$$= \left(\sum_{d=1}^{\infty} \chi_D(d)\, d^{r-t-s}\right) \left(\sum_{N=1}^{\infty} G(N)\, \sigma_{t-s}(N)\, N^{-t}\right)$$
$$= \left(\sum_{d=1}^{\infty} \chi_D(d)\, d^{r-t-s}\right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} G(m\,n)\, m^{-s} n^{-t}\right)$$
$$= \sum_{M=1}^{\infty} \sum_{N=1}^{\infty} \Big(\sum_{d|(M,N)} \chi_D(d)\, d^r\, G(MN/d^2)\Big) M^{-s} N^{-t},$$

and this proves the lemma.

We have now completed the calculation of $(T_M T_N^c)_{\tilde{X}}$ in all cases. Before stating the final result, we rewrite (59) in a form similar to (50):

$$(T_M T_N)_\infty = \tfrac{1}{2} \sum_{d|(M,N)} (d\chi_p(d) + d\chi_p(N/d))\, I_p(MN/d^2) \qquad \text{if } \nu_p(N) \leqq \nu_p(M). \tag{61}$$

To see that this holds, write $N = p^\nu N_0$, $M = p^\nu M_0$ with $p \nmid N_0$. Then

$$\sum_{d|(M,N)} d\chi_p(d)\, I_p(MN/d^2) = \sum_{d_0|(M_0,N_0)} d_0\, \chi_p(d_0)\, I_p(p^{2\nu} M_0 N_0/d_0^2),$$
$$\sum_{d|(M,N)} d\chi_p(N/d)\, I_p(MN/d^2) = \sum_{d_0|(M_0,N_0)} d_0\, p^\nu\, \chi_p(N_0/d_0)\, I_p(M_0 N_0/d_0^2).$$

These expressions differ only by a factor $\chi_p(N_0)$, since clearly $I_p(p^{2\nu} n) = p^\nu I_p(n)$ for any n. Thus if $\chi_p(N_0) = 1$, (61) reduces to (59), while if $\chi_p(N_0) = -1$ both sides of (61) are zero (the left-hand side because N is not a norm).

Summing up, we have proved:

Theorem 4. *Let M, N be positive integers, $\nu_p(N) \leqq \nu_p(M)$. Then the intersection number of the homology classes T_M^c and T_N^c on the compact surface $\tilde{X}$ is given by*

$$T_M T_N^c = \tfrac{1}{2} \sum_{d|(M,N)} (d\chi_p(d) + d\chi_p(N/d))\,(H_p(MN/d^2) + I_p(MN/d^2)),$$

where H_p and I_p are the functions defined in Equations (3) *and* (4) *of the Introduction.*

Chapter 2: Modular Forms Whose Fourier Coefficients Involve Class Numbers

Notation. We again fix a real quadratic field K. The discriminant of K is denoted D; the other notations concerning K ($\mathcal{O}$, x', $x \gg 0$, $\mathrm{N}(x)$, $\mathrm{Tr}(x)$) are the same as in Chapter 1. As before, $\mathfrak{H}$ denotes the upper half-plane; $\mathbb{R}_+$ and $\mathbb{R}_-$ denote the sets of real numbers $\geqq 0$ and $\leqq 0$, respectively, $\mathbb{N}$ the set of integers $\geqq 0$. For $z \in \mathbb{C}$, $n \in \mathbb{Z}$, we write $\mathrm{e}(z)$ for $e^{2\pi i z}$ and $z^{n/2}$ for $|z|^{n/2}\, e^{in\arg(z)/2}$ with $-\pi < \arg(z) \leqq \pi$.

For $k > 0$ even, $M_k(\Gamma_0(D), \chi_D)$ denotes the vector space of modular forms of weight k, level D and "Nebentypus" χ_D, i.e. of functions $f\colon \mathfrak{H} \to \mathbb{C}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = \chi_D(a)(cz+d)^k f(z) \qquad \left(z \in \mathfrak{H},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)\right) \tag{1}$$

and which are holomorphic on $\mathfrak{H}$ and at the cusps of $\Gamma_0(D)$. The (infinite-dimensional) vector space of functions $f\colon \mathfrak{H} \to \mathbb{C}$ satisfying (1), with no holomorphy conditions, is denoted $M_k^*(\Gamma_0(D), \chi_D)$; such functions will be called "non-analytic modular forms" (of weight k, level D and Nebentypus).

2.1. *The Modular Form* $\varphi_D(z)$

Let D be the discriminant of a real quadratic field. For $N \in \mathbb{N}$, set

$$c(N) = H_D(N) + I_D(N),$$

where

$$H_D(N) = \sum_{\substack{t \in \mathbb{Z} \\ t^2 \leqq 4N \\ t^2 \equiv 4N \pmod{D}}} H\left(\frac{4N - t^2}{D}\right)$$

($H(n)$ being the class number function defined in 1.2) and

$$I_D(N) = \frac{1}{\sqrt{D}} \sum_{\substack{\lambda \in \mathcal{O},\, \lambda \gg 0 \\ \lambda\lambda' = N}} \min(\lambda, \lambda').$$

We saw in Chapter 1 that, at least if D is prime, $c(N)$ $(N>0)$ represents the intersection number $T_1 T_N^c$ on an appropriate compactification of the Hilbert modular surface $\mathfrak{H}^2/SL_2(\mathcal{O})$, $H_D(N)$ being the actual intersection number of the curves T_1 and T_N on this surface and $I_D(N)$ the contribution from the cusps. For $N=0$, $c(N) = -1/12$ is half the volume of the curve T_1. The main result of this chapter is that the numbers $c(N)$ are the Fourier coefficients of a modular form in $M_2(\Gamma_0(D), \chi_D)$.

Theorem 1. *The function*

$$\varphi_D(z) = \sum_{N=0}^{\infty} c(N)\, e^{2\pi i N z} \qquad (z \in \mathfrak{H})$$

is a modular form of weight 2 *and Nebentypus* χ_D *for* $\Gamma_0(D)$.

This theorem is similar to various classical class number identities of Kronecker, Hurwitz and others (see bibliography) in which various expressions involving class numbers are shown to be equal to Fourier coefficients of modular forms. One such result, for example, due to Hurwitz [17], says that the expression $c(N)$ in the case $D=1$, i.e. the number

$$H_1(N) + \sum_{\substack{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z} \times \mathbb{Z} \\ \lambda_1, \lambda_2 > 0 \\ \lambda_1 \lambda_2 = N}} \min(\lambda_1, \lambda_2),$$

is equal to $2\sigma_1(N)$ if $N>0$, where $\sigma_1(N)$ as usual denotes the sum of the positive divisors of N; thus $\varphi_1(z)$ is $-\frac{1}{12}$ times the normalized Eisenstein series

$$E_2(z) = 1 - 24 \sum_{N=1}^{\infty} \sigma_1(N)\, e^{2\pi i N z}.$$

(This is of course not a special case of Theorem 1.) We now describe briefly some other related results and generalizations of Theorem 1.

For any positive even integer k, we set

$$c_k(N)=\sum_t p_k(t,N)\,H\left(\frac{4N-t^2}{D}\right)+\frac{1}{\sqrt{D}}\sum_\lambda \min(\lambda,\lambda')^{k-1},$$

where the summations are the same as in the definition of $c(N)$ and $p_k(t,N)$ is the polynomial defined by

$$p_k(t,N)=(\rho_+^{k-1}-\rho_-^{k-1})/(\rho_+-\rho_-),\qquad \rho_\pm=\tfrac{1}{2}(t\pm\sqrt{t^2-4N}).$$

Then the function $\varphi_{D,k}(z)=\sum c_k(N)\,e^{2\pi iNz}$ is a modular form of weight k for the group $\Gamma_0(D)$ and character χ_D and in fact a cusp form for $k>2$. If $k>2$ and $D=1$, then the Selberg-Eichler trace formula [32] tells us that

$$c_k(N)=-2\,\mathrm{Tr}(T(N),S_k(SL_2\mathbb{Z})),$$

where $T(N)$ is a Hecke operator on the space of cusp forms of weight k for the full modular group, so the function $\varphi_{1,k}(z)$ is -2 times the sum of all normalized Hecke eigenfunctions in this space. The proofs of these results, as well as new proofs of the results of Cohen mentioned below and various generalizations, will be given in [36].

