Before we prove this result, we first recall the construction of these forms. Suppose that λ is an integer, and that $k := \lambda + \frac{1}{2}$. For each $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4)$, let

$$j(A,z) := \left(\frac{\gamma}{\delta}\right)\epsilon_\delta^{-1}(\gamma z + \delta)^{\frac{1}{2}}$$

be the usual factor of automorphy for half-integral weight modular forms. If $f : \mathbb{H} \to \mathbb{C}$ is a function, then for $A \in \Gamma_0(4)$ we let

$$(f \mid_k A)(z) := j(A,z)^{-2\lambda-1} f(Az). \tag{2.5}$$

As usual, let $z = x + iy$ be the standard variable on $\mathbb{H}$. For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, we let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{2.6}$$

where $M_{\nu,\mu}(z)$ is the standard M-Whittaker function which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) u = 0.$$

If m is a positive integer, then define $\varphi_{-m,s}(z)$ by

$$\varphi_{-m,s}(z) := \mathcal{M}_s(-4\pi m y) e(-mx),$$

and consider the Poincaré series

$$\mathcal{F}_\lambda(-m,s;z) := \sum_{A \in \Gamma_\infty \backslash \Gamma_0(4)} (\varphi_{-m,s} \mid_k A)(z). \tag{2.7}$$

It is easy to verify that $\varphi_{-m,s}(z)$ is an eigenfunction, with eigenvalue

$$s(1-s) + (k^2 - 2k)/4, \tag{2.8}$$

of the weight k hyperbolic Laplacian

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Since $\varphi_{-m,s}(z) = O\left(y^{\mathrm{Re}(s) - \frac{k}{2}}\right)$ as $y \to 0$, it follows that $\mathcal{F}_\lambda(-m,s;z)$ converges absolutely for $\mathrm{Re}(s) > 1$, is a $\Gamma_0(4)$-invariant eigenfunction of the Laplacian, and is real analytic.

These series provide examples of weak Maass forms of half-integral weight. Following Bruinier and Funke [5], we make the following definition.

Definition 2.2 *A weak Maass form of weight k for the group $\Gamma_0(4)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following*:

(1) *For all $A \in \Gamma_0(4)$ we have*

$$(f\mid_k A)(z) = f(z).$$

(2) *We have $\Delta_k f = 0$.*
(3) *The function $f(z)$ has at most linear exponential growth at all the cusps.*

Remark If a weak Maass form $f(z)$ is holomorphic on $\mathbb{H}$, then it is a weakly holomorphic modular form.

In view of (2.8), it follows that the special s-values at $k/2$ and $1 - k/2$ of $\mathcal{F}_\lambda(-m, s; z)$ are weak Maass forms of weight $k = \lambda + \frac{1}{2}$ when the defining series is absolutely convergent.

If $\lambda \notin \{0, 1\}$ and $m \geq 1$ is an integer for which $(-1)^{\lambda+1} m \equiv 0, 1 \pmod 4$, then we recall the definition

$$F_\lambda(-m; z) := \begin{cases} \frac{3}{2}\mathcal{F}_\lambda\left(-m, \frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \geq 2, \\ \frac{3}{2(1-k)\Gamma(1-k)}\mathcal{F}_\lambda\left(-m, 1-\frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \leq -1. \end{cases} \tag{2.9}$$

By the discussion above, it follows that $F_\lambda(-m; z)$ is a weak Maass form of weight $k = \lambda + \frac{1}{2}$ on $\Gamma_0(4)$. If $\lambda = 1$ and m is a positive integer for which $m \equiv 0, 1 \pmod 4$, then define $F_1(-m; z)$ by

$$F_1(-m; z) := \frac{3}{2}\mathcal{F}_1\left(-m, \frac{3}{4}; z\right) \mid \mathrm{pr}_1 + 24\delta_{\square,m} G(z). \tag{2.10}$$

The function $G(z)$ is given by the Fourier expansion

$$G(z) := \sum_{n=0}^{\infty} H(n) q^n + \frac{1}{16\pi\sqrt{y}} \sum_{n=-\infty}^{\infty} \beta(4\pi n^2 y) q^{-n^2},$$

where $H(0) = -1/12$ and

$$\beta(s) := \int_1^\infty t^{-\frac{3}{2}} \mathrm{e}^{-st} \mathrm{d}t.$$

Proposition 3.6 of [7] proves that each $F_1(-m; z)$ is in $M^!_{\frac{3}{2}}$. These series form the basis given in (1.5).

Remark Note that the integral $\beta(s)$ is easily reformulated in terms of the incomplete Gamma-function. We make this observation since the non-holomorphic parts of the $F_\lambda(-m;z)$, for $\lambda \leq -7$ and $\lambda = -5$, will be described in such terms.

Remark We may define the series $F_0(-m;z) \in M^!_{\frac{1}{2}}$ using an argument analogous to Proposition 3.6 of [7]. Instead, we simply note that the existence of the basis (1.6) of $M^!_{\frac{1}{2}}$, together with the duality of Theorem 4 [23] and an elementary property of Kloosterman sums (see Proposition 3.1), gives a direct realization of the Fourier expansions of $F_0(-m;z)$ in terms of the expansions of the $F_1(-n;z)$ described above.

To compute the Fourier expansions of these weak Maass forms, we require some further preliminaries. For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, we let

$$\mathcal{W}_s(y) := |y|^{-\frac{k}{2}} W_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{2.11}$$

where $W_{\nu,\mu}$ denotes the usual W-Whittaker function. For $y > 0$, we shall require the following relations:

$$\mathcal{M}_{\frac{k}{2}}(-y) = \mathrm{e}^{\frac{y}{2}}, \tag{2.12}$$

$$\mathcal{W}_{1-\frac{k}{2}}(y) = \mathcal{W}_{\frac{k}{2}}(y) = \mathrm{e}^{-\frac{y}{2}}, \tag{2.13}$$

and

$$\mathcal{W}_{1-\frac{k}{2}}(-y) = \mathcal{W}_{\frac{k}{2}}(-y) = \mathrm{e}^{\frac{y}{2}}\Gamma\left(1-k, y\right), \tag{2.14}$$

where

$$\Gamma(a,x) := \int_x^\infty \mathrm{e}^{-t} t^a \frac{\mathrm{d}t}{t}$$

is the incomplete Gamma-function. For $z \in \mathbb{C}$, the functions $M_{\nu,\mu}(z)$ and $M_{\nu,-\mu}(z)$ are related by the identity

$$W_{\nu,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\nu)} M_{\nu,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\nu)} M_{\nu,-\mu}(z).$$

From these facts, we easily find, for $y > 0$, that

$$\mathcal{M}_{1-\frac{k}{2}}(-y) = (k-1)\mathrm{e}^{\frac{y}{2}}\Gamma(1-k,y) + (1-k)\Gamma(1-k)\mathrm{e}^{\frac{y}{2}}. \tag{2.15}$$

Proof of Theorem 2.1 Although the conclusions of Theorem 2.1 (1) and (3) were obtained previously by Bruinier, Jenkins and the second author in [7], for completeness we consider the calculation for general $\lambda \notin \{0,1\}$. In particular, lies in $S_{\kappa,L}$. This shows that $M_{\kappa,L} = M^{\mathrm{Eis}}_{\kappa,L} \oplus S_{\kappa,L}$ and (1.20) is a formula for the codimension of $S_{\kappa,L}$ in $M_{\kappa,L}$.

