Definition 7.3. Assuming the notation and hypotheses in Lemma 7.2, we refer to

$$f^{+}(z) := \sum_{n \gg -\infty} c_f^{+}(n) q^n$$

as the *holomorphic part of* $f(z)$, and we refer to

$$f^{-}(z) := \sum_{n<0} c_f^{-}(n) \Gamma(k-1, 4\pi|n|y) q^n$$

as the *non-holomorphic part of* $f(z)$.

Remark 17. A harmonic Maass form with trivial non-holomorphic part is a weakly holomorphic modular form. We shall make use of this fact as follows. If $f_1, f_2 \in H_{2-k}(\Gamma)$ are two harmonic Maass forms with equal non-holomorphic parts, then $f_1 - f_2 \in M_{2-k}^{!}(\Gamma)$.

7.3. **The ξ-operator and period integrals of cusp forms.** Harmonic Maass forms are related to classical modular forms thanks to the properties of differential operators. The first nontrivial relationship depends on the differential operator

$$\xi_w := 2iy^w \cdot \overline{\frac{\partial}{\partial \overline{z}}}. \tag{7.7}$$

The following lemma[15], which is a straightforward refinement of a proposition of Bruinier and Funke (see Proposition 3.2 of [63]), shall play a central role throughout this paper.

Lemma 7.4. *If* $f \in H_{2-k}(N, \chi)$, *then*

$$\xi_{2-k} : H_{2-k}(N, \chi) \longrightarrow S_k(N, \overline{\chi})$$

is a surjective map. Moreover, assuming the notation in Definition 7.3, we have that

$$\xi_{2-k}(f) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} \overline{c_f^{-}(-n)} n^{k-1} q^n.$$

Thanks to Lemma 7.4, we are in a position to relate the non-holomorphic parts of harmonic Maass forms, the expansions

$$f^{-}(z) := \sum_{n<0} c_f^{-}(n) \Gamma(k-1, 4\pi|n|y) q^n,$$

with "period integrals" of modular forms. This observation was critical in Zwegers's work on Ramanujan's mock theta functions.

To make this connection, we must relate the Fourier expansion of the cusp form $\xi_{2-k}(f)$ with $f^{-}(z)$. This connection is made by applying the simple integral identity

$$\int_{-\overline{z}}^{i\infty} \frac{e^{2\pi i n \tau}}{(-i(\tau+z))^{2-k}} \, d\tau = i(2\pi n)^{1-k} \cdot \Gamma(k-1, 4\pi n y) q^{-n}. \tag{7.8}$$

This identity follows by the direct calculation

$$\int_{-\overline{z}}^{i\infty} \frac{e^{2\pi i n \tau}}{(-i(\tau+z))^{2-k}} \, d\tau = \int_{2iy}^{i\infty} \frac{e^{2\pi i n(\tau - z)}}{(-i\tau)^{2-k}} \, d\tau = i(2\pi n)^{1-k} \cdot \Gamma(k-1, 4\pi n y)\, q^{-n}.$$

[15]The formula for $\xi_{2-k}(f)$ corrects a typographical error in [63].

In this way, we may think of the non-holomorphic parts of weight $2-k$ harmonic Maass forms as period integrals of weight k cusp forms, where one applies (7.8) to

$$\int_{-\overline{z}}^{i\infty} \frac{\sum_{n=1}^{\infty} a(n)e^{2\pi i n\tau}}{(-i(\tau+z))^{2-k}}\, d\tau,$$

where $\sum_{n=1}^{\infty} a(n)q^n$ is a weight k cusp form. In short, $f^-(z)$ is the period integral of the cusp form $\xi_{2-k}(f)$.

In addition to this important observation, we require the following fact concerning the nontriviality of certain principal parts of harmonic Maass forms.