The other result related to Theorem 1 which we would like to discuss is due to Cohen [23]. Let r be an odd positive integer, and define an arithmetical function $H(r,N)$ ($N\in\mathbb{N}$) by $H(r,0)=\zeta(1-2r)$, $H(r,N)=0$ for $N\equiv 1$ or $2 \pmod 4$, and

$$H(r,N)=L(1-r,\chi_\Delta)\sum_{d|f} d^{2r-1}\prod_{p|d}(1-\chi_\Delta(p)\,p^{-r})$$

for $N>0$, $N\equiv 0$ or $3 \pmod 4$, where $\Delta<0$ is the discriminant of $\mathbb{Q}(\sqrt{-N})$ and f is defined by $-N=\Delta f^2$. This function generalizes the class number function $H(N)=H(1,N)$ (cf. Eq. (15) and Proposition 1 of 1.2). We set

$$H_D(r,N)=\sum_t H\left(r,\frac{4N-t^2}{D}\right),$$

the summation being the same as before. Then Cohen shows that, for $r>1$, the function $\sum_{N=0}^{\infty} H_D(r,N)\,e^{2\pi iNz}$ belongs to $M_{r+1}(\Gamma_0(D),\chi_D)$. Thus for $r>1$ no corrective term like our $\sum\min(\lambda,\lambda')$ is needed. As with the case $r=1$, there is a generalization in which the terms $H\left(r,\frac{4N-t^2}{D}\right)$ in the sum above are weighted with a certain homogeneous polynomial in t and N, leading to modular forms (in fact cusp forms) of higher weight. Unfortunately, Cohen's proof does not work for $r=1$, although the starting point for both proofs, as we shall see, is the same.

The basic idea is to express the numbers $H(N)$ as Fourier coefficients of a modular form of half-integral weight. This suggestion was already made by Hecke [20] as a way of explaining the classical class number relations like the above-mentioned theorem of Hurwitz concerning $H_1(N)$. Hecke pointed out that, by the formula of Gauss and Hermite, the number $r_3(N)$ of representations of N as a sum of three squares can be expressed in terms of class numbers:

$$r_3(N)=\begin{cases} 12H(N) & \text{if } N\equiv 1 \text{ or } 2 \pmod 4,\\ 24H(4N) & \text{if } N\equiv 3 \pmod 8,\\ 0 & \text{if } N\equiv 7 \pmod 8,\\ r_3(N/4) & \text{if } N\equiv 0 \pmod 4.\end{cases}$$

On the other hand, $r_3(N)$ is the N-th Fourier coefficient of $\theta(z)^3$, where

$$\theta(z)=\sum_{t\in\mathbb{Z}} q^{t^2} \qquad (q=e^{2\pi i z})$$

is a modular form of weight one-half; thus one should expect that the function

$$\mathscr{H}(z)=\sum_{N=0}^{\infty} H(N)\, q^N \qquad (z\in\mathfrak{H})$$

is a modular form of weight 3/2, and then the number $H_1(N)$ would be the $4N$-th Fourier coefficient of the modular form $\mathscr{H}(z)\,\theta(z)$ of weight 2.

At the time of appearance of Hecke's paper, no satisfactory theory of modular forms of half-integral weight was known; such a theory has now been provided by Shimura ([33, 34]). However, one still cannot carry out Hecke's suggestion directly because, as we shall see, the function $\mathscr{H}(z)$ does *not* in fact transform like a modular form of weight 3/2. For $r>1$ odd, on the other hand, Cohen proves that the function $\sum_{N=0}^{\infty} H(r,N)\,q^N$ is a modular form of weight $r+\frac{1}{2}$ (for $\Gamma_0(4)$) in the sense of Shimura, namely equal to the linear combination

$$\frac{\zeta(1-2r)}{2^{2r+1}}\{(1-i)\,E_{r+1/2}(z)-i\,F_{r+1/2}(z)\}$$

of the two Eisenstein series

$$E_{r+1/2}(z)=\sum_{\substack{m=1\\ m\ \mathrm{odd}}}^{\infty}\ \sum_{\substack{n=-\infty\\ (n,m)=1}}^{\infty} \frac{\left(\frac{n}{m}\right)\left(\frac{-1}{m}\right)^{1/2}}{(mz+n)^{r+1/2}},$$

$$F_{r+1/2}(z)=z^{-r-1/2}\,E_{r+1/2}\left(\frac{-1}{4z}\right),$$

whose Fourier coefficients were calculated by Shimura in the papers cited.

For $r=1$ we should like to apply the same idea and show that $\mathscr{H}(z)$ is equal to the linear combination

$$\mathscr{F}(z)=-\tfrac{1}{96}\{(1-i)\,E_{3/2}(z)-i\,F_{3/2}(z)\} \tag{1}$$

of the two Eisenstein series of weight 3/2. However, the series defining $E_{r+1/2}(z)$ diverges for $r=1$. To overcome this difficulty, we use the well-known device of Hecke [25]: we introduce the series

$$E_{3/2,s}(z)=\sum_{\substack{m>0\\(m,2n)=1}}\sum\frac{\left(\frac{n}{m}\right)\left(\frac{-1}{m}\right)^{1/2}}{(mz+n)^{3/2}\,|mz+n|^{2s}}\qquad(z\in\mathfrak{H},\ s\in\mathbb{C}),\tag{2}$$

which converges absolutely for $\mathrm{Re}(s)>\frac{1}{4}$ and transforms by

$$E_{3/2,s}\left(\frac{az+b}{cz+d}\right)=\left(\frac{c}{d}\right)\left(\frac{-1}{d}\right)^{1/2}(cz+d)^{3/2}\,|cz+d|^{2s}E_{3/2,s}(z)$$

for $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(4)$ $\left(\text{for the definitions of } \left(\frac{c}{d}\right), \left(\frac{-1}{d}\right)^{1/2} \text{etc. cf. [33]}\right)$. This function is analytic in s, and by analytic continuation we obtain a function $E_{3/2}(z)=E_{3/2,0}(z)$ which is possibly not holomorphic in z but a least satisfies the transformation equation of a modular form of weight 3/2. We proceed similarly for $F_{3/2}(z)$ and then define $\mathscr{F}(z)$ by (1). The function $\mathscr{F}(z)$ is periodic of period 1 and hence has a Fourier expansion $\sum f_N\,e^{2\pi iNz}$, the coefficients f_N possibly being functions of $y=\mathrm{Im}(z)$. We will calculate these Fourier coefficients in the next section, finding that the N-th coefficient is equal to $H(N)$ (independent of y) for N positive and to 0 for N negative except for $N=-u^2$, $u\in\mathbb{Z}$. Thus $\mathscr{F}(z)$ is the sum of $\mathscr{H}(z)$ and a certain non-analytic expression involving the powers q^{-u^2}. In Section 2.3 we construct a theta series of weight 2 which will cancel the contribution from this non-analytic piece and create the term $\sum\min(\lambda,\lambda')$ in the formula for $c(N)$. The proof of Theorem 1 will be completed in Section 2.4.

2.2. *The Eisenstein Series of Weight $\frac{3}{2}$*

At the end of the last section we defined a function $\mathscr{F}(z)$ which transforms under $\Gamma_0(4)$ like a modular form of weight $\frac{3}{2}$, and explained a reason for expecting a relationship between $\mathscr{F}(z)$ and the function $\mathscr{H}(z)=\sum H(N)\,q^N$. In this section we will prove the following result.