In later applications we will mainly be interested in the Eisenstein series $E_0(\tau)$ which we simply denote by $E(\tau)$. In the same way we write $q(\gamma, n)$ for the Fourier coefficients $q_0(\gamma, n)$ of $E(\tau)$.

Let us finally cite the following result of [BK]:

Proposition 1.7. *The coefficients $q(\gamma, n)$ of $E(\tau)$ are rational numbers.*

1.3 Non-holomorphic Poincaré series

As in the previous section we assume that L is an even lattice of signature $(b^+, b^-) = (2, l)$, $(1, l-1)$, or $(0, l-2)$ with $l \geq 3$. Put $k = 1 - l/2$ and $\kappa = 1 + l/2$. We now construct certain vector valued Maass-Poincaré series for $\mathrm{Mp}_2(\mathbb{Z})$ of weight k. Series of a similar type are well known and appear in many places in the literature (see for instance [He, Ni, Fa]).

Let $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ be the usual Whittaker functions as defined in [AbSt] Chap. 13 p. 190 or [E1] Vol. I Chap. 6 p. 264. They are linearly independent solutions of the Whittaker differential equation

$$\frac{d^2 w}{dz^2} + \left(-\frac{1}{4} + \frac{\nu}{z} - \frac{\mu^2 - 1/4}{z^2} \right) w = 0. \tag{1.21}$$

The functions $M_{\nu,\mu}(z)$ and $M_{\nu,-\mu}(z)$ are related by the identity

$$W_{\nu,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \nu)} M_{\nu,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \nu)} M_{\nu,-\mu}(z) \tag{1.22}$$

([AbSt] p. 190 (13.1.34)). This implies in particular $W_{\nu,\mu}(z) = W_{\nu,-\mu}(z)$. As $z \to 0$ one has the asymptotic behavior

$$M_{\nu,\mu}(z) \sim z^{\mu + 1/2} \qquad (\mu \notin -\tfrac{1}{2}\mathbb{N}), \tag{1.23}$$

$$W_{\nu,\mu}(z) \sim \frac{\Gamma(2\mu)}{\Gamma(\mu - \nu + 1/2)} z^{-\mu + 1/2} \qquad (\mu \geq 1/2). \tag{1.24}$$

If $y \in \mathbb{R}$ and $y \to \infty$ one has

$$M_{\nu,\mu}(y) = \frac{\Gamma(1 + 2\mu)}{\Gamma(\mu - \nu + 1/2)} e^{y/2} y^{-\nu} (1 + O(y^{-1})), \tag{1.25}$$

$$W_{\nu,\mu}(y) = e^{-y/2} y^{\nu} (1 + O(y^{-1})). \tag{1.26}$$

For convenience we put for $s \in \mathbb{C}$ and $y \in \mathbb{R}_{>0}$:

$$\mathcal{M}_s(y) = y^{-k/2} M_{-k/2,\, s-1/2}(y). \tag{1.27}$$

In the same way we define for $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$:

$$\mathcal{W}_s(y) = |y|^{-k/2} W_{k/2\,\mathrm{sgn}(y),\, s-1/2}(|y|). \tag{1.28}$$

If $y < 0$ then equation (1.22) implies that

$$\mathcal{M}_s(|y|) = \frac{\Gamma(1+k/2-s)}{\Gamma(1-2s)} \mathcal{W}_s(y) - \frac{\Gamma(1+k/2-s)\Gamma(2s-1)}{\Gamma(1-2s)\Gamma(s+k/2)} \mathcal{M}_{1-s}(|y|). \tag{1.29}$$

The functions $\mathcal{M}_s(y)$ and $\mathcal{W}_s(y)$ are holomorphic in s. Later we will be interested in certain special s-values. For $y > 0$ we have

$$\mathcal{M}_{k/2}(y) = y^{-k/2} M_{-k/2,\, k/2-1/2}(y) = e^{y/2}, \tag{1.30}$$

$$\mathcal{W}_{1-k/2}(y) = y^{-k/2} W_{k/2,\, 1/2-k/2}(y) = e^{-y/2}. \tag{1.31}$$

Using the standard integral representation

$$\Gamma(1/2-\nu+\mu) W_{\nu,\,\mu}(z) = e^{-z/2} z^{\mu+1/2} \int_0^\infty e^{-tz} t^{-1/2-\nu+\mu} (1+t)^{-1/2+\nu+\mu}\, dt$$

($\Re(\mu-\nu) > -1/2$, $\Re(z) > 0$) of the W-Whittaker function ([E1] Vol. I p. 274 (18)), we find for $y < 0$:

$$\begin{aligned}
\mathcal{W}_{1-k/2}(y) &= |y|^{-k/2} W_{-k/2,\, 1/2-k/2}(|y|) \\
&= e^{-|y|/2} |y|^{1-k} \int_0^\infty e^{-t|y|} (1+t)^{-k}\, dt \\
&= e^{|y|/2} |y|^{1-k} \int_1^\infty e^{-t|y|} t^{-k}\, dt.
\end{aligned}$$

If we insert the definition of the incomplete Gamma function (cf. [AbSt] p. 81)

$$\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1}\, dt, \tag{1.32}$$

we obtain for $y < 0$ the identity

$$\mathcal{W}_{1-k/2}(y) = e^{-y/2} \Gamma(1-k, |y|). \tag{1.33}$$

The usual Laplace operator of weight k (cf. [Ma1])

Proof. By symmetry we can assume without loss of generality that $c(\mathfrak{a}) \geqslant c(\mathfrak{b})$. If $\omega = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ and $\omega' = \left(\begin{smallmatrix} * & * \\ c' & d' \end{smallmatrix}\right)$ with $0 < c, c' \leqslant X$ are both in $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$, then $\omega'' = \omega'\omega^{-1} = \left(\begin{smallmatrix} * & * \\ c'' & * \end{smallmatrix}\right)$ with $c'' = c'd - cd'$ is in $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$. If $c'' = 0$ then the cusps $\mathfrak{a}, \mathfrak{b}$ are equivalent, so equal, $\omega'' = \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$, $c' = c$ and $d' = d$. If $c'' \neq 0$ then $|c''| \geqslant c(\mathfrak{a})$ and

$$\left|\frac{d'}{c'} - \frac{d}{c}\right| \geqslant \frac{c(\mathfrak{a})}{cc'} \geqslant \frac{c(\mathfrak{a})}{cX}. \tag{2.37}$$

Summing this inequality over $0 < c \leqslant X$ and $0 \leqslant d \leqslant c$, where d'/c' is chosen to be the successive point to d/c, we get (2.36).

