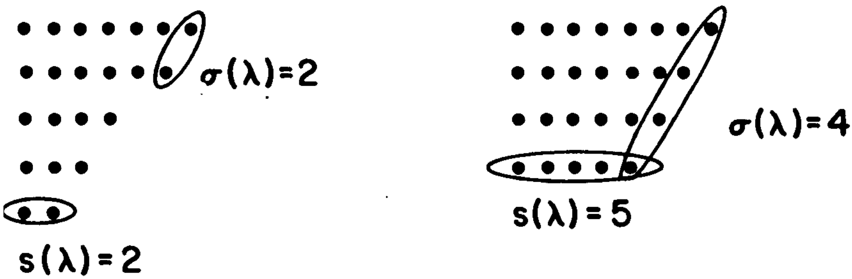
THEOREM 1.6. *Let $p_e(\mathscr{D}, n)$ (resp. $p_o(\mathscr{D}, n)$) denote the number of partitions of n into an even (resp. odd) number of distinct parts. Then*

$$p_e(\mathscr{D}, n) - p_o(\mathscr{D}, n) = \begin{cases} (-1)^m & \text{if } n = \frac{1}{2}m(3m \pm 1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We shall attempt to establish a one-to-one correspondence between the partitions enumerated by $p_e(\mathscr{D}, n)$ and those enumerated by $p_o(\mathscr{D}, n)$. For most integers n our attempt will be successful; however, whenever n is one of the pentagonal numbers $\frac{1}{2}m(3m \pm 1)$, a single exceptional case will arise.

To begin with, we note that each partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n has a smallest part $s(\lambda) = \lambda_r$; also, we observe that the largest part λ_1 of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is the first of a sequence of, say, $\sigma(\lambda)$ consecutive integers that are parts of λ (formally $\sigma(\lambda)$ is the largest j such that $\lambda_j = \lambda_1 - j + 1$). Graphically the parameters $s(\lambda)$ and $\sigma(\lambda)$ are easily described:

λ = (76432) λ = (8765)

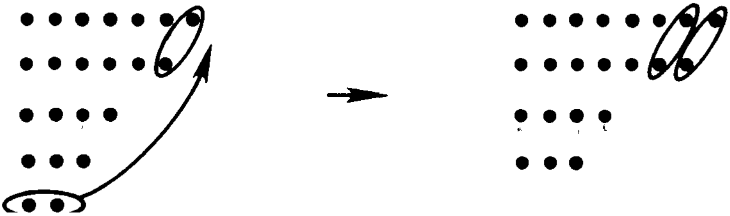


We transform partitions as follows.

Case 1. $s(\lambda) \leqslant \sigma(\lambda)$. In this event, we add one to each of the $s(\lambda)$ largest parts of λ and we delete the smallest part. Thus

$$\lambda = (76432) \to \lambda' = (8743);$$

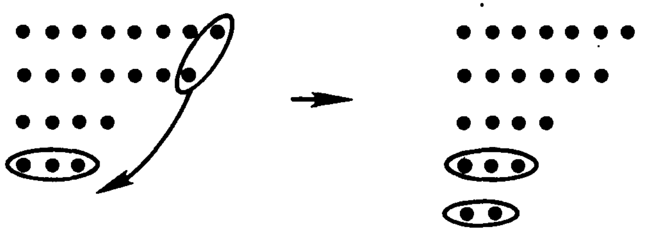
that is



Case 2. $s(\lambda) > \sigma(\lambda)$. In this event, we subtract one from each of the $\sigma(\lambda)$ largest parts of λ and insert a new smallest part of size $\sigma(\lambda)$. Thus

$$\lambda = (8743) \rightarrow (76432);$$

that is



The foregoing procedure in either case changes the parity of the number of parts of the partition, and noting that exactly one case is applicable to any partition λ, we see directly that the mapping establishes a one-to-one correspondence. However, there are certain partitions for which the mapping will not work. The example $\lambda = (8765)$ is a case in point. Case 2 should be applicable to it; however, the image partition is *no longer* one with distinct parts. Indeed, Case 2 breaks down in precisely those cases when the partition has r parts, $\sigma(\lambda) = r$ and $s(\lambda) = r + 1$, in which case the number being partitioned is

$$(r + 1) + (r + 2) + \cdots + 2r = \tfrac{1}{2}r(3r + 1).$$

On the other hand, Case 1 breaks down in precisely those cases when the partition has r parts, $\sigma(\lambda) = r$ and $s(\lambda) = r$, in which case the number being partitioned is

$$r + (r + 1) + \cdots + (2r - 1) = \tfrac{1}{2}r(3r - 1).$$

Consequently, if n is not a pentagonal number, $p_e(\mathscr{D}, n) = p_o(\mathscr{D}, n)$; if $n = \tfrac{1}{2}r(3r \pm 1)$, $p_e(\mathscr{D}, n) = p_o(\mathscr{D}, n) + (-1)^r$. ■

COROLLARY 1.7 (Euler's pentagonal number theorem).

$$\prod_{n=1}^{\infty} (1 - q^n) = 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}(1 + q^m)$$

$$= \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}. \qquad (1.3.1)$$

Proof. Clearly

$$\sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)} = 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)} + \sum_{m=-1}^{-\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}$$

$$= 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)} + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m+1)}$$

$$= 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}(1 + q^m)$$

$$= 1 + \sum_{n=1}^{\infty} (p_e(\mathscr{D}, n) - p_o(\mathscr{D}, n))q^n,$$

by Theorem 1.6.