Theorem 2. *For $z\in\mathfrak{H}$, we have*

$$\mathscr{F}(z)=\mathscr{H}(z)+y^{-1/2}\sum_{f=-\infty}^{\infty}\beta(4\pi f^2y)\,q^{-f^2},$$

where $y=\mathrm{Im}(z)$, $q=\mathrm{e}(z)$ and $\beta(x)$ is defined by

$$\beta(x)=\frac{1}{16\pi}\int_1^\infty u^{-3/2}\,e^{-xu}\,du\qquad(x\geqq 0).$$

Before proving this, we mention two corollaries. The first is a description of the way $\mathscr{H}(z)$ transforms under $\Gamma_0(4)$.

Corollary. *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, $c \neq 0$,

$$\left(\frac{c}{d}\right)\left(\frac{-1}{d}\right)^{-1/2}(cz+d)^{3/2}\mathscr{H}\left(\frac{az+b}{cz+d}\right)-\mathscr{H}(z)=\frac{1+i}{16\pi}\int_{d/c}^{i\infty}\frac{\theta(t)\,dt}{(t+z)^{3/2}},$$

where $\theta(t)=\sum_{f\in\mathbb{Z}} \mathrm{e}(f^2 t)$ *and the integral is taken along a vertical path in the upper half-plane.*

Indeed, by the theorem,

$$\mathscr{F}(z)-\mathscr{H}(z)=\frac{1}{16\pi}y^{-1/2}\int_1^\infty u^{-3/2}\,\theta(2iuy-z)\,du=\frac{1+i}{16\pi}\int_{-\bar z}^{i\infty}(z+v)^{-3/2}\,\theta(v)\,dv,$$

the integral being taken along the vertical path $v=2iuy-z$, $1\leqq u<\infty$. Denote the latter integral by $\psi(z)$; then, substituting $v=\dfrac{at-b}{-ct+d}$, we find

$$\begin{aligned}\psi\left(\frac{az+b}{cz+d}\right)&=\int_{-\bar z}^{d/c}\left(\frac{az+b}{cz+d}+\frac{at-b}{-ct+d}\right)^{-3/2}\theta\left(\frac{at-b}{-ct+d}\right)\frac{dt}{(ct-d)^2}\\&=\left(\frac{-c}{d}\right)\left(\frac{-1}{d}\right)^{-1/2}(cz+d)^{3/2}\int_{-\bar z}^{d/c}(z+t)^{-3/2}\theta(t)dt,\end{aligned}$$

where in the second line we have used our knowledge of the behaviour of $\theta(t)$ under $\Gamma_0(4)$. Thus

$$\left(\frac{c}{d}\right)\left(\frac{-1}{d}\right)^{-1/2}(cz+d)^{-3/2}\psi\left(\frac{az+b}{cz+d}\right)-\psi(z)=-\int_{d/c}^{i\infty}(z+t)^{-3/2}\,\theta(t)\,dt.$$

The expression on the left, with ψ replaced by $\mathscr{F}$, is zero because $\mathscr{F}$ transforms under $\Gamma_0(4)$ like a modular form of weight 3/2. The Corollary now follows from the identity $\mathscr{F}-\mathscr{H}=\dfrac{1+i}{16\pi}\psi$.

We should mention that one result concerning the behaviour of $\mathscr{H}$ under modular transformations was already known, namely the identity

$$(2z/i)^{-3/2}\mathscr{H}\left(\frac{-1}{4z}\right)+\mathscr{H}(z)=-\tfrac{1}{24}\theta(z)^3-\sqrt{\frac{z}{8i}}\int_{-\infty}^{\infty}\mathrm{e}(\xi^2 z)\frac{1+\mathrm{e}(2\xi z)}{1-\mathrm{e}(2\xi z)}\,\xi\,d\xi,$$

found by Eichler [21].

The other consequence of Theorem 2 was pointed out to us by H. Cohen, namely, a "modular" proof of the Gauss-Hermite formula quoted in Section 2.1. To see that $r_3(8N+3)=24H(8N+3)$, for example, we observe that

$$\sum H(8N+3)\,q^N=\tfrac{1}{8}\sum_{r \,(\mathrm{mod}\, 8)}\mathrm{e}(-3r/8)\,\mathscr{H}\left(\frac{z+r}{8}\right)=\tfrac{1}{8}\sum_{r\,(\mathrm{mod}\,8)}\mathrm{e}(-3r/8)\,\mathscr{F}\left(\frac{z+r}{8}\right),$$

the terms involving q^{-f^2} all dropping out because $-f^2$ is never congruent to 3 modulo 8. Therefore the function $\sum H(8N+3)\,q^N$ is a (holomorphic) modular form of weight 3/2 for some congruence group (in fact for $\Gamma_0(2)$), and since

$\frac{1}{24}\sum r_3(8N+3)\,q^N$ is also such a form, one can prove the equality of the two functions by comparing finitely many of their coefficients. A similar argument works for coefficients belonging to the sequence $4N+2$ or $4N+3$ or to any other arithmetical progression not containing the negatives of any squares.

We now give the proof of Theorem 2. Set

$$\mathscr{F}_s(z) = -\tfrac{1}{96}\{(1-i)\,E_{3/2,s}(z) - i\,F_{3/2,s}(z)\} \qquad (\mathrm{Re}(s)>\tfrac{1}{4}),$$

where $E_{3/2,s}(z)$ is the function defined by Eq. (2) of 2.1 and

$$F_{3/2,s}(z) = z^{-3/2}\,|z|^{-2s}\,E_{3/2,s}(-1/4z).$$

Then $\mathscr{F}_s(z)$ has an analytic continuation to the whole s-plane with $\mathscr{F}_0(z)=\mathscr{F}(z)$, and on the other hand $\mathscr{F}_s$ is periodic in z with period 1 and therefore has a Fourier development of the form

$$\mathscr{F}_s(z) = \sum_{N=-\infty}^{\infty} f_N(s,y)\,q^N$$

with $f_N(s,y)$ analytic in s. Theorem 2 will follow if we show

$$f_N(0,y) = \begin{cases} H(N) & \text{if } N>0,\\ -\dfrac{1}{12}+\dfrac{1}{8\pi}\,y^{-1/2} & \text{if } N=0,\\ 2y^{-1/2}\,\beta(4\pi f^2 y) & \text{if } N=-f^2,\ f>0,\\ 0 & \text{if } N<0,\ -N \neq \text{square}. \end{cases}$$

We begin by finding the Fourier expansion of $E_{3/2,s}(z)$. Write

$$E_{3/2,s}(z) = \sum_{\substack{m=1\\ m\,\mathrm{odd}}}^{\infty} \left(\frac{-1}{m}\right)^{1/2} m^{-3/2-2s} \sum_{n\,(\mathrm{mod}\,m)} \left(\frac{n}{m}\right) \sum_{h\in\mathbb{Z}} \left(z+\frac{n}{m}+h\right)^{-3/2} \left|z+\frac{n}{m}+h\right|^{-2s}.$$

By the Poisson summation formula,

$$\sum_{h\in\mathbb{Z}} (z+h)^{-3/2}\,|z+h|^{-2s} = \sum_{N=-\infty}^{\infty} \alpha_N(s,y)\,\mathrm{e}(Nz) \qquad (z\in\mathfrak{H})$$

with

$$\begin{aligned}\alpha_N(s,y) &= \int_{iy-\infty}^{iy+\infty} z^{-3/2}\,|z|^{-2s}\,\mathrm{e}(-Nz)\,dz\\ &= y^{-1/2-2s}\,e^{2\pi N y}\int_{-\infty}^{\infty} (v+i)^{-3/2}\,(v^2+1)^{-s}\,\mathrm{e}(-Nyv)\,dv\end{aligned}$$