Applying (2.36) for $X = c(\mathfrak{a}, \mathfrak{b})$, we infer the inequality (2.34) from the trivial bound

$$\left|\left\{d \,(\mathrm{mod}\ c) : \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\right\}\right| \geqslant 1, \qquad \text{if } c \in \mathcal{C}(\mathfrak{a}, \mathfrak{b}).$$

Proposition 2.9. *Let $\mathfrak{a}$ be a cusp for Γ, $z \in \mathbb{H}$ and $Y > 0$. We have*

$$|\{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma : \operatorname{Im} \sigma_{\mathfrak{a}}^{-1}\gamma z > Y\}| < 1 + \frac{10}{c(\mathfrak{a})Y}. \tag{2.38}$$

Proof. Conjugating the group, we can assume that $\mathfrak{a} = \infty$, $\sigma_{\mathfrak{a}} = 1$ and $\Gamma_{\mathfrak{a}} = B$. Then the strip $P = \{z : 0 < x < 1, y > 0\}$ is a fundamental domain of $\Gamma_{\mathfrak{a}}$. Let D be the standard polygon of Γ, so D consists of points in P of deformation less than 1. For the proof we may assume that $z \in D$, so $|cz+d| \geqslant 1$ for any $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c > 0$. Since $\operatorname{Im} \gamma z = y|cz+d|^{-2} > Y$, this implies $y > Y$, $c < (yY)^{\frac{-1}{2}}$ and $|cx+d| < (y/Y)^{\frac{1}{2}}$. By the last inequality and the spacing property (2.37) we estimate the number of pairs $\{c, d\}$ with $C \leqslant c < 2C$ by

$$1 + \frac{8C}{c(\mathfrak{a})}\left(\frac{y}{Y}\right)^{\frac{1}{2}} \leqslant \frac{10C}{c(\mathfrak{a})}\left(\frac{y}{Y}\right)^{\frac{1}{2}}.$$

Adding these bounds for $C = 2^{-n}(yY)^{\frac{-1}{2}}$ with $n \geqslant 1$, we get $10/c(\mathfrak{a})Y$. This is an estimate for the number of relevant γ's not in $\Gamma_{\mathfrak{a}}$. Finally adding 1 to account for $\Gamma_{\mathfrak{a}}$, we obtain (2.38).

As an example consider the Hecke congruence group $\Gamma_0(q)$ (note that $-1 \in \Gamma_0(q)$) and the cusps at $\infty, 0$. The scaling matrices are $\sigma_\infty = \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ and $\sigma_0 = \left(\begin{smallmatrix} & -1/\sqrt{q} \\ \sqrt{q} & \end{smallmatrix}\right)$. We have $\sigma_\infty^{-1}\Gamma_0(q)\sigma_\infty = \sigma_0^{-1}\Gamma_0(q)\sigma_0 = \Gamma_0(q)$, and the set $\sigma_\infty^{-1}\Gamma_0(q)\sigma_0 = \sigma_0^{-1}\Gamma_0(q)\sigma_\infty$ consists of matrices of type

$$\begin{pmatrix} \alpha\sqrt{q} & \beta/\sqrt{q} \\ \gamma\sqrt{q} & \delta\sqrt{q} \end{pmatrix} \qquad \text{with } \alpha, \beta, \gamma, \delta \in \mathbb{Z},\ \alpha\delta q - \beta\gamma = 1. \tag{2.39}$$

Hence $c(\infty) = c(0) = q$ and $c(\infty, 0) = c(0, \infty) = \sqrt{q}$.

Remark. Propositions 2.7, 2.8 and 2.9 are formulated for a Fuchsian group of motions rather than for a discrete group of matrices. Remember to take into account the factor $-1 = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ when translating the correspondence between the linear fractional transformations and their matrix representations (if -1 belongs to the group of matrices).

2.6. Multiplier systems

For a complex number $z \neq 0$ we choose its argument in $(-\pi, \pi]$. We denote the principal branch of the logarithm by $\log z$, so it is real for positive z and

$$\log z = \log |z| + i \arg z \quad \text{if } z \in \mathbb{C}^*.$$

Then we define the power z^s for any $s \in \mathbb{C}$ by

$$z^s = \exp(s \log z).$$

Let $A, B \in SL_2(\mathbb{R})$. By the chain rule

$$j_{AB}(z) = j_A(Bz) j_B(z)$$

it follows that the expression

$$2\pi\omega(A, B) = -\arg j_{AB}(z) + \arg j_A(Bz) + \arg j_B(z) \tag{2.40}$$

does not depend on $z \in \mathbb{H}$. More precisely, $\omega(A, B)$ takes only three values $-1, 0, 1$, because $|\omega(A, B)|$ is an integer $\leqslant \frac{3}{2}$. By examining numerous cases one can establish the following properties (cf. [**Pet**] or [**Ran**]):

$$\omega(AB, C) + \omega(A, B) = \omega(A, BC) + \omega(B, C) \tag{2.41}$$

$$\omega(A, B) = \omega(B, A) \quad \text{if } A, B \text{ commute} \tag{2.42}$$

$$\omega(DA, B) = \omega(A, BD) = \omega(A, B) \tag{2.43}$$

$$\omega(AD, B) = \omega(A, DB) \tag{2.44}$$

$$\omega(A^{-1}DA, B) + \omega(A, A^{-1}DAB) = \omega(A, B) \tag{2.45}$$

(2.46) $$\omega(A, D) = \omega(D, B) = 0$$

(2.47) $$\omega(ADA^{-1}, A) = \omega(A, A^{-1}DA) = 0$$

where A, B, C are arbitrary and $D = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$. Here (2.45) is just a special case of (2.41) with A, B, C replaced by A, $A^{-1}DA$, B respectively. Moreover we have

$$\omega(AD, A^{-1}) = \omega(A, DA^{-1}) = \omega(A, A^{-1}) = 0$$

except for $A = \begin{pmatrix} * & * \\ 0 & d \end{pmatrix}$ with $d < 0$, in which case $\omega(A, A^{-1}) = 1$.