Lemma 7.5. *If $f \in H_{2-k}(\Gamma)$ has the property that $\xi_{2-k}(f) \neq 0$, then the principal part of f is nonconstant for at least one cusp.*

Sketch of the proof. This lemma follows from the work of Bruinier and Funke [63]. Using their pairing $\{\bullet, \bullet\}$, one finds that $\{\xi_{2-k}f, f\} \neq 0$ thanks to its interpretation in terms of Petersson norms. On the other hand, Proposition 3.5 of [63] expresses this quantity in terms of the principal part of f and the coefficients of the cusp form $\xi_{2-k}(f)$. An inspection of this formula reveals that at least one principal part of f must be nonconstant. $\square$

7.4. **The D-operator.** In addition to the differential operator ξ_{2-k}, which defines the surjective map

$$\xi_{2-k} : H_{2-k}(N,\chi) \longrightarrow S_k(N,\overline{\chi}),$$

we consider the differential operator

$$D := \frac{1}{2\pi i} \cdot \frac{d}{dz}. \tag{7.9}$$

We have the following theorem for integer weights.

Theorem 7.6. *Suppose that $2 \leq k \in \mathbb{Z}$ and $f \in H_{2-k}(N)$, then*

$$D^{k-1}(f) \in M_k^!(N).$$

Moreover, assuming the notation in (7.6), we have

$$D^{k-1}f = D^{k-1}f^+ = \sum_{n \gg -\infty} c_f^+(n) n^{k-1} q^n.$$

To prove this theorem, we must first recall some further differential operators, the Maass raising and lowering operators (for example, see [63, 71]) R_k and L_k. They are defined by

$$R_k = 2i\frac{\partial}{\partial z} + ky^{-1} = i\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + ky^{-1},$$
$$L_k = -2iy^2\frac{\partial}{\partial \overline{z}} = -iy^2\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

With respect to the Petersson slash operator (7.4), these operators satisfy the intertwining properties

$$R_k(f \mid_k \gamma) = (R_k f) \mid_{k+2} \gamma,$$
$$L_k(f \mid_k \gamma) = (L_k f) \mid_{k-2} \gamma,$$

meromorphic continuation in s to the whole complex plane. If we form a vector of Eisenstein series, indexed by the cusps, then the vector valued automorphic form will have a functional equation $s \to 1 - s$.

3.9 Maass raising and lowering operators

The Maass raising and lowering operators are differential operators found by [Maass, 1953] which have the property that when they are applied to an automorphic function of weight k as in Definition 3.5.2 then they produce a new automorphic function whose weight is either raised or lowered by 2. Without further ado, let's define these differential operators.

Definition 3.9.1 (Maass raising operator) Let $k \in \mathbb{Z}$. We define the Maass raising operator R_k to be the differential operator

$$R_k := iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{k}{2} = (z - \bar{z})\frac{\partial}{\partial z} + \frac{k}{2}.$$

Definition 3.9.2 (Maass lowering operator) Let $k \in \mathbb{Z}$. We define the Maass lowering operator L_k to be the differential operator

$$L_k := -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{k}{2} = -(z - \bar{z})\frac{\partial}{\partial \bar{z}} - \frac{k}{2}.$$

The following identities may be easily verified.

$$L_k = \overline{R_{-k}}, \qquad R_k = \overline{L_{-k}}, \tag{3.9.3}$$

$$\Delta_k = -L_{k+2}R_k - \frac{k}{2}\left(1 + \frac{k}{2}\right) = -R_{k-2}L_k + \frac{k}{2}\left(1 - \frac{k}{2}\right) \tag{3.9.4}$$

$$\Delta_{k+2}R_k = R_k\Delta_k, \qquad \Delta_{k-2}L_k = L_k\Delta_k. \tag{3.9.5}$$