(the last formula is obtained by the substitution $z=(v+i)\,y$), and inserting this into the formula for $E_{3/2,s}$ we find

$$\begin{aligned}E_{3/2,s}(z) &= \sum_{\substack{m=1\\ m\,\mathrm{odd}}}^{\infty} m^{-1-2s} \sum_{N=-\infty}^{\infty} \gamma_m(-N)\,\alpha_N(s,y)\,\mathrm{e}(Nz)\\ &= \sum_{N=-\infty}^{\infty} E_{-N}^{\mathrm{odd}}(1+2s)\,\alpha_N(s,y)\,e^{2\pi i N z},\end{aligned}$$

where $\gamma_m(-N)$ denotes the Gauss sum

$$\gamma_m(-N)=\left(\frac{-1}{m}\right)^{1/2} m^{-1/2} \sum_{n \,(\mathrm{mod}\, m)} \left(\frac{n}{m}\right) \mathrm{e}(nN/m) \qquad (m \text{ odd})$$

and $E^{\mathrm{odd}}_{-N}(s)$ the Dirichlet series

$$E^{\mathrm{odd}}_{-N}(s)=\sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} \gamma_m(-N)\, m^{-s}.$$

A similar calculation for $F_{3/2,s}(z)$ gives

$$F_{3/2,s}(z)=2^{2s+3}\, i+(1+i)\sum_{N=-\infty}^{\infty} E^{\mathrm{even}}_{-N}(1+2s)\,\alpha_N(s,y)\, e^{2\pi i N z}$$

with

$$E^{\mathrm{even}}_{-N}(s)=\sum_{\substack{m=1\\ m \text{ even}}}^{\infty} \gamma_m(-N)(m/2)^{-s},$$

the Gauss sum $\gamma_m(-N)$ now being defined by

$$\gamma_m(-N)=m^{-1/2}\sum_{n \,(\mathrm{mod}\, 2m)} \left(\frac{m}{n}\right) i^{n/2}\, \mathrm{e}(Nn/2m) \qquad (m \text{ even}).$$

(The constant term $2^{2s+3}i$ comes from the term $m=1$, $n=0$ in (2).) Thus the Fourier expansion of $\mathscr{F}_s(z)$ is

$$\mathscr{F}_s(z)=-\frac{2^{2s}}{12}-\frac{1-i}{48}\sum_{N=-\infty}^{\infty} E_{-N}(1+2s)\,\alpha_N(s,y)\, q^N$$

with

$$E_{-N}(s)=\tfrac{1}{2}\left(E^{\mathrm{odd}}_{-N}(s)+E^{\mathrm{even}}_{-N}(s)\right).$$

The Gauss sums $\gamma_m(-N)$ and the Dirichlet series E^{odd}_{-N}, E^{even}_{-N} and E_{-N} are evaluated in [15], §4, Theorems 2 and 3. It turns out that $E_{-N}(s)=0$ identically if N is congruent to 1 or 2 (mod 4), while if $N \neq 0$ is congruent to 0 or 3 (mod 4), then

$$E_{-N}(s)=\zeta(2s)^{-1} L(s,\chi_d) \sum_{\substack{a,c>0\\ ac \mid f}} \mu(a)\,\chi_d(a)\, c^{-2s+1} a^{-s}, \tag{3}$$

where d is the discriminant of $\mathbb{Q}(\sqrt{-N})$, χ_d the associated character, and f the number defined by $-N=df^2$. Finally, for $N=0$ we have

$$E_0(s)=\zeta(2s-1)/\zeta(2s).$$

We are interested in finding the value of the Fourier coefficients

$$f_N(s,y)=-\frac{1-i}{48}E_{-N}(1+2s)\,\alpha_N(s,y)+\begin{cases}-2^{2s}/12 & \text{if } N=0\\ 0 & \text{if } N\neq 0\end{cases}$$

at $s=0$.

The integral defining $\alpha_N(s,y)$ can be evaluated easily at $s=0$:

$$\alpha_N(0,y)=\begin{cases}-4\pi(1+i)\,N^{1/2} & \text{if } N>0,\\ 0 & \text{if } N\leqq 0.\end{cases}$$

If N is positive, formula (3) shows that $E_{-N}(s)$ is holomorphic at $s=1$ with

$$\begin{aligned}E_{-N}(1)&=\frac{6}{\pi^2}L(1,\chi_d)\sum_{ac|f}\mu(a)\,\chi_d(a)/a\,c\\ &=\frac{6}{\pi}N^{-1/2}L(0,\chi_d)\sum_{k|f}k\prod_{p|k}(1-\chi_d(p)/p)\\ &=\frac{6}{\pi}N^{-1/2}H(N)\end{aligned}$$

(we have used the functional equation of $L(s,\chi_d)$), so $f_N(0,y)=H(N)$ as claimed. If N is negative but $-N$ is not a square, then the number d in (3) is the discriminant of a real quadratic field and so $L(s,\chi_d)$ and hence $E_{-N}(s)$ are holomorphic at $s=1$; this, together with $\alpha_N(0,y)=0$, implies that $f_N(0,y)=0$. It remains to treat the case $N=-f^2$.

First, if $N=-f^2$, $f>0$, then

$$E_{-N}(s)=\zeta(2s)^{-1}\zeta(s)\sum_{ac|f}\mu(a)\,c^{-2s+1}a^{-s}$$

has a pole of residue $\zeta(2)^{-1}$ at $s=1$, so

$$f_N(0,y)=-\frac{1-i}{16\pi^2}\alpha'_N(0,y),$$

where

$$\begin{aligned}\alpha'_N(0,y)&=\frac{\partial}{\partial s}\alpha_N(s,y)\bigg|_{s=0}\\ &=-y^{-1/2}e^{2\pi N y}\int_{-\infty}^{\infty}(v+i)^{-3/2}\log(v^2+1)\,\mathrm{e}(-Nyv)\,dv\\ &=-(2iy)^{-1/2}\int_{1/2-i\infty}^{1/2+i\infty}u^{-3/2}e^{4\pi Nyu}\log\{4u(1-u)\}\,du\end{aligned}$$

(the last equation is obtained by substituting $v=2iu-i$). We deform the path of integration in the last integral to a path in the cut plane $\mathbb{C}-[1,\infty)$ which circles the cut clockwise from $i\varepsilon+\infty$ to $\frac{1}{2}$ to $i\varepsilon-\infty$. Across the cut, $\log\{4u(1-u)\}$ jumps by $2\pi i$ and the other terms in the integrand are continuous. Therefore

$$\alpha'_N(0,y)=-2\pi i(2iy)^{-1/2}\int_1^{\infty}u^{-3/2}e^{-4\pi|N|yu}\,du=-16\pi^2(1+i)\,y^{-1/2}\beta(4\pi|N|\,y)$$

and hence

$$f_N(0,y)=2y^{-1/2}\beta(4\pi f^2 y)\qquad (N=-f^2<0)$$

as claimed. Finally, for $N=0$ we have

$$f_0(0,y)=-\frac{1}{12}-\frac{1-i}{32\pi^2}\alpha_0'(0,y)=-\frac{1}{12}-\frac{1}{8\pi}y^{-1/2},$$

because $E_0(s)=\zeta(2s-1)/\zeta(2s)$ has a pole of residue $3/\pi^2$ at $s=1$. This completes the proof of Theorem 2.

