(\frac{2\pi}{27\alpha}\right)}\,[1-\tfrac{5}{72}\alpha+\tfrac{17}{1152}\alpha^2-\dots],$$

$$J_2(\alpha)=\sqrt{\left(\frac{\pi}{6\alpha}\right)}\,[1-\tfrac{4}{3}\alpha+\tfrac{44}{9}\alpha^2-\dots].$$

It can be proved that these expansions possess the property that (for α complex) the error due to stopping at any term never exceeds in absolute value the first term neglected; in addition, for α positive, the error is of the same sign as that term*.

I now revert to the construction of the set of transformation formulae; there is no difficulty in verifying that

$$q^{-\frac{1}{24}}f(q)-2\sqrt{\left(\frac{2\pi}{\alpha}\right)}\,q_1^{\frac{4}{3}}\,\omega(q_1^2)=2\sqrt{\left(\frac{6\alpha}{\pi}\right)}\,J_1(\alpha)=\frac{4\beta\sqrt{3}}{\pi}\,J_2(\beta),$$

$$q^{-\frac{1}{24}}f(-q)+\sqrt{\left(\frac{\pi}{\alpha}\right)}\,q_1^{-\frac{1}{24}}f(-q_1)=2\sqrt{\left(\frac{6\alpha}{\pi}\right)}\,J(\alpha)=\frac{2\beta\sqrt{6}}{\pi}\,J(\beta).$$

$$q^{-\frac{1}{24}}\phi(q)-2\sqrt{\left(\frac{\pi}{\alpha}\right)}\,q_1^{-\frac{1}{24}}\psi(q_1)=\sqrt{\left(\frac{6\alpha}{\pi}\right)}\,J(\alpha)=\frac{\beta\sqrt{6}}{\pi}\,J(\beta),$$

$$q^{-\frac{1}{24}}\phi(-q)-\sqrt{\left(\frac{2\pi}{\alpha}\right)}\,q_1^{\frac{1}{3}}\,\nu(-q_1)=\sqrt{\left(\frac{6\alpha}{\pi}\right)}\,J_1(\alpha)=\frac{2\beta\sqrt{3}}{\pi}\,J_2(\beta),$$

$$q^{-\frac{1}{24}}\psi(-q)-\sqrt{\left(\frac{\pi}{2\alpha}\right)}\,q_1^{\frac{1}{3}}\,\nu(q_1)=-\sqrt{\left(\frac{3\alpha}{2\pi}\right)}\,J_1(\alpha)=-\frac{\beta\sqrt{3}}{\pi}\,J_2(\beta),$$

$$q^{\frac{2}{3}}\,\omega(q)-\sqrt{\left(\frac{\pi}{4\alpha}\right)}\,q_1^{-\frac{1}{12}}f(q_1^2)=-\sqrt{\left(\frac{3\alpha}{\pi}\right)}\,J_2(\tfrac{1}{2}\alpha)=-\frac{2\beta\sqrt{3}}{\pi}\,J_1(2\beta).$$

The transformation formula for $\omega(-q)$ is a little more troublesome; we need the two relations

$$f(q^8)+2q\,\omega(q)+2q^3\,\omega(-q^4)=\vartheta_3(0,q)\,\vartheta_3^2(0,q^2)\prod_{n=1}^{\infty}(1-q^{4n-2})^{-2},$$

$$f(q^8)+q\,\omega(q)-q\omega(-q)=\vartheta_3(0,q^4)\,\vartheta_3^2(0,q^2)\prod_{n=1}^{\infty}(1-q^{4n-2})^{-2}.$$

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* This property is established by the method given by G. N. Watson, *Compositio Math.*, 1 (1934), 39–68 (64–65). It is the fact that these expansions are asymptotic (and not terminating series) which shows that mock ϑ-functions are of a more complex character than ordinary ϑ-functions.

From these relations we have

$$q_1^{\frac{8}{3}}\,\omega(-q_1^4)=\tfrac{1}{2}q_1^{-\frac{1}{3}}\vartheta_3(0,\,q_1)\,\vartheta_3^2(0,\,q_1^2)\prod_{n=1}^{\infty}(1-q_1^{4n})^{-2}-\tfrac{1}{2}q_1^{-\frac{1}{3}}f(q_1^8)-q_1^{\frac{2}{3}}\,\omega(q_1)$$

$$=\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{-\frac{1}{12}}\vartheta_3(0,\,q)\,\vartheta_3^2(0,\,q^{\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)^{-2}$$

$$-\left[\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{\frac{1}{6}}\,\omega(q^{\frac{1}{4}})+4\sqrt{\left(\frac{3\beta}{\pi}\right)}\,J_1(8\beta)\right]$$

$$-\left[\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{-\frac{1}{12}}f(q^2)-\sqrt{\left(\frac{3\beta}{\pi}\right)}\,J_2(\tfrac{1}{2}\beta)\right]$$

$$=-\sqrt{\left(\frac{\pi}{4\beta}\right)}\,q^{\frac{1}{6}}\,\omega(-q^{\frac{1}{4}})+\sqrt{\left(\frac{3\beta}{\pi}\right)}\,[J_2(\tfrac{1}{2}\beta)-4J_1(8\beta)].$$