For any real number k we define the factor system of weight k by setting

(2.48) $$w(A, B) = e(k\omega(A, B)).$$

Note that $w(A, B)$ depends on $k \pmod 1$ and $w(A, B) = 1$ if k is an integer. We have

(2.49) $$w(A, B) j_{AB}(z)^k = j_A(Bz)^k j_B(z)^k.$$

For any $A \in SL_2(\mathbb{R})$ we define the "slash" operator $|A$ acting on functions $f : H \to \mathbb{C}$ by

(2.50) $$f_{|A}(z) = j_A(z)^{-k} f(Az).$$

This satisfies the rule of composition

(2.51) $$f_{|AB} = w(A, B)(f_{|A})_{|B}.$$

Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup. A multiplier system of weight k for Γ is a function $\vartheta : \Gamma \to \mathbb{C}$ such that

(2.52) $$|\vartheta(\gamma)| = 1$$

(2.53) $$\vartheta(\gamma_1\gamma_2) = w(\gamma_1, \gamma_2)\vartheta(\gamma_1)\vartheta(\gamma_2).$$

Since $w(-1, -1) = e(k)$, the above properties imply that if -1 belongs to Γ, then $\vartheta(-1) = \pm e(-k/2)$. Throughout we shall require that

(2.54) $$\vartheta(-1) = e(-k/2) \quad \text{if } -1 \in \Gamma,$$

which is called the consistency condition (the other choice yields the zero automorphic form only).

Chapter 3

The Eisenstein and the Poincaré Series

3.1. General Poincaré series

A very important class of automorphic forms is constructed by the method of averaging. Suppose $\mathfrak{a}$ is a cusp for Γ which is singular with respect to a multiplier system ϑ of weight k. Let $p : \mathbb{H} \to \mathbb{C}$ be a holomorphic function which is periodic of period 1. Define π: $\Gamma \times \mathbb{H} \to \mathbb{C}$ by

$$\pi(\gamma, z) = \bar{\vartheta}(\gamma)\bar{w}\left(\sigma_{\mathfrak{a}}^{-1}, \gamma\right) j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} p\left(\sigma_{\mathfrak{a}}^{-1}\gamma z\right). \tag{3.1}$$

Actually $\pi(\gamma, z)$ depends only on the coset $\Gamma_{\mathfrak{a}}\gamma$. To prove this, consider $\pi(\gamma', z)$, where $\gamma' = \eta\gamma$ with $\eta \in \Gamma_{\mathfrak{a}}$, so $\eta = \sigma_{\mathfrak{a}}\beta\sigma_{\mathfrak{a}}^{-1}$, where β is an integral translation. We have

$$\begin{aligned} p\left(\sigma_{\mathfrak{a}}^{-1}\gamma' z\right) &= p\left(\beta\sigma_{\mathfrak{a}}^{-1}\gamma z\right) = p\left(\sigma_{\mathfrak{a}}^{-1}\gamma z\right) \\ j_{\sigma_{\mathfrak{a}}^{-1}\gamma'}(z)^{-k} &= j_{\beta\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} = j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} \\ \vartheta(\gamma') &= \vartheta(\eta\gamma) = w(\eta, \gamma)\vartheta(\gamma) \end{aligned}$$

by (2.53) and

$$w(\eta, \gamma)w\left(\sigma_{\mathfrak{a}}^{-1}, \gamma'\right) = w\left(\sigma_{\mathfrak{a}}^{-1}, \gamma\right)$$

by (2.45). Collecting these results, we arrive at $\pi(\gamma', z) = \pi(\gamma, z)$. This property allows us to write without ambiguity the infinite series

$$\mathbf{P}_{\mathfrak{a}}(z) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \pi(\gamma, z) \tag{3.2}$$

provided it converges absolutely. For example, if $p(z)$ is bounded the series (3.2) is majorized by

$$\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} |j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)|^{-k} = y^k \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\operatorname{Im} \sigma_{\mathfrak{a}}^{-1}\gamma z)^{\frac{k}{2}},$$

and this converges absolutely if $k > 2$ by virtue of Proposition 2.9.

For any singular cusp $\mathfrak{b}$ we deduce the following:

$$\begin{aligned}
\mathbf{P}_{\mathfrak{a}|\sigma_{\mathfrak{b}}}(z) &= j_{\sigma_{\mathfrak{b}}}(z)^{-k}\mathbf{P}_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z)\\
&= j_{\sigma_{\mathfrak{b}}}(z)^{-k} \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \pi(\gamma, \sigma_{\mathfrak{b}} z)\\
&= j_{\sigma_{\mathfrak{b}}}(z)^{-k} \sum_{\gamma\in B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} \pi\left(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}} z\right)\\
&= j_{\sigma_{\mathfrak{b}}}(z)^{-k} \sum_{\gamma\in B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} \bar{\vartheta}\left(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}\right) \bar{w}\left(\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}\right) j_{\gamma\sigma_{\mathfrak{b}}^{-1}}(\sigma_{\mathfrak{b}} z)^{-k} p(\gamma z).
\end{aligned}$$

Since $j_\sigma(z)^{-k} j_{\gamma\sigma^{-1}}(\sigma z)^{-k} = \bar{w}(\gamma\sigma^{-1}, \sigma) j_\gamma(z)^{-k}$, this gives

$$\mathbf{P}_{\mathfrak{a}|\sigma_{\mathfrak{b}}}(z) = \sum_{\gamma\in B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} \bar{\vartheta}_{\mathfrak{a}\mathfrak{b}}(\gamma) j_\gamma(z)^{-k} p(\gamma z) \tag{3.3}$$

where

$$\vartheta_{\mathfrak{a}\mathfrak{b}}(\gamma) = \vartheta\left(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}\right) w\left(\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}\right) w\left(\gamma\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}}\right). \tag{3.4}$$

Using the relation (2.41), one can derive a handful of expressions for $\vartheta_{\mathfrak{a}\mathfrak{b}}(\gamma)$. For example, we have

$$\vartheta_{\mathfrak{a}\mathfrak{b}}(\gamma) w(\sigma_{\mathfrak{a}}, \gamma) = \vartheta\left(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}\right) w\left(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}}\right). \tag{3.5}$$

Hence, in particular, for $\mathfrak{a} = \mathfrak{b}$ the series (3.3) becomes

$$\mathbf{P}_{\mathfrak{a}|\sigma_{\mathfrak{a}}}(z) = \sum_{\gamma'\in\Gamma'_\infty\backslash\Gamma'} \bar{\vartheta}'(\gamma') j_{\gamma'}(z)^{-k} p(\gamma' z) \tag{3.6}$$

where ϑ' is the multiplier system for the conjugate group $\Gamma' = \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$ given by

$$\vartheta'(\gamma') w(\sigma_{\mathfrak{a}}, \gamma') = \vartheta(\gamma) w(\gamma, \sigma_{\mathfrak{a}}) \quad \text{if } \gamma' = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}. \tag{3.7}$$

The function $\mathbf{P}_{\mathfrak{a}}(z)$ defined by (3.2) is called the Poincaré series associated with the cusp $\mathfrak{a}$ and the generating function p ($\mathfrak{a}$ is required to be singular with respect to the multiplier system, and p is periodic such that the series (3.2) converges absolutely).