2.3. A Theta-Series Attached to an Indefinite Quadratic Form

If F is a positive definite quadratic form in $2k$ variables, and $L\subset\mathbb{R}^{2k}$ some lattice on which F takes integral values, then the associated theta-series $\sum_{\lambda\in L} e^{2\pi i z F(\lambda)}$ $(z\in\mathfrak{H})$ is a modular form of weight k (of some level and for some quadratic character depending on F and L); similarly, the series $\sum p(\lambda)\,e^{2\pi i z F(\lambda)}$, where $p(\lambda)$ is a homogeneous polynomial of degree m which is spherical with respect to F, is a modular form of weight $k+m$ (for precise statements and proofs see [30] or [24]). If, on the other hand, F is an *indefinite* form, then these series diverge because $|e^{2\pi i z F(\lambda)}|$ grows exponentially in $|\lambda^2|$ in the cone $F(\lambda)<0$. To obtain a convergent series, we can either allow the coefficient $p(\lambda)$ to be a non-analytic (or piecewise analytic) homogeneous function of λ which is identically zero for $F(\lambda)<0$, or allow $p(\lambda)=p_z(\lambda)$ to depend on z in such a way that $p_z(\lambda)$ is much smaller than $e^{-2\pi i z F(\lambda)}$ as $|\lambda|\to\infty$. If the function $p_z(\lambda)$ is chosen in such a way that the Fourier transform (with respect to λ) of $p_z(\lambda)\,e^{2\pi i z F(\lambda)}$ equals $z^{-r}p_{-1/z}(\lambda)\,e^{-2\pi i F(\lambda)/z}$ for some r, then the same proof as in the classical case (namely, by application of the Poisson summation formula) shows that $\sum p_z(\lambda)\,e^{2\pi i z F(\lambda)}$ is a modular form of weight r. In this section we shall construct such a modular form, of weight 2, associated to the norm form of the quadratic field K, i.e. F will be the indefinite form $F(\lambda_1,\lambda_2)=\lambda_1\lambda_2$ on $\mathbb{R}^2$ and L will be the lattice $\mathcal{O}$, embedded in $\mathbb{R}^2$ by $x\mapsto(x,x')$. Our coefficient function $p_z(\lambda_1,\lambda_2)$ will be

$$\frac{2}{\sqrt{y}}\beta(\pi y(\lambda_1-\lambda_2)^2)-\begin{cases}\frac{1}{2}\min(|\lambda_1|,|\lambda_2|) & \text{if } \lambda_1\lambda_2>0,\\ 0 & \text{if } \lambda_1\lambda_2\leqq 0,\end{cases}$$

where

$$\beta(x)=\frac{1}{16\pi}\int_1^\infty u^{-3/2}\,e^{-xu}\,du \qquad (\operatorname{Re}x\geqq 0)$$

is the function defined in Theorem 2.

We will need some properties of the function $\beta(x)$ and of the related function

$$f(a,x)=\int_x^\infty e^{-u^2+2\pi i a u}\frac{du}{u} \qquad (a\in\mathbb{C},\ x\in\mathbb{C}-\mathbb{R}_-).$$

These functions are related to the standard "complementary error function"

$$\operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du \qquad (x\in\mathbb{C})$$

by the formulas

$$\beta(x)=\frac{1}{8\pi}(e^{-x}-\sqrt{\pi x}\,\operatorname{erfc}(\sqrt{x})) \tag{4}$$

(which also gives the analytic continuation of $\beta(x)$ to $\mathbb{C}-\mathbb{R}_-$) and

$$\frac{\partial f(a,x)}{\partial a}=i\pi^{3/2}\,e^{-\pi^2 a^2}\operatorname{erfc}(x+i\pi a).$$

The properties we need are summarized in the following lemma.

Lemma 1. *The functions* β, f *and* erfc *satisfy the identities*

a) $\int_{-\infty}^{\infty}\beta(c\,t^2)\,e^{2\pi i t x}\,dt=\frac{1}{16}\pi^{-5/2}\,c^{1/2}\,\frac{1-e^{-\pi^2x^2/c}}{x^2}$,

b) $\int_{-\infty}^{\infty}\frac{1-e^{-\alpha x^2}}{x^2}\,e^{2\pi i t x}\,dx=16\pi^{3/2}\,\alpha^{1/2}\,\beta(\pi^2t^2/\alpha)$,

c) $\operatorname{erfc}(x)+\operatorname{erfc}(-x)=2$,

d) $f(-a,-x)-f(a,x)=i\pi\operatorname{erfc}(\pi a)$ $(x\in\mathfrak{H})$.

Proof. a) is obtained easily by substituting the definition of β into the integral and inverting the order of integration, and b) by differentiating both sides of the identity with respect to α; alternatively, b) can be considered as the inverse Fourier transform formula to a). Formula c) is standard and easy: one sees by differentiating the left-hand side that it is a constant, and the constant is found by setting $x=0$. Finally, the function $f(-a,-x)-f(a,x)$ is defined for $a\in\mathbb{C}$, $x\in\mathbb{C}-\mathbb{R}=\mathfrak{H}\cup-\mathfrak{H}$, and its derivative with respect to x is identically zero, so for fixed a this function has a constant value $\varphi_+(a)$ for x in the upper half-plane and a constant value $\varphi_-(a)$ for x in the lower half-plane. Differentiating with respect to a, we find

$$\frac{d}{da}\varphi_{\pm}(a)=-2i\pi^{3/2}\,e^{-\pi^2a^2},$$

so

$$\varphi_{\pm}(a)=i\pi\operatorname{erfc}(\pi a)+c_{\pm}$$

for some complex constants c_+ and c_-. Interchanging the roles of (a,x) and $(-a,-x)$ in the definition of $\varphi_{\pm}(a)$ leads to

$$\varphi_{\pm}(-a)=-\varphi_{\mp}(a),$$

which, together with c) of the lemma, implies $c_++c_-=-2\pi i$. Finally, it follows by the calculus of residues that

$$\lim_{\varepsilon\searrow 0}(f(a,x-i\varepsilon)-f(a,x+i\varepsilon))=2\pi i$$

for x real and negative, and this implies $c_+-c_-=2\pi i$. Hence $c_+=0$ and $c_-=-2\pi i$, so $\varphi_{\pm}(a)=\pm i\pi\operatorname{erfc}(\pm\pi a)$.

We are now in a position to construct our non-analytic theta-series.

Proposition 1. *For $z\in\mathfrak{H}$, define continuous complex-valued functions U_z, V_z and W_z on $\mathbb{R}^2$ by*

$$U_z(\lambda,\lambda')=2y^{-1/2}\beta(\pi y(\lambda-\lambda')^2)\,\mathrm{e}(\lambda\lambda' z),$$

$$V_z(\lambda,\lambda')=\begin{cases}\frac{1}{2}\min(|\lambda|,|\lambda'|)\,\mathrm{e}(\lambda\lambda' z) & \text{if } \lambda\lambda'>0,\\ 0 & \text{if } \lambda\lambda'\leqq 0,\end{cases}$$

$$W_z(\lambda,\lambda')=U_z(\lambda,\lambda')-V_z(\lambda,\lambda'),$$

where $y=\operatorname{Im}(z)$. Let $\tilde W$ denote the Fourier transform of W:

$$\tilde W_z(\mu,\mu')=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_z(\lambda,\lambda')\,\mathrm{e}(-\lambda\mu-\lambda'\mu')\,d\lambda\,d\lambda'.$$

Then

$$\tilde W_z(\mu,\mu')=z^{-2}\,W_{-1/z}(\mu,\mu'). \tag{5}$$

Corollary. *The function*

$$\mathscr{W}(z)=\sum_{\lambda\in\mathcal{O}} W_z(\lambda,\lambda')\qquad (z\in\mathfrak{H})$$

is a non-analytic modular form of weight 2, level D and Nebentypus χ_D.