Hence, replacing q_1 by $q^{\frac{1}{4}}$, we get, as the last of the required transformations,

$$q^{\frac{2}{3}}\,\omega(-q)+\sqrt{\left(\frac{\pi}{a}\right)}\,q_1^{\frac{2}{3}}\,\omega(-q_1)=2\sqrt{\left(\frac{3a}{\pi}\right)}\,J_3(a),$$

where

$$J_3(a)=\tfrac{1}{4}J_2(\tfrac{1}{8}a)-J_1(2a)$$

$$=\tfrac{1}{4}\int_0^{\infty}e^{-\frac{3}{16}ay^2}\,\frac{\cosh\frac{1}{8}ay}{\cosh\frac{3}{8}ay}\,dy-\int_0^{\infty}e^{-3ax^2}\,\frac{\sinh 2ax}{\sinh 3ax}\,dx$$

$$=\int_0^{\infty}e^{-3ax^2}\left\{\frac{\cosh\frac{1}{2}ax}{\cosh\frac{3}{2}ax}-\frac{\sinh 2ax}{\sinh 3ax}\right\}dx,$$

i.e.

$$J_3(a)=\int_0^{\infty}e^{-3ax^2}\,\frac{\sinh ax}{\sinh 3ax}\,dx.$$

It is easy to prove that

$$J_3(\beta)=\left(\frac{a}{\pi}\right)^{\frac{3}{2}}J_3(a),$$

and that $J_3(a)$ possesses the asymptotic expansion

$$J_3(a)=\tfrac{1}{6}\sqrt{\left(\frac{\pi}{3a}\right)}\,[1-\tfrac{2}{9}a+\tfrac{17}{108}a^2-\ldots],$$

for small values of a, this expansion having the same general properties as the asymptotic expansions previously obtained.

Now that I have no more to say about the functions of order 3, I conclude with a brief mention of the functions of orders 5 and 7. The basic hypergeometric series which has been used hitherto is of no avail for these functions, and other means must be sought to establish Ramanujan's relations which connect functions of order 5. After spending a fortnight on fruitless attempts, I proceeded to attack the problem by the most elementary methods available, namely applications of Euler's formulae mingled with rearrangements of repeated series; and within the day I had proved not only the five relations set out by Ramanujan but also five other relations whose existence he had merely stated. My proofs of these relations are all so long that I took the trouble to analyse one of the longest in the hope of being able to say that it involved "thirty-nine steps"; it was, however, disappointing to a student of John Buchan to find that a moderately liberal count revealed only twenty-four.

The functions of order 7 seem to possess fewer features of interest, though a study of their behaviour near the unit circle by the process of estimating the sum of those terms of the series by which they are defined which are in the neighbourhood of the greatest terms has raised one question for which it was fascinating to seek the answer.

The study of Ramanujan's work and of the problems to which it gives rise inevitably recalls to mind Lamé's remark that, when reading Hermite's papers on modular functions, "on a la chair de poule". I would express my own attitude with more prolixity by saying that such a formula as

$$\int_0^\infty e^{-3\pi x^2} \frac{\sinh \pi x}{\sinh 3\pi x}\, dx = \frac{1}{e^{\frac{2}{3}\pi}\sqrt{3}} \sum_{n=0}^{\infty} \frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2(1+e^{-3\pi})^2 \ldots (1+e^{-(2n+1)\pi})^2}$$

gives me a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of the four statues representing "Day", "Night", "Evening", and "Dawn" which Michelangelo has set over the tombs of Giuliano de' Medici and Lorenzo de' Medici.

Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine where

> "Pale, beyond porch and portal,
> Crowned with calm leaves, she stands
> Who gathers all things mortal
> With cold immortal hands".

MOCK ϑ-FUNCTIONS AND REAL ANALYTIC MODULAR FORMS

S.P. ZWEGERS

Abstract. In this paper we examine three examples of Ramanujan's third order mock ϑ-functions and relate them to Rogers' false ϑ-series and to a real-analytic modular form of weight 1/2.