Furthermore, the raising and lowering operators $R_k, -L_{k+2}$ are adjoint operators with respect to the Petersson inner product (see Definition 3.5.5) and satisfy (see [Roelcke, 1966], [Bump, 1997])

$$\iint\limits_{\Gamma_0(N)\backslash\mathfrak{h}} \big(R_k f\big)(z)\cdot\overline{g(z)}\,\frac{dxdy}{y^2} = \iint\limits_{\Gamma_0(N)\backslash\mathfrak{h}} f(z)\cdot\overline{\big(-L_{k+2}\,g\big)(z)}\,\frac{dxdy}{y^2} \tag{3.9.6}$$

which can be succinctly written in the form

$$\big\langle R_k f, g\big\rangle = \big\langle f, (-L_{k+2}g)\big\rangle,$$

where $f \in \mathcal{A}^*_{k,\chi}\,(\Gamma_0(N))$ and $g \in \mathcal{A}^*_{k+2,\chi}\,(\Gamma_0(N))$.

Proposition 3.9.7 **(R_k raises weights by 2, L_k lowers weights by 2)** *Fix $k, N \in \mathbb{Z}$ (with $N \geq 1$) and fix a character χ* (mod N). *Let $\mathcal{A}^*_{k,\chi}(\Gamma_0(N))$ be the $\mathbb{C}$-vector space of automorphic functions of weight k and character χ for $\Gamma_0(N)$ as in Definition 3.5.2. If $f \in \mathcal{A}^*_{k,\chi}(\Gamma_0(N))$ then*

$$R_k f \in \mathcal{A}^*_{k+2,\chi}(\Gamma_0(N)), \qquad L_k f \in \mathcal{A}^*_{k-2,\chi}(\Gamma_0(N)). \tag{3.9.8}$$

Furthermore, if $\Delta_k f = \lambda f$ for some eigenvalue $\lambda \in \mathbb{C}$, then

$$\Delta_{k+2}\big(R_k f\big) = \lambda\big(R_k f\big), \qquad \Delta_{k-2}\big(L_k f\big) = \lambda\big(L_k f\big). \tag{3.9.9}$$

Proof First, note that (3.9.9) follows from (3.9.5).

Next, we will prove that

$$\begin{aligned} \Big(\big(R_k f\big)\,\big|_{k+2}\,\alpha\Big)\,(z) &= R_k\Big(\big(f\big|_k\,\alpha\big)\,(z)\Big), \\ \Big(\big(L_k f\big)\,\big|_{k-2}\,\alpha\Big)\,(z) &= L_k\Big(\big(f\big|_k\,\alpha\big)\,(z)\Big) \end{aligned} \tag{3.9.10}$$

for any $\alpha \in \Gamma_0(N)$ and any smooth function $f : \mathfrak{h} \to \mathbb{C}$.

It is easy to see that (3.9.10) implies (3.9.8). For example, since we assume that $f \in \mathcal{A}^*_{k,\chi}(\Gamma_0(N))$, one obtains immediately that $\big(f\big|_k\,\alpha\big)\,(z) = \chi(d) f(z)$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Consequently $\big((R_k f)\,\big|_{k+2}\,\alpha\big)\,(z) = \chi(d)(R_k f)(z)$. We shall now prove (3.9.10) for the Maass raising operator R_k. The proof is very similar for the lowering operator and we leave the details to the reader.

Let $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Using the identity $c(z - \bar{z}) = (cz + d) - (c\bar{z} + d)$, and the fact that $\frac{\partial}{\partial z}\bar{z} = 0$, we compute

$$\begin{aligned} R_k\Big(\big(f\big|_k\,\alpha\big)\,(z)\Big) &= \left((z - \bar{z})\frac{\partial}{\partial z} + \frac{k}{2}\right)\left[\left(\frac{c\bar{z} + d}{cz + d}\right)^{\frac{k}{2}} f\left(\frac{az + b}{cz + d}\right)\right] \\ &= \left(\frac{c\bar{z} + d}{cz + d}\right)^{\frac{k}{2}} \cdot \left[\left(-\frac{k}{2}c(z - \bar{z}) \cdot \frac{1}{cz + d} + \frac{k}{2}\right) f(\alpha z) + \frac{z - \bar{z}}{(cz + d)^2} f'(\alpha z)\right] \\ &= \frac{k}{2}\left(\frac{c\bar{z} + d}{cz + d}\right)^{\frac{k+2}{2}} f(\alpha z) + \frac{z - \bar{z}}{(cz + d)^2}\left(\frac{c\bar{z} + d}{cz + d}\right)^{\frac{k}{2}} f'(\alpha z). \end{aligned} \tag{3.9.11}$$