Proof. We calculate the Fourier transforms of U_z and V_z separately. First, in the integral defining $\tilde U_z$ we substitute $\lambda'=\lambda+t$ and use a) of Lemma 1:

$$\begin{aligned}\tilde U_z(\mu,\mu')&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}(-\lambda\mu-\lambda'\mu')\,U_z(\lambda,\lambda')\,d\lambda'\,d\lambda\\ &=2y^{-1/2}\int_{-\infty}^{\infty}\mathrm{e}(\lambda(z\lambda-\mu-\mu'))\int_{-\infty}^{\infty}\beta(\pi y t^2)\,\mathrm{e}((z\lambda-\mu')t)\,dt\,d\lambda\\ &=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\frac{1-e^{-\pi(z\lambda-\mu')^2/y}}{(z\lambda-\mu')^2}\,\mathrm{e}(z\lambda^2-\mu\lambda-\mu'\lambda)\,d\lambda.\end{aligned}$$

Substituting $\lambda=u+\mu'/z$ and applying b) of Lemma 1, we find

$$\begin{aligned}\tilde U_z(\mu,\mu')&=\frac{1}{8\pi^2z^2}\,\mathrm{e}(-\mu\mu'/z)\int_{-\infty}^{\infty}\frac{e^{2\pi i z u^2}-e^{-\pi|z|^2u^2/y}}{u^2}\,\mathrm{e}((\mu'-\mu)u)\,du\\ &=2z^{-2}\,\mathrm{e}(-\mu\mu'/z)\{|z|y^{-1/2}\beta(\pi y(\mu-\mu')^2/|z|^2)-w^{1/2}\beta(\pi(\mu-\mu')^2/w)\}\\ &=z^{-2}U_{-1/z}(\mu,\mu')+8w^{-3/2}\,\mathrm{e}(-\mu\mu'/z)\,\beta(\pi(\mu-\mu')^2/w),\end{aligned}$$

where in the last two lines we have set $w=2z/i$, so $|\arg(w)|<\frac{\pi}{2}$. For $\tilde V_z$ we find

$$\tilde V_z(\mu,\mu')=T(\mu,\mu')+T(-\mu,-\mu')+T(\mu',\mu)+T(-\mu',-\mu),$$

where

$$T(\mu,\mu')=\tfrac{1}{2}\iint_{0\leqq\lambda\leqq\lambda'}\lambda\,\mathrm{e}(z\lambda\lambda'-\lambda\mu-\lambda'\mu')\,d\lambda'\,d\lambda.$$

In this expression we perform the λ'-integration and substitute $\lambda = u + \mu'/z$ to obtain

$$\begin{aligned}
T(\mu,\mu') &= \frac{i}{4\pi}\int_0^\infty \frac{\lambda}{z\lambda-\mu'}\,\mathrm{e}(z\lambda^2-\mu\lambda-\mu'\lambda)\,d\lambda\\
&= \frac{i}{4\pi z}\,\mathrm{e}(-\mu\mu'/z)\int_{-\mu'/z}^{\infty}\frac{u+\mu'/z}{u}\,\mathrm{e}(zu^2-u\mu+u\mu')\,du\\
&= \frac{1}{4\pi}\,w^{-3/2}\,\mathrm{e}(-(\mu-\mu')^2/4z)\,\mathrm{erfc}\left(i(\mu'+\mu)\sqrt{\frac{\pi}{w}}\right)\\
&\quad+\frac{i\mu'}{4\pi}\,z^{-2}\,\mathrm{e}(-\mu\mu'/z)\,f\left(\frac{\mu'-\mu}{\sqrt{\pi w}},\,2i\mu'\sqrt{\frac{\pi}{w}}\right).
\end{aligned}$$

Adding to this the corresponding formula for $T(-\mu,-\mu')$ and using c) and d) of Lemma 1, we find

$$\begin{aligned}
&T(\mu,\mu')+T(-\mu,-\mu')\\
&=\frac{1}{2\pi}\,w^{-3/2}\,\mathrm{e}(-(\mu-\mu')^2/4z)+\tfrac14|\mu'|\,z^{-2}\,\mathrm{e}(-\mu\mu'/z)\,\mathrm{erfc}\left(\mathrm{sign}(\mu')(\mu'-\mu)\sqrt{\frac{\pi}{w}}\right).
\end{aligned}$$

Adding to this the formula obtained by interchanging μ and μ', and using again c) of the lemma as well as Equation (4), we find after a short calculation

$$\tilde V_z(\mu,\mu') = z^{-2}\,V_{-1/z}(\mu,\mu') + 8\,w^{-3/2}\,\mathrm{e}(-\mu\mu'/z)\,\beta(\pi(\mu-\mu')^2/w).$$

Comparing this with the result for $\tilde U_z$ we obtain Equation (5).

The proof of the corollary is now essentially the same as the standard proof that theta-series associated to definite quadratic forms are modular forms, as given in [30], Chapter VI or [24], pages 81–87. We recall briefly how the argument goes. As well as the series $\mathscr{W}(z)$, one must consider the sums

$$\mathscr{W}_\nu(z) = \sum_{\lambda\in\mathcal{O}} W_z(\lambda+\nu,\lambda'+\nu')$$

over the translated lattices $\mathcal{O}+\nu$, where ν belongs to the inverse different $\mathfrak{d}^{-1} = (1/\sqrt{D})$. Clearly $\mathscr{W}_\nu$ depends only on the residue class of $\nu \pmod{\mathcal{O}}$, so there are only D distinct functions $\mathscr{W}_\nu$, with $\mathscr{W}_0 = \mathscr{W}$. Then

$$\mathscr{W}_\nu(z+1) = \sum_{\lambda\in\mathcal{O}} W_z(\lambda+\nu,\lambda'+\nu')\,\mathrm{e}(\mathrm{N}(\lambda+\nu)) = \mathrm{e}(\mathrm{N}\nu)\,\mathscr{W}_\nu(z),$$

since $\mathrm{N}(\lambda+\nu)-\mathrm{N}(\nu)\in\mathbb{Z}$ for $\lambda\in\mathcal{O}$, $\nu\in\mathfrak{d}^{-1}$. On the other hand, by the Poisson summation formula

$$\mathscr{W}_\nu(z) = D^{-1/2}\sum_{\mu\in\mathfrak{d}^{-1}} \tilde W_z(\mu,\mu')\,\mathrm{e}(\mathrm{Tr}\,\mu\nu),$$

and combining this with (5) we find

$$z^{-2}\,\mathscr{W}_\nu(-1/z) = D^{-1/2}\sum_{\mu\in\mathfrak{d}^{-1}}\mathrm{e}(\mathrm{Tr}\,\mu\nu)\,W_z(\mu,\mu') = D^{-1/2}\sum_{\mu\in\mathfrak{d}^{-1}/\mathcal{O}}\mathrm{e}(\mathrm{Tr}\,\mu\nu)\,\mathscr{W}_\mu(z).$$

Thus we have

$$\mathscr{W}_\nu|T = \mathrm{e}(\mathrm{N}\nu)\,\mathscr{W}_\nu,\qquad \mathscr{W}_\nu|J = D^{-1/2}\sum_\mu \mathrm{e}(\mathrm{Tr}\,\mu\nu)\,\mathscr{W}_\mu, \tag{6}$$

where T and J are the matrices $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ in $\Gamma=SL_2(\mathbb{Z})/\{\pm 1\}$ and $\mathscr{W}_\nu\Big|\begin{pmatrix}a&b\\c&d\end{pmatrix}$ denotes the function $(cz+d)^{-2}\mathscr{W}_\nu\left(\frac{az+b}{cz+d}\right)$. Since T and J generate Γ, we obtain a representation of Γ in the space generated by the D functions $\mathscr{W}_\nu$. The first step in the proof of the Corollary is to show that the Equations (6) imply $\mathscr{W}_\nu|A=\mathscr{W}_\nu$ für A in the principal congruence group $\Gamma(D)$. The argument is given in [24], pages 85–87. Now to show that $\mathscr{W}_0$ transforms under $\Gamma_0(D)$ like a modular form of Nebentypus, we take $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(D)$ and set $R=T^aJT^dJT^aJ$. Then $R\equiv\begin{pmatrix}a&0\\0&d\end{pmatrix}\pmod D$, so, choosing $x\in\mathbb{Z}$ with $dx\equiv b\pmod D$, we find $A=A'T^xR$ with $A'\in\Gamma(D)$ and hence $\mathscr{W}_0|A=\mathscr{W}_0|R$. But from (6) we find

$$\mathscr{W}_\nu|T^aJ=D^{-1/2}\,\mathrm{e}(a\,\mathrm{N}(\nu))\sum_\mu \mathscr{W}_\mu\,\mathrm{e}(\mathrm{Tr}\,\mu\nu)$$

and hence

$$\begin{aligned}\mathscr{W}_\nu|R&=((\mathscr{W}_\nu|T^aJ)|T^dJ)|T^aJ\\&=D^{-3/2}\sum_\mu\sum_\lambda\sum_\kappa \mathrm{e}(a\mathrm{N}\nu+d\mathrm{N}\mu+a\mathrm{N}\lambda+\mathrm{Tr}(\mu\nu+\lambda\mu+\kappa\lambda))\mathscr{W}_\kappa\\&=D^{-3/2}\sum_\kappa\mathscr{W}_\kappa\sum_\lambda \mathrm{e}(\mathrm{Tr}\,\lambda'(\kappa-a\nu'))\sum_\mu \mathrm{e}(d\,\mathrm{N}(\mu+a\lambda'+a\nu')).\end{aligned}$$