1. Introduction

Mock ϑ-functions were introduced by S. Ramanujan in the last letter he wrote to G.H. Hardy, dated January, 1920. For a photocopy of the mathematical part of this letter see [Ra, pp. 127–131] (also reproduced in [A2]). In this letter he provided a list of 17 mock ϑ-functions (4 of "order three", 10 of "order five" and 3 of "order seven"), together with identities they satisfy.

In [AH] we find a definition of the concept of a mock ϑ-function. Slightly rephrased it reads: a mock ϑ-function is a function f of the complex variable q, defined by a q-series of a particular type (Ramanujan calls this the Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions:

(1) infinitely many roots of unity are exponential singularities,
(2) for every root of unity ξ there is a ϑ-function $\vartheta_\xi(q)$ such that the difference $f(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially,
(3) there is no ϑ-function that works for all ξ, i.e. f is not the sum of two functions, one of which is a ϑ-function and the other a function which is bounded in all roots of unity.

(When Ramanujan refers to ϑ-functions, he means sums, products, and quotients of series of the form $\sum_{n\in\mathbf{Z}} \epsilon^n q^{an^2+bn}$ with $a, b \in \mathbf{Q}$ and $\epsilon = -1, 1$).

The 17 functions given by Ramanujan indeed satisfy condition (1) and (2) (see [W1], [W2] and [S]). However no proof has ever been given that they also satisfy condition (3). Watson (see [W1]) proved a very weak form of condition (3) for the "third order" mock ϑ-functions, namely, that they are not equal to ϑ-functions.

In section 3 we will see that condition (3) is not satisfied if we weaken it slightly. Indeed, we shall discuss a vector-valued third order mock ϑ-function F for which there is a real analytic modular form H such that $F - H$ is bounded in all roots of unity.

Before that, we discuss in the next section a connection between mock ϑ-functions and Rogers' false ϑ-series. Again we look at the behaviour of a mock theta function when q approaches a root of unity radially. But now we extend the function across the unit circle.

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2. False ϑ-series

We will consider the mock ϑ-function ν, which is not mentioned in Ramanujan's letter, but which was found by Watson in [W1], and can also be found in Ramanujan's "lost" notebook [Ra]:

$$\begin{aligned}\nu(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}} \\ &= \frac{1}{1+q} + \frac{q^2}{(1+q)(1+q^3)} + \frac{q^6}{(1+q)(1+q^3)(1+q^5)} + \dots\end{aligned} \tag{2.1}$$

We can easily see that the defining sum for ν converges not only for $|q| < 1$, but also for $|q| > 1$. We will now study the function that is defined by the sum outside the unit disk. In order to do so, we replace q by q^{-1} in the sum, take $|q| < 1$ and call this new function ν_-. We get

$$\begin{aligned}\nu_-(q) &= \sum_{n=0}^{\infty} \frac{q^{n+1}}{(-q;q^2)_{n+1}} \\ &= \frac{q}{1+q} + \frac{q^2}{(1+q)(1+q^3)} + \frac{q^3}{(1+q)(1+q^3)(1+q^5)} + \dots\end{aligned} \tag{2.2}$$

In Ramanujan's "lost" notebook [Ra] we find the following identity for $|q| < 1$ (which was proved by Andrews in [A1]):

$$\nu_-(q) = \sum_{n=0}^{\infty} (-1)^n q^{6n^2+4n+1}(1+q^{4n+2}) \tag{2.3}$$

$$= \left(\sum_{n=0}^{\infty} - \sum_{n=-\infty}^{-1} \right) (-1)^n q^{6n^2+4n+1} = q^{\frac{1}{3}} \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{-3}{n} \right) q^{\frac{2}{3}n^2} \tag{2.4}$$

From these identities we see that ν_- has a very simple power series expansion. This expansion looks very much like a ϑ-function, only the signs are somewhat different. Rogers uses the term *false ϑ-series* for this type of functions (see [Ro, pp. 328]).

The following proposition (see [LZ]) shows that for every root of unity ξ the function ν_- is bounded as $q \to \xi$ radially. We can even compute the complete asymptotic expansion.