To complete the proof we must show that

$$1 + \sum_{n=1}^{\infty} (p_e(\mathscr{D}, n) - p_o(\mathscr{D}, n))q^n = \prod_{n=1}^{\infty} (1 - q^n).$$

Now

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \cdots (-1)^{a_1+a_2+a_3+\cdots} q^{a_1\cdot 1 + a_2\cdot 2 + a_3\cdot 3\ldots},$$

as in the proof of (1.2.4) in Theorem 1.1. Note now that each partition with distinct parts is counted with a weight $(-1)^{a_1+a_2+a_3+\cdots}$, which is $+1$ if the partition has an even number of parts and -1 if the partition has an odd number of parts. Consequently

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \cdots (-1)^{a_1+a_2+a_3+\cdots} q^{a_1\cdot 1 + a_2\cdot 2 + a_3\cdot 3 + \cdots}$$

$$= 1 + \sum_{n=1} (p_e(\mathscr{D}, n) - p_o(\mathscr{D}, n))q^n,$$

and so we have the desired result. ∎

Corollary 1.8 (Euler). *If* $n > 0$, *then*

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7)$$
$$+ \cdots + (-1)^m p(n - \tfrac{1}{2}m(3m-1))$$
$$+ (-1)^m p(n - \tfrac{1}{2}m(3m+1)) + \cdots = 0, \qquad (1.3.2)$$

where we recall that $p(M) = 0$ *for all negative* M.

Proof. Let a_n denote the left-hand side of (1.3.2). Then clearly

$$\sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} p(n)q^n \cdot \left[1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}(1 + q^m)\right]$$

$$= \prod_{n=1}^{\infty} (1 - q^n)^{-1} \cdot \prod_{n=1}^{\infty} (1 - q^n)$$

$$= 1$$

where the penultimate equation follows immediately by (1.2.3) and Corollary 1.7. Hence, $a_n = 0$ for $n > 0$. ■

Corollary 1.8 provides an extremely efficient algorithm for computing $p(n)$ that we shall discuss further in Chapter 14.

Examples

1. (Subbarao) The number of partitions of n in which each part appears two, three, or five times equals the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12.

2. The number of partitions of n in which only odd parts may be repeated equals the number of partitions of n in which no part appears more than three times.

3. The number of partitions of n in which only parts $\not\equiv 0 \pmod{2^m}$ may be repeated equals the number of partitions of n in which no part appears more than $2^{m+1} - 1$ times.

4. (Ramanujan) The number of partitions of n with unique smallest part and largest part at most twice the smallest part equals the number of partitions of n in which the largest part is odd and the smallest part is larger than half the largest part.

5. Let $P_1(r; n)$ denote the number of partitions of n into parts that are either even and not congruent to $4r - 2 \pmod{4r}$ or odd and congruent to $2r - 1$ or $4r - 1 \pmod{4r}$. Let $P_2(r; n)$ denote the number of partitions of n in which only even parts may be repeated and all odd parts are congruent to $2r - 1$ modulo $2r$. Then $P_1(r; n) = P_2(r; n)$.

Comment on Examples 6–7. P. A. MacMahon introduced what he termed "modular" partitions. Given the positive integers k and n, there exist (by the Euclidean algorithm) $h \geqslant 0$ and $0 < j \leqslant k$ such that

$$n = kh + j.$$

The "modular" partitions are a modification of the Ferrers graph so that n is represented by a row of h k's and one j. Thus the representation of $8 + 8 + 7 + 7 + 6 + 5 + 2$ to the modulus 2 is

$$\begin{array}{l} 2\,2\,2\,2 \\ 2\,2\,2\,2 \\ 2\,2\,2\,1 \\ 2\,2\,2\,1 \\ 2\,2\,2 \\ 2\,2\,1 \\ 2 \end{array}$$

Note that the ordinary Ferrers graph is just the modular representation with modulus 1.