Replacing $\mu+a\lambda'+a\nu'$ by μ in the inner sum, we see that this sum is equal to the standard Gauss sum

$$\sum_{\mu\in\mathfrak{d}^{-1}/\mathcal{O}}\mathrm{e}(d\,\mathrm{N}\mu)=D^{1/2}\chi_D(d).$$

Hence

$$\mathscr{W}_\nu|R=D^{-1}\chi_D(d)\sum_\kappa\mathscr{W}_\kappa\sum_\lambda\mathrm{e}(\mathrm{Tr}\,\lambda'(\kappa-a\nu'))=\chi_D(d)\,\mathscr{W}_{a\nu'}$$

(the inner sum is zero if $\kappa\neq a\nu'$). In particular, taking $\nu=0$ we find $\mathscr{W}_0|R=\chi_D(d)\,\mathscr{W}_0$, and this completes the proof that $\mathscr{W}_0|A=\chi_D(d)\,\mathscr{W}_0$ for $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(D)$.

2.4. *Proof of Theorem 1*

We proved in Section 2.2 the identity

$$\mathscr{F}(z)=\sum_{N=0}^{\infty}H(N)\,q^N+y^{-1/2}\sum_{u\in\mathbb{Z}}\beta(4\pi u^2y)\,q^{-u^2},$$

where as usual $q=\mathrm{e}(z)$ and $y=\mathrm{Im}(z)$ and $\mathscr{F}$ is a function satisfying

$$\mathscr{F}\left(\frac{az+b}{cz+d}\right)=\left(\frac{-1}{d}\right)^{1/2}\left(\frac{c}{d}\right)(cz+d)^{3/2}\mathscr{F}(z)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ (and of course $c \neq 0$). The function $\theta(z) = \sum_{t \in \mathbb{Z}} q^{t^2}$, on the other hand, satisfies

$$\theta\left(\frac{az+b}{cz+d}\right) = \left(\frac{-1}{d}\right)^{-1/2} \left(\frac{c}{d}\right) (cz+d)^{1/2}\, \theta(z)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$. It follows immediately that

$$\begin{aligned} &\mathscr{F}\left(D\frac{az+b}{cz+d}\right) \theta\left(\frac{az+b}{cz+d}\right) \\ &\quad = \mathscr{F}\left(\frac{aDz+bD}{(c/D)\,Dz+d}\right) \theta\left(\frac{az+b}{cz+d}\right) \\ &\quad = \left(\frac{-1}{d}\right)^{1/2} \left(\frac{c/D}{d}\right) (cz+d)^{3/2}\, \mathscr{F}(Dz) \left(\frac{-1}{d}\right)^{-1/2} \left(\frac{c}{d}\right) (cz+d)^{1/2}\, \theta(z) \\ &\quad = \left(\frac{D}{d}\right) (cz+d)^2\, \mathscr{F}(Dz)\, \theta(z) \end{aligned}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4D)$, i.e. the function

$$\mathscr{F}(Dz)\,\theta(z) = \sum_{N=-\infty}^{\infty} c_N(y)\, q^N,$$

$$c_N(y) = \sum_{\substack{t^2 \leq N \\ t^2 \equiv N \,(\mathrm{mod}\, D)}} H\left(\frac{N-t^2}{D}\right) + D^{-1/2} y^{-1/2} \sum_{\substack{t, u \in \mathbb{Z} \\ t^2 - Du^2 = N}} \beta(4\pi u^2 y),$$

belongs to $M_2^*(\Gamma_0(4D), \chi_D)$. We claim that the function $\sum c_{4N}(\frac{1}{4}y)\, q^N$ belongs to $M_2^*(\Gamma_0(D), \chi_D)$. The corresponding statement in [23] (Proposition 5.1) is proved by appealing to Lemmas 1 and 4 of [29]; however, since this latter paper treats only analytic modular forms, we give the proof of the special assertion we need:

Lemma 2. *If $f(z) = \sum_{n \in \mathbb{Z}} a_n(y)\, q^n$ is in $M_k^*(\Gamma_0(4D), \chi_D)$, and $a_n(y) = 0$ for all $n \equiv 2 \pmod 4$, then the function*

$$h(z) = \tfrac{1}{4} \sum_{r=1}^{4} f\left(\frac{z+r}{4}\right) = \sum_{n \in \mathbb{Z}} a_{4n}(\tfrac{1}{4}y)\, q^n$$

is in $M_k^*(\Gamma_0(D), \chi_D)$.

Proof. We prove the lemma in two steps, first showing that the function

$$g(z) = \tfrac{1}{2}\left(f\left(\frac{z}{2}\right) + f\left(\frac{z+1}{2}\right)\right) = \sum a_{2n}(\tfrac{1}{2}y)\, q^n$$

is in $M_k^*(\Gamma_0(2D), \chi_D)$. Set $X_0=\begin{pmatrix}1&0\\0&2\end{pmatrix}$, $X_1=\begin{pmatrix}1&1\\0&2\end{pmatrix}$ and let $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ be any matrix in $\Gamma_0(2D)$. If b is even, then the matrices

$$X_0AX_0^{-1}=\begin{pmatrix}a&\frac{1}{2}b\\2c&d\end{pmatrix},\quad X_1AX_1^{-1}=\begin{pmatrix}a+c&\frac{1}{2}(b+d-a-c)\\2c&d-c\end{pmatrix}$$

are in $\Gamma_0(4D)$, so

$$\begin{aligned}g(Az)&=\tfrac{1}{2}f(X_0AX_0^{-1}(X_0z))+\tfrac{1}{2}f(X_1AX_1^{-1}(X_1z))\\&=\tfrac{1}{2}\chi_D(d)(cz+d)^kf(X_0z)+\tfrac{1}{2}\chi_D(d-c)(cz+d)^kf(X_1z)\\&=\chi_D(d)(cz+d)^kg(z),\end{aligned}$$

the last equality holding because $\chi_D(d-c)=\chi_D(d)$. If b is odd, then the matrices

$$X_0AX_1^{-1}=\begin{pmatrix}a&\frac{1}{2}(b-a)\\2c&d-c\end{pmatrix},\quad X_1AX_0^{-1}=\begin{pmatrix}a+c&\frac{1}{2}(b+d)\\2c&d\end{pmatrix}$$

are in $\Gamma_0(4D)$, so

$$\begin{aligned}g(Az)&=\tfrac{1}{2}f(X_0AX_1^{-1}(X_1z))+\tfrac{1}{2}f(X_1AX_0^{-1}(X_0z))\\&=\tfrac{1}{2}\chi_D(d-c)(cz+d)^kf(X_1z)+\tfrac{1}{2}\chi_D(d)(cz+d)^kf(X_0z)\\&=\chi_D(d)(cz+d)^kg(z).\end{aligned}$$

This proves the assertion concerning g. Under the assumptions made on f, $a_{2n}(\frac{1}{2}y)=0$ for n odd, so

$$g(z)=g(z+\tfrac{1}{2})=\sum a_{4n}(\tfrac{1}{2}y)\,q^{2n}=h(2z).$$

Let $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(D)$. The formulas just given for the four matrices $X_\alpha AX_\beta^{-1}$ $(\alpha,\beta=0,1)$ show that at least one of these matrices lies in $\Gamma_0(2D)$; then

$$h(Az)=g(X_\alpha AX_\beta^{-1}(X_\beta z))=\chi_D(d)(cz+d)^kg(X_\beta z)=\chi_D(d)(cz+d)^kh(z).$$