Proposition 2.1. *Let $C : \mathbf{Z} \to \mathbf{C}$ be a periodic function with mean value 0. Then the associated L-series $L(s, C) = \sum_{n=1}^{\infty} C(n)n^{-s}$ ($\mathrm{Re}(s) > 1$) extends holomorphically to $\mathbf{C}$. The two functions $\sum_{n=1}^{\infty} C(n)e^{-nt}$ and $\sum_{n=1}^{\infty} C(n)e^{-n^2t}$ ($t > 0$) have the asymptotic expansions*

$$\begin{aligned}\sum_{n=1}^{\infty} C(n)e^{-nt} &\sim \sum_{r=0}^{\infty} L(-r, C)\frac{(-t)^r}{r!} \\ \sum_{n=1}^{\infty} C(n)e^{-n^2t} &\sim \sum_{r=0}^{\infty} L(-2r, C)\frac{(-t)^r}{r!}\end{aligned} \tag{2.5}$$

as $t \searrow 0$. The numbers $L(-r, C)$ are given explicitly by

$$L(-r, C) = -\frac{M^r}{r+1} \sum_{n=1}^{M} C(n)B_{r+1}\left(\frac{n}{M}\right) \qquad (r = 0, 1, \dots) \tag{2.6}$$

where $B_k(x)$ denotes the k^{th} Bernoulli polynomial and M is any period of the function C.

In order to get the asymptotic expansion of ν_- as $q \to \xi$ radially, with ξ a root of unity, we write $q = \xi e^{-t}$. Thus we have to find the asymptotic expansion of $\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{-3}{n}\right)\xi^{\frac{2}{3}n^2}e^{-\frac{2}{3}tn^2}$ as $t \searrow 0$. We can now use the proposition provided we check that $C(n) := (-1)^{n+1}\left(\frac{-3}{n}\right)\xi^{\frac{2}{3}n^2}$ is a periodic function with mean value 0. Indeed, if K is the order of ξ then $6K$ is a period for C, while $C(6K-n) = -C(n)$. Hence the mean value of C is zero.

The behaviour of ν outside the unit circle is thus completely known. A question that now arises is whether the behaviour of ν outside the unit circle is related to the behaviour of ν inside the unit circle. Numerical computations in this and related examples led me to the following:

Conjecture 2.2. *If ξ is a root of unity where ν is bounded (as $q \to \xi$ radially inside the unit circle), for example $\xi = 1$, then ν is C^∞ over the line radially through ξ.*

If ξ is a root of unity where ν is not bounded, for example $\xi = -1$, then the asymptotic expansion of the bounded term in condition (2) in the introduction is the same as the asymptotic expansion of ν as $q \to \xi$ radially outside the unit circle.

Let us proceed a bit, assuming this conjecture. Let $\tilde{\nu}$ be a function which is defined in- and outside the unit circle and also at all roots of unity, such that (a) $\tilde{\nu}$ is holomorphic in- and outside the unit circle, (b) $\tilde{\nu}$ is C^∞ over all radial lines through roots of unity and (c) $\tilde{\nu} = \nu$ outside the unit circle. If we can find such a function $\tilde{\nu}$, then $\nu - \tilde{\nu}$ is zero outside the unit circle, it has asymptotic expansion zero for $q \to \xi$ if ξ is a root of unity where ν is bounded, and the bounded term in condition (2) for mock ϑ- functions also has asymptotic expansion zero for $q \to \xi$. Because of this one might expect $\nu - \tilde{\nu}$ to be modular. If indeed this is the case we have written ν as the sum of two functions $\nu - \tilde{\nu}$ and $\tilde{\nu}$, one of which is a ϑ-function and the other a function which is bounded in all roots of unity. This contradicts condition (3) in the definition of a mock ϑ-function.

Ramanujan probably had this idea in mind when he wrote in his letter to Hardy: "... I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singulartities of the original function. Also I have shown that if *it is necessarily so* then it leads to the following assertion—viz. it is possible to construct two power series in x, namely $\sum a_n x^n$ and $\sum b_n x^n$, both of which have *essential singularities* on the unit circle, are convergent when $|x| < 1$, and tend to *finite limits at every point* $x = e^{2i\pi r/s}$, and that at the same time the limit of $\sum a_n x^n$ at the point $x = e^{2i\pi r/s}$ is equal to the limit of $\sum b_n x^n$ at the point $x = e^{-2i\pi r/s}$."

Although it's possible to construct two such power series (see [A2, pp. 284]), it might not be possible to construct a function $\tilde{\nu}$ that satisfies the conditions (a), (b) and (c).

3. Mock ϑ-Functions and Real Analytic Modular Forms

In this section we will consider the following third order mock ϑ-functions:

$$\begin{aligned} f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \\ &= 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots \\ \omega(q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} \\ &= \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} + \frac{q^{12}}{(1-q)^2(1-q^3)^2(1-q^5)^2} + \dots \end{aligned} \tag{3.1}$$

Ramanujan mentioned f in his letter, and ω can be found in [W1] and [Ra].