In a similar manner we have

$$\begin{aligned} \Big((R_k f)\Big|_{k+2}\,\alpha\Big)\,(z) &= \left(\frac{c\bar{z} + d}{cz + d}\right)^{\frac{k+2}{2}} \cdot \left((w - \bar{w})\frac{\partial}{\partial w} + \frac{k}{2}\right) f(w)\,\bigg|_{w = \frac{az+b}{cz+d}} \\ &= \left(\frac{c\bar{z} + d}{cz + d}\right)^{\frac{k+2}{2}} \cdot \left(\frac{k}{2} f(w) + (w - \bar{w}) f'(w)\right)\,\bigg|_{w = \frac{az+b}{cz+d}}. \end{aligned} \tag{3.9.12}$$

One immediately observes that (3.9.11) and (3.9.12) are the same because

$$(w-\bar{w})\Big|_{w=\frac{az+b}{cz+d}} = \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{cz+d} = \frac{z-\bar{z}}{(cz+d)^2}. \qquad \square$$

Proposition 3.9.13 (Action of Maass operators on Whittaker functions) *Let $k \in \mathbb{Z}$ and let R_k, L_k, be the Maass raising and lowering operators, respectively, as in Definitions 3.9.1, 3.9.2. Let $r \in \mathbb{R}$ with $r > 0$. Then the action of the Maass operators R_k, L_k on the Fourier-Whittaker expansion (Theorem 3.7.4) is given by*

$$R_k\Big(W_{\frac{k}{2},\nu}(4\pi ry)\cdot e^{2\pi irx}\Big) = -W_{\frac{k+2}{2},\nu}(4\pi ry)\cdot e^{2\pi irx},$$

$$L_k\Big(W_{\frac{k}{2},\nu}(4\pi ry)\cdot e^{2\pi irx}\Big) = -\left(\nu^2-\left(\frac{k-1}{2}\right)^2\right)W_{\frac{k-2}{2},\nu}(4\pi ry)\cdot e^{2\pi irx}$$

If $r < 0$, the action is given by

$$R_k\Big(W_{-\frac{k}{2},\nu}(4\pi|r|y)\cdot e^{2\pi irx}\Big) = -\left(\nu^2-\left(\frac{k+1}{2}\right)^2\right)W_{-\frac{k+2}{2},\nu}(4\pi|r|y)\cdot e^{2\pi irx},$$

$$L_k\Big(W_{-\frac{k}{2},\nu}(4\pi|r|y)e^{2\pi irx}\Big) = -W_{-\frac{k-2}{2},\nu}(4\pi|r|y)\cdot e^{2\pi irx}.$$

Proof The proof follows from the Definitions 3.9.1, 3.9.2, and the recurrence relations (3.6.7) after a routine calculation. $\square$

3.10 The bottom of the spectrum

Fix integers k, N with $N \geq 1$, and let χ be a Dirichlet character (mod N). To recapitulate, we have been studying the Hilbert space of smooth functions $f : \mathfrak{h} \to \mathbb{C}$ which transform by

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)\left(\frac{cz+d}{|cz+d|}\right)^k f(z) \qquad (3.10.1)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$ and all $z \in \mathfrak{h}$. We have defined $\mathcal{L}^2(\Gamma_0(N)\backslash\mathfrak{h}, k, \chi)$ to be the space of all smooth functions satisfying (3.10.1) and the $\mathcal{L}^2$ condition

$$\iint_{\Gamma_0(N)\backslash\mathfrak{h}} |f(z)|^2 \frac{dxdy}{y^2} < \infty.$$