We now apply this lemma with $f(z)=\mathscr{F}(z)\,\theta(z)$, $a_n(y)=c_n(y)$. The fact that $c_N(y)=0$ for $N\equiv 2 \pmod 4$ follows from the fact $H(n)=0$ unless $n\equiv 0$ or $3 \pmod 4$ and from $D\equiv 0$ or $1 \pmod 4$. Thus $\sum c_{4N}(\frac{1}{4}y)q^N$ is in $M_{\frac{3}{2}}^*(\Gamma_0(D),\chi_D)$. But

$$\begin{aligned}c_{4N}(\tfrac{1}{4}y)&=\sum_{\substack{t^2\leqq 4N\\t^2\equiv 4N\,(\mathrm{mod}\,D)}}H\left(\frac{4N-t^2}{D}\right)+2D^{-1/2}y^{-1/2}\sum_{\substack{t,\,u\in\mathbf{Z}\\N=(t^2-Du^2)/4}}\beta(\pi Du^2y)\\&=H_D(N)+2D^{-1/2}y^{-1/2}\sum_{\substack{\lambda\in\mathcal{O}\\\lambda\lambda'=N}}\beta(\pi(\lambda-\lambda')^2y),\end{aligned}$$

so this means that the function

$$\sum_{N=0}^{\infty}H_D(N)\,q^N+2D^{-1/2}y^{-1/2}\sum_{\lambda\in\mathcal{O}}\beta(\pi(\lambda-\lambda')^2y)\,q^{\lambda\lambda'}$$

is in $M_2^*(\Gamma_0(D), \chi_D)$. On the other hand, by the Corollary to Proposition 1, we know that the function

$$-D^{-1/2}\mathscr{W}(z)=D^{-1/2}\sum_{\substack{\lambda\in\mathcal{O}\\ \lambda\gg 0}}\min(\lambda,\lambda')\,q^{\lambda\lambda'}-2D^{-1/2}y^{-1/2}\sum_{\lambda\in\mathcal{O}}\beta(\pi(\lambda-\lambda')^2y)\,q^{\lambda\lambda'}$$

is also in $M_2^*(\Gamma_0(D), \chi_D)$. Adding these two functions, we find that the function

$$\varphi_D(z)=\sum_{N=0}^{\infty}H_D(N)\,q^N+D^{-1/2}\sum_{\substack{\lambda\in\mathcal{O}\\ \lambda\gg 0}}\min(\lambda,\lambda')\,q^{\lambda\lambda'}$$

is in $M_2^*(\Gamma_0(D), \chi_D)$. Since the N-th Fourier coefficient of $\varphi_D(z)$ is independent of y and is $O(N^r)$ for some r (in fact, for $r=1$), the function $\varphi_D(z)$ is analytic in the upper half-plane and is $O(y^{-r})$ as $y\to 0$, which implies that $\varphi_D(z)$ is holomorphic at the cusps of $\Gamma_0(D)$. Hence $\varphi_D\in M_2(\Gamma_0(D), \chi_D)$.

Chapter 3: Modular Forms with Intersection Numbers as Fourier Coefficients

3.1. Modular Forms of Nebentypus and the Homology of the Hilbert Modular Surface

In Chapter 3 we return to our Hilbert modular surface $X=\mathfrak{H}^2/SL_2(\mathcal{O})$, again supposing that the discriminant of the quadratic field K is a prime $p\equiv 1 \pmod 4$. The middle homology group $H_2(\tilde{X})$ (all homology and cohomology is with coefficients in $\mathbb{Q}$ unless otherwise stated) of the compactification $\tilde{X}=X\cup\bigcup S_k$ of X is the direct sum of $\mathrm{Im}(H_2(X)\to H_2(\tilde{X}))$ and the subspace generated by the homology classes of the curves S_k, the two subspaces being orthogonal complements of one another with respect to the intersection form. In the first component lie the homology classes T_N^c $(N=1,2,\dots)$ defined in 1.4 and one further important class which we now describe.

On $\mathfrak{H}\times\mathfrak{H}$ we have two differential forms $\omega_j=-\dfrac{1}{2\pi}y_j^{-2}\,dx_j\wedge dy_j$ $(j=1,2)$, where $z_j=x_j+iy_j$ $(j=1,2)$ are the coordinates. Each ω_j is an $SL_2(\mathbb{R})$-invariant form on $\mathfrak{H}$, so ω_1 and ω_2 are $SL_2(\mathcal{O})$-invariant; they can therefore be considered as differential forms on the smooth non-compact surface X' obtained from X by removing the finitely many singular points (quotient singularities). The sum $\omega_1+\omega_2$ is the first Chern form c_1 on X', while the product $\omega=\omega_1\wedge\omega_2$ is the second Chern form (Gauss-Bonnet form) c_2; clearly $c_1\wedge c_1=2c_2$. Then ([4], 1.3, Eq. (9))

$$\int_{X'}c_1\wedge c_1=2\int_{X'}\omega=4\zeta_K(-1), \tag{1}$$

where $\zeta_K(s)$ $(s\in\mathbb{C})$ denotes the Dedekind zeta-function of K. On the other hand, in ([4], p. 229) it is shown how the forms ω_j can be modified by coboundaries to obtain differential forms *with compact support* in X'; extending these forms to $\tilde{X}\supset X'$ by 0 on the complement of X', we obtain differential forms on $\tilde{X}$ representing cohomology classes in $\mathrm{Im}(H_c^2(X')\to H^2(\tilde{X}))$, or equivalently (by called either the *Sturm bound* or the *Hecke bound* in the literature. A more modern reference is Section 9.4 of [Stein (2007)].

Theorem 3.13 (Sturm [Sturm (1987)]). *Let $\Gamma \in \mathrm{SL}_2(\mathbf{Z})$ be a congruence subgroup of index M and let $f \in M_k(\Gamma)$ be a modular form. If*

$$\nu_\infty(f) > M \cdot \frac{k}{12}$$

then f is identically zero.

Proof. If $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ then we can prove this immediately by considering the residue theorem (3.1), as we did above; if f has a zero of order greater than $k/12$ at ∞, then it must be zero, because the right-hand side of the residue formula (3.1) is $k/12$, and all of the elements of the left-hand side are non-negative.

We now use the fact that Γ is a subgroup of finite index of $\mathrm{SL}_2(\mathbf{Z})$; this means that we can write

$$\mathrm{SL}_2(\mathbf{Z}) = \bigcup_{i=1}^{M} \Gamma\gamma_i,$$

for some finite set of $\gamma_i \in \mathrm{SL}_2(\mathbf{Z})$. Without loss of generality, we can assume that $\gamma_1 = I$. We define a modular form F by

$$F := f \cdot \prod_{i=2}^{M} f|[\gamma_i]_k;$$

we will now show that F is a modular form for $\mathrm{SL}_2(\mathbf{Z})$, so we reduce it to a case that we have solved. We note that each of the $f|[\gamma_i]$ is a modular form of weight k for a suitable congruence subgroup, by a variant of the argument before Proposition 2.20.

To show that F is a modular form for $\mathrm{SL}_2(\mathbf{Z})$, we need to show that it transforms correctly under the action of $\mathrm{SL}_2(\mathbf{Z})$. This will follow because allowing an element $g \in \mathrm{SL}_2(\mathbf{Z})$ to act on the right permutes the γ_i, so the product is left unchanged. This means that F is not just a modular form for Γ, but a modular form for the full modular group.

We see that F has weight kM, because it is the product of M weight k modular forms, so we now apply the theorem in the level 1 case to obtain the bound. □

Remark 3.14. We see that the bound is in fact sharp in general for $\mathrm{SL}_2(\mathbf{Z})$; if we consider the modular form Δ^i, we see that it has a unique zero of exact multiplicity i at ∞, so we cannot replace the strict inequality with a non-strict inequality.