Definition 3.1. Define $F = (f_0, f_1, f_2)^T$ by:

$$\begin{aligned} f_0(\tau) &= q^{-\frac{1}{24}} f(q) \\ f_1(\tau) &= 2q^{\frac{1}{3}} \omega(q^{\frac{1}{2}}) \\ f_2(\tau) &= 2q^{\frac{1}{3}} \omega(-q^{\frac{1}{2}}), \end{aligned} \tag{3.2}$$

with $q = e^{2\pi i\tau}$, $\tau \in \mathcal{H}$.

In [W1] Watson gave the modular transformation properties of f and ω. If we rewrite them in terms of F we get

Lemma 3.2. *For $\tau \in \mathcal{H}$ we have*

$$F(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} F(\tau) \tag{3.3}$$

and

$$\frac{1}{\sqrt{-i\tau}} F(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} F(\tau) + R(\tau), \tag{3.4}$$

with $\zeta_n = e^{2\pi i/n}$, $R(\tau) = 4\sqrt{3}\sqrt{-i\tau}(j_2(\tau), -j_1(\tau), j_3(\tau))^T$, where

$$\begin{aligned} j_1(\tau) &= \int_0^\infty e^{3\pi i\tau x^2} \frac{\sin 2\pi\tau x}{\sin 3\pi\tau x} dx \\ j_2(\tau) &= \int_0^\infty e^{3\pi i\tau x^2} \frac{\cos \pi\tau x}{\cos 3\pi\tau x} dx \\ j_3(\tau) &= \int_0^\infty e^{3\pi i\tau x^2} \frac{\sin \pi\tau x}{\sin 3\pi\tau x} dx. \end{aligned} \tag{3.5}$$

Proof. The transformation formula for $\tau \to \tau+1$ is trivial.

If we take the first formula from the set of transformation formulae on p. 78 in [W1], with $\alpha = -2\pi i\tau$, and multiply both sides by -1, we get

$$\frac{1}{\sqrt{-i\tau}} f_1(-1/\tau) - f_0(\tau) = -\frac{4\sqrt{3}}{\sqrt{-i\tau}} J_1(-2\pi i\tau) = -\frac{4\sqrt{3}}{\sqrt{-i\tau}} j_1(\tau), \tag{3.6}$$

which is the second component of equation (3.4).

If we take the last formula from the set of transformation formulae on p. 78 in [W1], with $\alpha = -\pi i\tau$, and multiply both sides by -2, we get

$$\frac{1}{\sqrt{-i\tau}} f_0(-1/\tau) - f_1(\tau) = \frac{2\sqrt{3}}{\sqrt{-i\tau}} J_2(-\frac{\pi i\tau}{2}) = \frac{4\sqrt{3}}{\sqrt{-i\tau}} j_2(\tau), \tag{3.7}$$

where we have replaced x by $2x$ in the integral. This equation is the first component of equation (3.4).

If we take the formula on the middle of p. 79 in [W1], with $\alpha = -\pi i\tau$, and multiply both sides by 2, we get

$$\frac{1}{\sqrt{-i\tau}} f_2(-1/\tau) + f_2(\tau) = \frac{4\sqrt{3}}{\sqrt{-i\tau}} J_3(-\pi i\tau) = \frac{4\sqrt{3}}{\sqrt{-i\tau}} j_3(\tau), \tag{3.8}$$

which is the third component of equation (3.4). □

In a moment we will define a (nonholomorphic) function G that satisfies the same modular transformation properties as F. Before that, we rewrite R in terms of period integrals of the following theta functions of weight 3/2:

$$\begin{aligned} g_0(z) &= \sum_{n\in\mathbf{Z}} (-1)^n (n+1/3) e^{3\pi i (n+\frac{1}{3})^2 z} \\ g_1(z) &= -\sum_{n\in\mathbf{Z}} (n+1/6) e^{3\pi i (n+\frac{1}{6})^2 z} \\ g_2(z) &= \sum_{n\in\mathbf{Z}} (n+1/3) e^{3\pi i (n+\frac{1}{3})^2 z}. \end{aligned} \tag{3.9}$$

These theta functions have the following modular transformation properties, which can be verified using standard methods:

$$\begin{pmatrix} g_0(z+1) \\ g_1(z+1) \\ g_2(z+1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \zeta_6 \\ 0 & \zeta_{24} & 0 \\ \zeta_6 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_0(z) \\ g_1(z) \\ g_2(z) \end{pmatrix} \tag{3.10}$$

and

$$\begin{pmatrix} g_0(-1/z) \\ g_1(-1/z) \\ g_2(-1/z) \end{pmatrix} = -(-iz)^{3/2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} g_0(z) \\ g_1(z) \\ g_2(z) \end{pmatrix}. \tag{3.11}$$

From these transformation properties and the Fourier expansions, we see that the g_j's are cusp forms.