Proposition 3.1. *The Poincaré series $\mathbf{P}_{\mathfrak{a}}(z)$ is an automorphic form,*

$$\mathbf{P}_{\mathfrak{a}|\tau}(z) = \vartheta(\tau)\mathbf{P}_{\mathfrak{a}}(z) \quad \text{if} \quad \tau\in\Gamma. \tag{3.8}$$

Proof. By conjugating the group we may assume that $\mathfrak{a} = \infty$ and $\sigma_\infty = 1$, in which case the series (3.2) looks simpler, namely

$$\mathbf{P}_\infty(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \bar{\vartheta}(\gamma) j_\gamma(z)^{-k} p(\gamma z). \tag{3.9}$$

Hence for $\tau \in \Gamma$

$$\begin{aligned}\mathbf{P}_\infty(\tau z) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \bar{\vartheta}(\gamma) j_\gamma(\tau z)^{-k} p(\gamma \tau z) \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \bar{\vartheta}(\gamma\tau^{-1}) j_{\gamma\tau^{-1}}(\tau z)^{-k} p(\gamma z).\end{aligned}$$

Here we have

$$\bar{\vartheta}(\gamma\tau^{-1}) = \bar{w}(\gamma, \tau^{-1}) \bar{\vartheta}(\gamma) \bar{\vartheta}(\tau^{-1}) = \bar{w}(\gamma, \tau^{-1}) w(\tau, \tau^{-1}) \vartheta(\tau) \bar{\vartheta}(\gamma),$$

$$j_{\gamma\tau^{-1}}(\tau z)^{-k} = w(\gamma, \tau^{-1}) j_\gamma(z)^{-k} j_{\tau^{-1}}(\tau z)^{-k} = w(\gamma, \tau^{-1}) \bar{w}(\tau, \tau^{-1}) j_\tau(z)^k j_\gamma(z)^{-k},$$

and by these expressions we arrive at (3.8). Clearly $\mathbf{P}_\mathfrak{a}(z)$ is holomorphic in $\mathbb{H}$. To prove the holomorphy at cusps we need to expand $\mathbf{P}_\mathfrak{a}(z)$ in Fourier series.

3.2. Fourier expansion of Poincaré series

Let $\mathfrak{a}, \mathfrak{b}$ be singular cusps for a multiplier system ϑ on Γ. We seek the Fourier expansion of $\mathbf{P}_\mathfrak{a}(z)$ at the cusp $\mathfrak{b}$, i.e. for the series (3.3). Applying the double coset decomposition (2.32), we split the series into

$$\mathbf{P}_{\mathfrak{a}|\sigma_\mathfrak{b}}(z) = \delta_{\mathfrak{a}\mathfrak{b}} p(z) + \sum_{1 \neq \gamma \in B \backslash \sigma_\mathfrak{a}^{-1} \Gamma \sigma_\mathfrak{b} / B} \bar{\vartheta}_{\mathfrak{a}\mathfrak{b}}(\gamma) I_\gamma(z)$$

where the first term comes from the contribution of $\gamma = 1$ (which exists only if $\mathfrak{a} = \mathfrak{b}$), and for any $\gamma = \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \sigma_\mathfrak{a}^{-1} \Gamma \sigma_\mathfrak{b}$ with $c > 0$ we have

$$\begin{aligned} I_\gamma(z) &= \sum_{\tau \in B} j_{\gamma\tau}(z)^{-k} p(\gamma\tau z) \\ &= \sum_{n \in \mathbb{Z}} (c(z+n)+d)^{-k} p\left(\frac{a}{c} - \frac{1}{c(c(z+n)+d)}\right).\end{aligned}$$

By Poisson's summation we get

$$I_\gamma(z) = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} (c(z+v)+d)^{-k} p\left(\frac{a}{c} - \frac{1}{c(c(z+v)+d)}\right) e(-nv)\,dv.$$

In the sequel we specialize the generating function to

$$p(z) = e(mz)$$

where m is a non-negative integer. For this function we can compute the Fourier integral quite explicitly. First by a linear change of variable we obtain

$$I_\gamma(z) = \sum_{n\in\mathbb{Z}} e\left(nz + \frac{ma+nd}{c}\right) \mathcal{J}_c(m,n)$$

where

$$\mathcal{J}_c(m,n) = \int_{-\infty+iy}^{\infty+iy} (cv)^{-k} e\left(\frac{-m}{c^2 v} - nv\right) dv.$$

Notice that this integral does not depend on y by Cauchy's theorem. If $n \leqslant 0$, then, moving the horizontal line of integration upwards, we see that the integral vanishes,

$$\mathcal{J}_c(m,n) = 0 \quad \text{if } n \leqslant 0.$$

If $n > 0$ but $m = 0$, then we have (see [**G-R**], 8.315.1)

$$\mathcal{J}_c(0,n) = \left(\frac{2\pi}{ic}\right)^k \frac{n^{k-1}}{\Gamma(k)}. \tag{3.10}$$

For $n > 0$ and $m > 0$ we have (see [**G-R**], 8.412.2)

$$\mathcal{J}_c(m,n) = \frac{2\pi}{i^k c}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{3.11}$$

where $J_\gamma(x)$ is the Bessel function of order ν, defined by

$$J_\nu(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\Gamma(\ell+1+\nu)} \left(\frac{x}{2}\right)^{\nu+2\ell}. \tag{3.12}$$

Exercise. Derive (3.11) from (3.10) using power series expansion for $e(z)$.

Collecting the above computations, we obtain the desired Fourier expansion for the Poincaré series generated by the function $p(z) = e(mz)$, namely

$$\mathbf{P}_{\mathfrak{a}|\sigma_\mathfrak{b}}(z) = \delta_{\mathfrak{ab}} e(mz) + \sum_{n=1}^{\infty} e(nz) \sum_{c>0} S_{\mathfrak{ab}}(m,n;c)\mathcal{J}_c(m,n)$$

where $\mathcal{J}_c(m,n)$ are given by (3.10)-(3.11), and $S_{\mathfrak{ab}}(m,n;c)$ is the Kloosterman sum defined by

$$S_{\mathfrak{ab}}(m,n;c) = \sum_{\gamma=\begin{pmatrix} a & * \\ c & d\end{pmatrix}\in B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/B} \bar{\vartheta}_{\mathfrak{ab}}(\gamma) e\left(\frac{ma+nd}{c}\right). \tag{3.13}$$

Recall that $\vartheta_{\mathfrak{ab}}(\gamma)$ is given in terms of the multiplier system by (3.5).

Since there are no negative terms in the Fourier expansion at any cusp, it proves that the Poincaré series is an automorphic form in our strict sense.