6. Let $W_1(r, m, n)$ denote the number of partitions of n into m parts, each large han 1, with exactly r odd parts, each distinct. Let $W_2(r, m, n)$ denote the umber of partitions of n with $2m$ as largest part and exactly r

Proof. Combinatorial reasoning of the type used in the previous three proofs shows that Fine's theorem is equivalent to the following assertion:

$$\sum_{j=0}^{\infty} \frac{t^{j+1}q^{2j+1}}{(tq;q^2)_{j+1}} = tq\sum_{j=0}^{\infty}(-q)_j q^j t^j.$$

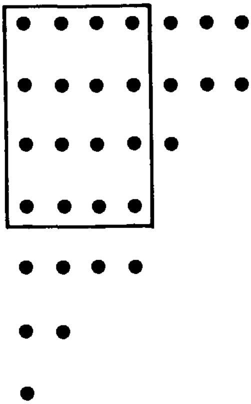
Now

$$\begin{aligned}
&tq\sum_{j=0}^{\infty}(-q)_j q^j t^j\\
&= tq\sum_{j=0}^{\infty}\frac{(q^2;q^2)_j q^j t^j}{(q)_j}\\
&= tq(q^2;q^2)_\infty \sum_{j=0}^{\infty}\frac{t^j q^j}{(q)_j}\frac{1}{(q^{2j+2};q^2)_\infty}\\
&= tq(q^2;q^2)_\infty \sum_{j=0}^{\infty}\frac{t^j q^j}{(q)_j}\sum_{m=0}^{\infty}\frac{q^{2jm+2m}}{(q^2;q^2)_m} \qquad \text{(by (2.2.5))}\\
&= tq(q^2;q^2)_\infty \sum_{m=0}^{\infty}\frac{q^{2m}}{(q^2;q^2)_m}\sum_{j=0}^{\infty}\frac{t^j q^{j(2m+1)}}{(q)_j}\\
&= tq(q^2;q^2)_\infty \sum_{m=0}^{\infty}\frac{q^{2m}}{(q^2;q^2)_m(tq^{2m+1})_\infty} \qquad \text{(by (2.2.5))}\\
&= \frac{tq(q^2;q^2)_\infty}{(tq)_\infty}\sum_{m=0}^{\infty}\frac{(tq;q^2)_m(tq^2;q^2)_m q^{2m}}{(q^2;q^2)_m}\\
&= \frac{tq(q^2;q^2)_\infty}{(tq)_\infty}\frac{(tq^2;q^2)_\infty(tq^3;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(q^2;q^2)_m t^m q^{2m}}{(q^2;q^2)_m(tq^3;q^2)_m}\\
&\qquad \text{(by Corollary 2.3)}\\
&= \sum_{m=0}^{\infty}\frac{t^{m+1}q^{2m+1}}{(tq;q^2)_{m+1}}.
\end{aligned}$$

■

We conclude this chapter with a look at a property of partitions called the Durfee square, and we utilize it to provide a new proof of (2.2.9).

Combinatorial Proof of Eq. (2.2.9). To each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ we may assign a parameter $d(\lambda)$ as the number of λ_j such that $\lambda_j \geqslant j$. Let us see what $d(\lambda)$ measures in the graphical representation of λ. Suppose $\lambda = (124^2 57^2)$, then $d(\lambda) = 4$, the graphical representation is



and as we have indicated, $d(\lambda)$ measures the largest square of nodes contained in the partition λ. This square is called the *Durfee square* (after W. P. Durfee), and $d(\lambda)$ is called the *side of the Durfee square*. It is clear from the graphical representation that if $\lambda \vdash n$ and $d(\lambda) = s$, then the partition λ may be uniquely written as $(s^s) + \lambda' + \lambda''$ where (s^s) counts the nodes in the Durfee square, λ' represents the nodes below the Durfee square (and is therefore some partition all of whose parts are $\leqslant s$), and λ'' represents the conjugate of the nodes to the right of the Durfee square and so λ'' is also some partition whose parts are $\leqslant s$. In the foregoing example the partition $\lambda = (124^2 57^2)$ is uniquely written as $(4^4) + (124) + (2^2 3)$. Since partitions with parts $\leqslant s$ are generated by

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^s)} = \frac{1}{(q)_s}$$

(Theorem 1.1), we see that the set of all partitions with Durfee square of side s is generated by

$$q^{s^2}\frac{1}{(q)_s}\cdot\frac{1}{(q)_s} = \frac{q^{s^2}}{(q)_s^{\,2}}.$$

Therefore

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n = \sum_{s=0}^{\infty}\frac{q^{s^2}}{(q)_s^{\,2}}. \qquad \blacksquare$$

Examples

1. The following generalization of Corollary 2.3 is valid for each integer $k \geqslant 1$,

$$\sum_{n=0}^{\infty}\frac{(a; q^k)_n(b)_{kn}t^n}{(q^k; q^k)_n(c)_{kn}} = \frac{(b)_\infty(at; q^k)_\infty}{(c)_\infty(t; q^k)_\infty}\sum_{n=0}^{\infty}\frac{(c/b)_n(t; q^k)_n b^n}{(q)_n(at; q^k)_n}.$$