Lemma 3.3. *For $\tau \in \mathcal{H}$ we have*

$$R(\tau) = -2i\sqrt{3} \int_0^{i\infty} \frac{g(z)}{\sqrt{-i(z+\tau)}}\, dz, \tag{3.12}$$

where g is the vector $(g_0, g_1, g_2)^T$, and we have to integrate each component of the vector.

Proof. (sketch)
If we replace τ by $-1/\tau$ in equation (3.4), multiply both sides by $\frac{1}{\sqrt{-i\tau}} \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$

and subtract equation (3.4), then we see that

$$R(\tau) = \frac{-1}{\sqrt{-i\tau}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} R(-1/\tau). \tag{3.13}$$

If we now take $\tau = it$ with $t \in \mathbf{R}$, $t > 0$, we have

$$R(it) = \frac{-1}{\sqrt{t}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} R(i/t) = \frac{4\sqrt{3}}{t} \begin{pmatrix} j_1(i/t) \\ -j_2(i/t) \\ j_3(i/t) \end{pmatrix}. \tag{3.14}$$

We now consider the first component:

$$\frac{4\sqrt{3}}{t} j_1(i/t) = \frac{4\sqrt{3}}{t} \int_0^\infty e^{-3\pi x^2/t} \frac{\sinh 2\pi x/t}{\sinh 3\pi x/t}\, dx = 4\sqrt{3} \int_0^\infty e^{-3\pi t y^2} \frac{\sinh 2\pi y}{\sinh 3\pi y}\, dy, \tag{3.15}$$

where we have substituted $x = ty$ in the integral.

From the theory of partial fraction decompositions (see [WW, pp. 134–136]) we get

$$\frac{\sinh 2\pi y}{\sinh 3\pi y} = -\frac{i\sqrt{3}}{6\pi} \sum_{n\in\mathbf{Z}} \frac{(-1)^n}{y - i(n+\frac{1}{3})} - \frac{i\sqrt{3}}{6\pi} \sum_{n\in\mathbf{Z}} \frac{(-1)^n}{-y - i(n+\frac{1}{3})}. \tag{3.16}$$

Using this we see

$$\begin{aligned} &\frac{4\sqrt{3}}{t} j_1(i/t) \\ &\quad = -\frac{2i}{\pi} \int_0^\infty e^{-3\pi t y^2} \left(\sum_{n\in\mathbf{Z}} \frac{(-1)^n}{y - i(n+\frac{1}{3})} + \sum_{n\in\mathbf{Z}} \frac{(-1)^n}{-y - i(n+\frac{1}{3})} \right) dy \\ &\quad = -\frac{2i}{\pi} \int_{-\infty}^\infty e^{-3\pi t y^2} \left(\sum_{n\in\mathbf{Z}} \frac{(-1)^n}{y - i(n+\frac{1}{3})} \right) dy \\ &\quad = -\frac{2i}{\pi} \sum_{n\in\mathbf{Z}} (-1)^n \int_{-\infty}^\infty \frac{e^{-3\pi t y^2}}{y - i(n+\frac{1}{3})}\, dy. \end{aligned} \tag{3.17}$$

It's not immediately clear that interchanging the order of integration and summation in the last equation is justified. However, it can be proven rigorously if we consider $\int_{-\infty}^\infty e^{-3\pi t y^2} \left(\sum_{n\in\mathbf{Z}} (-1)^n \left(\frac{1}{y - i(n+\frac{1}{3})} + \frac{1}{i(n+\frac{1}{3})} \right) \right) dy$ (here we can interchange the order of integration and summation because of absolute convergence).