For $m=0$ we denote $\mathbf{P}_{\mathfrak{a}}(z)$ by

$$E_{\mathfrak{a}}(z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \bar{\vartheta}(\gamma)\bar{w}\left(\sigma_{\mathfrak{a}}^{-1},\gamma\right) j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k}, \tag{3.14}$$

which is called the Eisenstein series of weight k. This has the Fourier expansion

$$j_{\sigma_{\mathfrak{b}}}(z)^{-k} E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z) = \delta_{\mathfrak{ab}} + \sum_{n=1}^{\infty} \eta_{\mathfrak{ab}}(n) e(nz) \tag{3.15}$$

with

$$\eta_{\mathfrak{ab}}(n) = \left(\frac{2\pi}{i}\right)^k \frac{n^{k-1}}{\Gamma(k)} \sum_{c>0} c^{-k} S_{\mathfrak{ab}}(0,n;c). \tag{3.16}$$

For $m>0$ we denote $\mathbf{P}_{\mathfrak{a}}(z)$ by

$$P_{\mathfrak{a}m}(z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \bar{\vartheta}(\gamma)\bar{w}\left(\sigma_{\mathfrak{a}}^{-1},\gamma\right) j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} e\left(m\sigma_{\mathfrak{a}}^{-1}\gamma z\right) \tag{3.17}$$

and call $P_{\mathfrak{a}m}(z)$ the m-th Poincaré series of weight k. This has the Fourier expansion

$$j_{\sigma_{\mathfrak{b}}}(z)^{-k} P_{\mathfrak{a}m}(\sigma_{\mathfrak{b}} z) = \sum_{n=1}^{\infty} p_{\mathfrak{ab}}(m,n) e(nz) \tag{3.18}$$

with

(3.19)

$$p_{\mathfrak{ab}}(m,n) = \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \left\{\delta_{\mathfrak{ab}}\delta_{mn} + 2\pi i^{-k} \sum_{c>0} c^{-1} S_{\mathfrak{ab}}(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right\}.$$

Since there is no constant term in the Fourier expansion (3.18), we obtain

Proposition 3.2. *For $m \geqslant 1$ the Poincaré series $P_{\mathfrak{a}m}(z)$ is a cusp form.*

Suppose that $k \in \frac{1}{2} + \mathbb{Z}$. We define a class of Poincaré series $P_k(s; z)$. For matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, with $c \geq 0$, define the character $\chi(\cdot)$ by

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{cases} e\left(-\frac{b}{24}\right) & \text{if } c = 0, \\ i^{-1/2}(-1)^{\frac{1}{2}(c+ad+1)} e\left(-\frac{a+d}{24c} - \frac{a}{4} + \frac{3dc}{8}\right) \cdot \omega_{-d,c}^{-1} & \text{if } c > 0, \end{cases} \tag{11.7}$$

where

$$\omega_{d,c} := e^{\pi i s(d,c)}. \tag{11.8}$$

Here $s(d,c)$ denotes the classical Dedekind sum.

Throughout, let $z = x + iy$, and for $s \in \mathbb{C}, k \in \frac{1}{2} + \mathbb{Z}$, and $y \in \mathbb{R} \setminus \{0\}$, and let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{11.9}$$

where $M_{\nu,\mu}(z)$ again is the M-Whittaker function. Furthermore, let

$$\varphi_{s,k}(z) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right) e\left(-\frac{x}{24}\right).$$

Using this notation, define the Poincaré series $P_k(s; z)$ by

$$P_k(s; z) := \frac{2}{\sqrt{\pi}} \sum_{M \in \Gamma_\infty \backslash \Gamma_0(2)} \chi(M)^{-1}(cz+d)^{-k} \varphi_{s,k}(Mz). \tag{11.10}$$

Here Γ_∞ again is the subgroup of translations in $\mathrm{SL}_2(\mathbb{Z})$.

The defining series is absolutely convergent for $P_k\left(1 - \frac{k}{2}; z\right)$ for $k < 1/2$, and is conditionally convergent when $k = 1/2$. We are interested in $P_{\frac{1}{2}}\left(\frac{3}{4}; z\right)$, which we define by analytically continuing the Fourier expansion. This argument is not straightforward (see Theorem 3.2 and Corollary 4.2 of [53]). Thanks to the properties of $M_{\nu,\mu}$, we find that $P_{\frac{1}{2}}\left(\frac{3}{4}; 24z\right)$ is a Maass form of weight 1/2 for $\Gamma_0(144)$ with Nebentypus χ_{12}.

A long calculation gives the following Fourier expansion

$$P_{\frac{1}{2}}\left(\frac{3}{4}; z\right) = \left(1 - \pi^{-\frac{1}{2}} \cdot \Gamma\left(\frac{1}{2}, \frac{\pi y}{6}\right)\right) \cdot q^{-\frac{1}{24}} + \sum_{n=-\infty}^{0} \gamma_y(n) q^{n-\frac{1}{24}} + \sum_{n=1}^{\infty} \beta(n) q^{n-\frac{1}{24}}, \tag{11.11}$$

where for positive integers n we have

$$\beta(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right). \tag{11.12}$$

The Poincaré series $P_{\frac{1}{2}}\left(\frac{3}{4}; z\right)$ was defined so that (11.12) coincides with the conjectured expressions for the coefficients $\alpha(n)$.

For convenience, we let

$$P(z) := P_{\frac{1}{2}}\left(\frac{3}{4}; 24z\right). \tag{11.13}$$

Canonically decompose $P(z)$ into a non-holomorphic and a holomorphic part

$$P(z) = P^-(z) + P^+(z). \tag{11.14}$$

In particular, we have that

$$P^+(z) = q^{-1} + \sum_{n=1}^{\infty} \beta(n) q^{24n-1}.$$

Since $P(z)$ and $D\left(\frac{1}{2}; z\right)$ are Maass forms of weight 1/2 for $\Gamma_0(144)$ with Nebentypus χ_{12}, (11.11) and (11.12) imply that the proof of the conjecture reduces to proving that these forms are equal. This conclusion is obtained after a lengthy and somewhat complicated argument. $\square$

11.2. **Exact formulas for harmonic Maass forms with weight $\leq 1/2$.** Generalizing the results of the previous section, Bringmann and the author have obtained exact formulas for the coefficients of the holomorphic parts of harmonic Maass forms with weight $2-k \leq 1/2$ [59]. Suppose that f is in $H_{2-k}(N, \chi)$, the space of weight $2-k$ harmonic Maass forms on $\Gamma_0(N)$ with Nebentypus character χ, where we assume that $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$. As usual, we denote its Fourier expansion by

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n. \tag{11.15}$$

It is our objective to determine exact formulas for the coefficients $c_f^+(n)$ of the holomorphic part of f.