A much simpler space than $\mathcal{L}^2(\Gamma_0(N)\backslash\mathfrak{h}, k, \chi)$ is the space $\mathcal{L}^2(\mathbb{Z}\backslash\mathbb{R})$ consisting of all smooth functions satisfying $f(x+1) = f(x)$, $(\forall x \in \mathbb{R})$ together with the $\mathcal{L}^2$ condition $\int_0^1 |f(x)|^2 < \infty$. We showed in Chapter 1 that every function in $\mathcal{L}^2(\mathbb{Z}\backslash\mathbb{R})$ has a Fourier expansion $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi inx}$, so that a basis for the space is given by the exponential functions $e^{2\pi inx}$ with $n \in \mathbb{Z}$. The exponential function is an eigenfunction of the Laplacian $-\frac{d^2}{dx^2}$ with eigenvalue $4\pi^2n^2$, i.e.,

$$-\frac{d^2}{dx^2}e^{2\pi inx} = 4\pi^2n^2\, e^{2\pi inx}.$$

The eigenvalues comprise the spectrum. The bottom of the spectrum is the smallest eigenvalue. In the case of $-\frac{d^2}{dx^2}$ acting on $\mathcal{L}^2(\mathbb{Z}\backslash\mathbb{R})$, the bottom of the spectrum is 0 and this corresponds to the constant eigenfunction.

Similarly, [Selberg, 1956] decomposed the space $\mathcal{L}^2(\Gamma_0(N)\backslash\mathfrak{h}, k, \chi)$ into eigenfunctions of $\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}$. Such an eigenfunction f satisfies the second order partial differential equation

$$\Delta_k f = \lambda f,$$

where $\lambda = \lambda(\nu) = \nu(1-\nu)$. This conforms with Definition 3.5.7.

Proposition 3.10.2 (Bottom of the spectrum) *Fix integers k and $N \geq 1$. Let χ be a Dirichlet character (mod N). The operator Δ_k acting on the Hilbert space $\mathcal{L}^2(\Gamma_0(N)\backslash\mathfrak{h}, k, \chi)$ has a self-adjoint extension and is bounded below by*

$$\lambda\left(\frac{|k|}{2}\right) := \frac{|k|}{2}\left(1 - \frac{|k|}{2}\right).$$

If there exist elements of $\mathcal{L}^2(\Gamma_0(N)\backslash\mathfrak{h}, k, \chi)$ which have eigenvalue $\lambda\left(\frac{|k|}{2}\right)$, then they are given by $y^{\left|\frac{k}{2}\right|}f(z)$ where f is a holomorphic modular form of weight k and character χ satisfying (3.3.5) if $k > 0$, or the complex conjugate of such a function if $k < 0$.

Proof For a proof of the standard fact that the Laplace operator Δ_k has a self-adjoint extension see [Iwaniec, 2002]. Now, consider a non-zero function $f \in \mathcal{L}^2(\Gamma_0(N)\backslash\mathfrak{h}, k, \chi)$ satisfying $\Delta_k f = \mu f$ for some eigenvalue $\mu \in \mathbb{C}$. Since Δ_k is a self-adjoint operator with respect to the Petersson inner product (see Definition 3.5.5) we have

$$\mu\langle f, f\rangle = \langle \Delta_k f, f\rangle = \langle f, \Delta_k f\rangle = \overline{\mu}\langle f, f\rangle.$$

Because $f \neq 0$, $\langle f, f\rangle > 0$ it follows that $\mu = \overline{\mu} \in \mathbb{R}$. To show that the classical holomorphic modular forms and their conjugates lie at the bottom of the spectrum, we require the Maass raising and lowering operators R_k, L_k defined in Definitions 3.9.1 and 3.9.2. There is a natural connection between L_k and holomorphic modular forms. Indeed, it follows easily from the expression for