We have for $r \in \mathbf{R}$, $r \neq 0$

$$\int_{-\infty}^\infty \frac{e^{-\pi t y^2}}{y - ir}\, dy = \pi i r \int_0^\infty \frac{e^{-\pi r^2 u}}{\sqrt{u+t}}\, du, \tag{3.18}$$

(both sides are solutions of $(-\frac{\partial}{\partial t} + \pi r^2) f(t) = \frac{\pi i r}{\sqrt{t}}$ and have the same limit 0 if $t \to \infty$, and hence are equal). If we use this with $r = (n + 1/3)$ and t replaced by $3t$, we obtain

$$\begin{aligned}\frac{4\sqrt{3}}{t}j_1(i/t) &= 2\sum_{n\in\mathbf{Z}}(-1)^n(n+1/3)\int_0^\infty \frac{e^{-\pi(n+1/3)^2u}}{\sqrt{u+3t}}\,du \\ &= 2\int_0^\infty \frac{\sum_{n\in\mathbf{Z}}(-1)^n(n+1/3)e^{-\pi(n+\frac{1}{3})^2u}}{\sqrt{u+3t}}\,du.\end{aligned}\tag{3.19}$$

Again it's not immediately clear that interchanging the order of integration and summation in the last step is justified. It can be proven rigorously by first using partial integration on the integral

$$\int_0^\infty \frac{e^{-\pi(n+1/3)^2u}}{\sqrt{u+3t}}\,du = \frac{1}{\pi(n+1/3)^2}\frac{1}{\sqrt{3t}} - \frac{1}{2\pi(n+1/3)^2}\int_0^\infty \frac{e^{-\pi(n+1/3)^2u}}{(u+3t)^{3/2}}\,du,\tag{3.20}$$

then interchanging the order of integration and summation, which is justified by absolute convergence, and finally using partial integration again. By partial integration we introduce some "boundary terms". To get rid of them we have to use Abel's theorem on continuity up to the circle of convergence, see [WW, pp. 57–58].

If we now substitute $u = -3iz$ in the integral we get

$$\frac{4\sqrt{3}}{t}j_1(i/t) = -2i\sqrt{3}\int_0^{i\infty}\frac{g_0(z)}{\sqrt{-i(z+it)}}\,dz,\tag{3.21}$$

so we have proven the first component of equation (3.12) for $\tau = it$. Since both sides are analytic on $\mathcal{H}$, the identity holds for all $\tau\in\mathcal{H}$.

The second and third component of equation (3.12) can be proven along the same lines. Here we have to use

$$\begin{aligned}\frac{\cosh\pi y}{\cosh 3\pi y} &= -\frac{i\sqrt{3}}{6\pi}\sum_{n\in\mathbf{Z}}{}^{*}\frac{1}{y-i(n+\frac{1}{6})} - \frac{i\sqrt{3}}{6\pi}\sum_{n\in\mathbf{Z}}{}^{*}\frac{1}{-y-i(n+\frac{1}{6})}\\ \frac{\sinh\pi y}{\sinh 3\pi y} &= -\frac{i\sqrt{3}}{6\pi}\sum_{n\in\mathbf{Z}}{}^{*}\frac{1}{y-i(n+\frac{1}{3})} - \frac{i\sqrt{3}}{6\pi}\sum_{n\in\mathbf{Z}}{}^{*}\frac{1}{-y-i(n+\frac{1}{3})},\end{aligned}\tag{3.22}$$

where $\sum_{n\in\mathbf{Z}}^{*}$ means $\lim_{m\to\infty}\sum_{n=-m}^{m}$. □

Definition 3.4. For $\tau\in\mathcal{H}\cup\mathbf{Q}$ we define

$$G(\tau) := 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty}\frac{(g_1(z),g_0(z),-g_2(z))^T}{\sqrt{-i(z+\tau)}}\,dz.\tag{3.23}$$

The integrals converge, even if $\tau\in\mathbf{Q}$, because the g_j's are cusp forms.

The function G satisfies the same modular transformation properties as F:

Lemma 3.5. *For $\tau\in\mathcal{H}$ we have*

$$G(\tau+1) = \begin{pmatrix}\zeta_{24}^{-1} & 0 & 0\\ 0 & 0 & \zeta_3\\ 0 & \zeta_3 & 0\end{pmatrix}G(\tau)\tag{3.24}$$

and

$$\frac{1}{\sqrt{-i\tau}}G(-1/\tau) = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}G(\tau) + R(\tau).\tag{3.25}$$

Proof. The first equation follows from equation (3.10) if we replace z by $z-1$ in the integral.