We now define the functions which are required for these exact formulas. Throughout, we let $k \in \frac{1}{2}\mathbb{Z}$, and we let χ be a Dirichlet character modulo N, where $4 \mid N$ whenever $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Using this character, for a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we let

$$\Psi_k(M) := \begin{cases} \chi(d) & \text{if } k \in \mathbb{Z}, \\ \chi(d) \left(\frac{c}{d}\right) \epsilon_d^{2k} & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases} \tag{11.16}$$

where ϵ_d is defined by (7.2), and where $\left(\frac{c}{d}\right)$ is the usual extended Legendre symbol. In addition, if $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then we let

$$\mu(T; z) := (cz+d)^{2-k}. \tag{11.17}$$

Moreover, for pairs of matrices $S, T \in \mathrm{SL}_2(\mathbb{Z})$, we then let

$$\sigma(T, S) := \frac{\mu(T; Sz)\mu(S; z)}{\mu(TS; z)}. \tag{11.18}$$

Using this notation, we now define certain generic Kloosterman sums which are naturally associated with cusps of $\Gamma_0(N)$.

Suppose that $\rho = \frac{a_\rho}{c_\rho} = L^{-1}\infty$, ($L \in \mathrm{SL}_2(\mathbb{Z})$) is a cusp of $\Gamma_0(N)$ with $c_\rho | N$ and $\gcd(a_\rho, N) = 1$. Let t_ρ and κ_ρ be the cusp width and parameter of ρ with respect to $\Gamma_0(N)$ (see 11.21). Suppose that $c > 0$ with $c_\rho | c$ and $\frac{N}{c_\rho} \nmid c$. Then for integers n and m we

CHAPTER XVI

THE CONFLUENT HYPERGEOMETRIC FUNCTION

16·1. *The confluence of two singularities of Riemann's equation.*

We have seen (§ 10·8) that the linear differential equation with two regular singularities only can be integrated in terms of elementary functions; while the solution of the linear differential equation with three regular singularities is substantially the topic of Chapter XIV. As the next type in order of complexity, we shall consider a modified form of the differential equation which is obtained from Riemann's equation by the confluence of two of the singularities. This confluence gives an equation with an irregular singularity (corresponding to the confluent singularities of Riemann's equation) and a regular singularity corresponding to the third singularity of Riemann's equation.

The confluent equation is obtained by making $c \to \infty$ in the equation defined by the scheme

$$P\left\{\begin{matrix} 0 & \infty & c & \\ \frac{1}{2}+m & -c & c-k & z \\ \frac{1}{2}-m & 0 & k & \end{matrix}\right\}.$$

The equation in question is readily found to be

$$\frac{d^2u}{dz^2}+\frac{du}{dz}+\left(\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right)u=0 \text{ .... (A).}$$

We modify this equation by writing $u=e^{-\frac{1}{2}z}W_{k,m}(z)$ and obtain as the equation* for $W_{k,m}(z)$

$$\frac{d^2W}{dz^2}+\left\{-\frac{1}{4}+\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right\}W=0 \text{ .... (B).}$$

The reader will verify that the singularities of this equation are at 0 and ∞, the former being regular and the latter irregular; and when $2m$ is *not an integer*, two integrals of equation (B) which are regular near 0 and valid for all finite values of z are given by the series

$$M_{k,m}(z)=z^{\frac{1}{2}+m}e^{-\frac{1}{2}z}\left\{1+\frac{\frac{1}{2}+m-k}{1\,!\,(2m+1)}z+\frac{(\frac{1}{2}+m-k)(\frac{3}{2}+m-k)}{2\,!\,(2m+1)(2m+2)}z^2+\dots\right\},$$

* This equation was given by Whittaker, *Bulletin American Math. Soc.* x. (1904), pp. 125–134.

$$M_{k,-m}(z)=z^{\frac{1}{2}-m}e^{-\frac{1}{2}z}\left\{1+\frac{\frac{1}{2}-m-k}{1!\,(1-2m)}z+\frac{(\frac{1}{2}-m-k)(\frac{3}{2}-m-k)}{2!\,(1-2m)(2-2m)}z^2+\dots\right\}.$$

These series obviously form a fundamental system of solutions.

[NOTE. Series of the type in { } have been considered by Kummer* and more recently by Jacobsthal† and Barnes‡; the special series in which $k=0$ had been investigated by Lagrange in 1762–1765 (*Oeuvres*, I. p. 480). In the notation of Kummer, modified by Barnes, they would be written ${}_1F_1\{\frac{1}{2}\pm m-k;\ \pm 2m+1;\ z\}$; the reason for discussing solutions of equation (B) rather than those of the equation $z\dfrac{d^2y}{dz^2}-(z-\rho)\dfrac{dy}{dz}-ay=0$, of which ${}_1F_1(a;\ \rho;\ z)$ is a solution, is the greater appearance of symmetry in the formulae, together with a simplicity in the equations giving various functions of Applied Mathematics (see § 16·2) in terms of solutions of equation (B).]

16·11. *Kummer's formulae.*

(I) We shall now shew that, if $2m$ is not a negative integer, then

$$z^{-\frac{1}{2}-m}M_{k,m}(z)=(-z)^{-\frac{1}{2}-m}M_{-k,m}(-z),$$

that is to say,

$$e^{-z}\left\{1+\frac{\frac{1}{2}+m-k}{1!\,(2m+1)}z+\frac{(\frac{1}{2}+m-k)(\frac{3}{2}+m-k)}{2!\,(2m+1)(2m+2)}z^2+\dots\right\}$$
$$=1-\frac{\frac{1}{2}+m+k}{1!\,(2m+1)}z+\frac{(\frac{1}{2}+m+k)(\frac{3}{2}+m+k)}{2!\,(2m+1)(2m+2)}z^2-\dots.$$

For, replacing e^{-z} by its expansion in powers of z, the coefficient of z^n in the product of absolutely convergent series on the left is

$$\frac{(-)^n}{n!}F\left(\frac{1}{2}+m-k,\ -n;\ 2m+1;\ 1\right)=\frac{(-)^n}{n!}\frac{\Gamma(2m+1)\,\Gamma(m+\frac{1}{2}+k+n)}{\Gamma(m+\frac{1}{2}+k)\,\Gamma(2m+1+n)},$$

by § 14·11, and this is the coefficient of z^n on the right§; we have thus obtained the required result.

This will be called *Kummer's first formula.*

(II) The equation

$$M_{0,m}(z)=z^{\frac{1}{2}+m}\left\{1+\sum_{p=1}^{\infty}\frac{z^{2p}}{2^{4p}\,.\,p!\,(m+1)(m+2)\dots(m+p)}\right\},$$

valid when $2m$ is not a negative integer, will be called *Kummer's second formula.*

To prove it we observe that the coefficient of $z^{n+m+\frac{1}{2}}$ in the product

$$z^{m+\frac{1}{2}}e^{-\frac{1}{2}z}\,{}_1F_1(m+\tfrac{1}{2};\ 2m+1;\ z),$$

* *Journal für Math.* XV. (1836), p. 139.