We have

$$\begin{aligned}\frac{1}{\sqrt{-i\tau}}G(-1/\tau) &= \frac{2i\sqrt{3}}{\sqrt{-i\tau}}\int_{1/\overline{\tau}}^{i\infty}\frac{(g_1(z),g_0(z),-g_2(z))^T}{\sqrt{-i(z-1/\tau)}}\,dz\\ &= 2i\sqrt{3}\int_0^{-\overline{\tau}}\frac{(g_1(-1/z),g_0(-1/z),-g_2(-1/z))^T}{\sqrt{1+\tau/z}}\frac{dz}{(-iz)^2},\end{aligned}\tag{3.26}$$

where we have replaced z by $-1/z$ in the integral. If we now use equation (3.11) we get

$$\frac{1}{\sqrt{-i\tau}}G(-1/\tau) = -2i\sqrt{3}\int_0^{-\overline{\tau}}\frac{g(z)}{\sqrt{-i(z+\tau)}}\,dz;\tag{3.27}$$

hence we get

$$\begin{aligned}&\frac{1}{\sqrt{-i\tau}}G(-1/\tau) - \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}G(\tau)\\ &= -2i\sqrt{3}\int_0^{-\overline{\tau}}\frac{g(z)}{\sqrt{-i(z+\tau)}}\,dz - 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty}\frac{g(z)}{\sqrt{-i(z+\tau)}}\,dz\\ &= -2i\sqrt{3}\int_0^{i\infty}\frac{g(z)}{\sqrt{-i(z+\tau)}}\,dz = R(\tau),\end{aligned}\tag{3.28}$$

by Lemma 3.3. □

Theorem 3.6. *The function H definied by*

$$H(\tau) = F(\tau) - G(\tau),\tag{3.29}$$

is a (vector-valued) real-analytic modular form of weight $1/2$, satisfying

$$\begin{aligned}H(\tau+1) &= \begin{pmatrix}\zeta_{24}^{-1}&0&0\\0&0&\zeta_3\\0&\zeta_3&0\end{pmatrix}H(\tau),\\ \frac{1}{\sqrt{-i\tau}}H(-1/\tau) &= \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}H(\tau),\end{aligned}\tag{3.30}$$

and H is an eigenfunction of the Casimir operator $\Omega_{1/2} = -4y^2\frac{\partial^2}{\partial\tau\partial\overline{\tau}} + iy\frac{\partial}{\partial\overline{\tau}} + \frac{3}{16}$ with eigenvalue $\frac{3}{16}$, where $\tau = x+iy$, $\frac{\partial}{\partial\tau} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial\overline{\tau}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

Proof. The modular transformation properties of H are a direct consequence of the transformation properties of F and G given in Lemma 3.2 and Lemma 3.5.

Since F is a holomorphic function of τ, we have $\frac{\partial}{\partial\overline{\tau}}F(\tau) = 0$; hence

$$\begin{aligned}\frac{\partial}{\partial\overline{\tau}}H(\tau) = -\frac{\partial}{\partial\overline{\tau}}G(\tau) &= -2i\sqrt{3}\frac{(g_1(-\overline{\tau}),g_0(-\overline{\tau}),-g_2(-\overline{\tau}))^T}{\sqrt{-i(\tau-\overline{\tau})}}\\ &= -\frac{i\sqrt{6}}{\sqrt{y}}(g_1(-\overline{\tau}),g_0(-\overline{\tau}),-g_2(-\overline{\tau}))^T.\end{aligned}\tag{3.31}$$

We see that $\sqrt{y}\frac{\partial}{\partial\bar\tau}H(\tau)$ is anti-holomorphic, so

$$\frac{\partial}{\partial\tau}\sqrt{y}\frac{\partial}{\partial\bar\tau}H(\tau) = 0. \tag{3.32}$$

We can write the operator $\Omega_{1/2} = -4y^2\frac{\partial^2}{\partial\tau\partial\bar\tau} + iy\frac{\partial}{\partial\bar\tau} + \frac{3}{16}$ as

$$\Omega_{1/2} = \frac{3}{16} - 4y^{3/2}\frac{\partial}{\partial\tau}\sqrt{y}\frac{\partial}{\partial\bar\tau}. \tag{3.33}$$

Hence

$$\Omega_{1/2}H = \frac{3}{16}H. \tag{3.34}$$

□

If we now write F as $H+G$, we get the following:

Corollary 3.7. *The vector-valued third order mock ϑ-function F can be written as the sum of a real analytic modular form H and a function G that is bounded in all rational points.*

4. Other mock ϑ-functions

In the previous section we have only dealt with the third order mock ϑ-functions f and ω. However, I have found similar results for most other mock ϑ-functions, and I expect that it can be done for all known ones. I hope to present these results, and the details omitted in the previous section, in my Ph.D.-thesis, which should appear somewhere near the end of 2002.

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School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland
E-mail address: `sander.zwegers@ucd.ie`