† *Math. Ann.* LVI. (1903), pp. 129–154.

‡ *Trans. Camb. Phil. Soc.* XX. (1908), pp. 253–279.

§ The result is still true when $m+\frac{1}{2}+k$ is a negative integer, by a slight modification of the analysis of § 14·11.

of which the second and third factors possess absolutely convergent expansions, is (§ 3·73)

$$\frac{(\frac{1}{2}+m)(\frac{3}{2}+m)\dots(n-m+\frac{1}{2})}{n!\,(2m+1)(2m+2)\dots(2m+n)}F(-n,\ -2m-n;\ -n+\tfrac{1}{2}-m;\ \tfrac{1}{2})$$

$$=\frac{(\frac{1}{2}+m)(\frac{3}{2}+m)\dots(n-m+\frac{1}{2})}{n!\,(2m+1)(2m+2)\dots(2m+n)}F(-\tfrac{1}{2}n,\ -m-\tfrac{1}{2}n;\ -n+\tfrac{1}{2}-m;\ 1),$$

by Kummer's relation*

$$F(2a,\ 2\beta;\ a+\beta+\tfrac{1}{2};\ x)=F\{a,\ \beta;\ a+\beta+\tfrac{1}{2};\ 4x(1-x)\},$$

valid when $0\leqslant x\leqslant\frac{1}{2}$; and so the coefficient of $z^{n+m+\frac{1}{2}}$ (by § 14·11) is

$$\frac{(\frac{1}{2}+m)(\frac{3}{2}+m)\dots(n-m+\frac{1}{2})}{n!\,(2m+1)(2m+2)\dots(2m+n)}\,\frac{\Gamma(-n+\frac{1}{2}-m)\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-m-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-\frac{1}{2}n)}$$

$$=\frac{\Gamma(\frac{1}{2}-m)\,\Gamma(\frac{1}{2})}{n!\,(2m+1)(2m+2)\dots(2m+n)\,\Gamma(\frac{1}{2}-m-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-\frac{1}{2}n)},$$

and when n is odd this vanishes; for even values of $n\,(=2p)$ it is

$$\frac{\Gamma(\frac{1}{2}-m)(-\frac{1}{2})(-\frac{3}{2})\dots(\frac{1}{2}-p)}{2p!\,2^{2p}(m+\frac{1}{2})(m+\frac{3}{2})\dots(m+p-\frac{1}{2})(m+1)(m+2)\dots(m+p)\,\Gamma(\frac{1}{2}-m-p)}$$

$$=\frac{1.3\dots(2p-1)}{2p!\,2^{3p}(m+1)(m+2)\dots(m+p)}=\frac{1}{2^{4p}.\,p!\,(m+1)(m+2)\dots(m+p)}.$$

16·12. *Definition† of the function $W_{k,m}(z)$.*

The solutions $M_{k,\pm m}(z)$ of equation (B) of § 16·1 are not, however, the most convenient to take as the standard solutions, on account of the disappearance of one of them when $2m$ is an integer.

The integral obtained by confluence from that of § 14·6, when multiplied by a constant multiple of $e^{\frac{1}{2}z}$, is‡

$$W_{k,m}(z)$$
$$=-\frac{1}{2\pi i}\Gamma\left(k+\frac{1}{2}-m\right)e^{-\frac{1}{2}z}z^k\int_{\infty}^{(0+)}(-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}\,dt.$$

It is supposed that $\arg z$ has its principal value and that the contour is so chosen that the point $t=-z$ is outside it. The integrand is rendered one-valued by taking $|\arg(-t)|\leqslant\pi$ and taking that value of $\arg(1+t/z)$ which tends to zero as $t\to 0$ by a path lying inside the contour.

Under these circumstances it follows from § 5·32 that the integral is an analytic function of z. To shew that it satisfies equation (B), write

$$v=\int_{\infty}^{(0+)}(-t)^{-k-\frac{1}{2}+m}(1+t/z)^{k-\frac{1}{2}+m}e^{-t}\,dt;$$

* See Chapter XIV, examples 12 and 13, p. 298.

† The function $W_{k,m}(z)$ was defined by means of an integral in this manner by Whittaker, *loc. cit.* p. 125.

‡ A suitable contour has been chosen and the variable t of § 14·6 replaced by $-t$.

and we have without difficulty*

$$\frac{d^2v}{dz^2}+\left(\frac{2k}{z}-1\right)\frac{dv}{dz}+\frac{\frac{1}{4}-m^2+k(k-1)}{z^2}v$$

$$=-\frac{(k-\frac{1}{2}+m)}{z^2}\int_{\infty}^{(0+)}\frac{d}{dt}\left\{t^{-k+\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{3}{2}+m}e^{-t}\right\}dt$$

$$=0,$$

since the expression in {} tends to zero as $t\to+\infty$; and this is the condition that $e^{-\frac{1}{2}z}z^k v$ should satisfy (B).

Accordingly the function $W_{k,m}(z)$ defined by the integral

$$-\frac{1}{2\pi i}\Gamma\left(k+\frac{1}{2}-m\right)e^{-\frac{1}{2}z}z^k\int_{\infty}^{(0+)}(-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt$$

is a solution of the differential equation (B).

The formula for $W_{k,m}(z)$ becomes nugatory when $k-\frac{1}{2}-m$ is a negative integer. To overcome this difficulty, we observe that *whenever*

$$R\left(k-\frac{1}{2}-m\right)\leqslant 0$$

and $k-\frac{1}{2}-m$ *is not an integer*, we may transform the contour integral into an infinite integral, after the manner of § 12·22; and so, when

$$R\left(k-\frac{1}{2}-m\right)\leqslant 0,$$

$$W_{k,m}(z)=\frac{e^{-\frac{1}{2}z}z^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^{\infty}t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt.$$

This formula suffices to define $W_{k,m}(z)$ in the critical cases when $m+\frac{1}{2}-k$ is a positive integer, and so $W_{k,m}(z)$ is defined for all values of k and m and all values of z except negative real values†.

Example. Solve the equation

$$\frac{d^2u}{dz^2}+\left(a+\frac{b}{z}+\frac{c}{z^2}\right)u=0$$

in terms of functions of the type $W_{k,m}(z)$, where a, b, c are any constants.

16·2. *Expression of various functions by functions of the type* $W_{k,m}(z)$.

It has been shewn‡ that various functions employed in Applied Mathematics are expressible by means of the function $W_{k,m}(z)$; the following are a few examples:

* The differentiations under the sign of integration are legitimate by § 4·44 corollary.

† When z is real and negative, $W_{k,m}(z)$ may be defined to be either $W_{k,m}(z+0i)$ or $W_{k,m}(z-0i)$, whichever is more convenient.

‡ Whittaker, *Bulletin American Math. Soc.* x; this paper contains a more complete account than is given here.