Other parameters besides the number of parts $\#(\lambda)$ of a partition λ will interest us from time to time; so we shall have occasion to consider other types of partition generating functions of several variables.

The preceding comments suggest the interest of considering infinite series and products in two (or more) variables. In the following section, we shall develop an elementary technique for proving many series and product identities. We shall obtain several classical theorems of great importance, such as Jacobi's triple product identity. As will become clear in Section 2.3, the results of Section 2.2 are quite useful in treating partition identities. It is possible, however, to skip Section 2.2 and read Section 2.3, referring back only for the statements of theorems. For the reader who needs series transformations to attack a partition problem, the first six examples at the end of this chapter form a good test of the techniques used in Section 2.2.

2.2 Elementary Series-Product Identities

We begin with a theorem due to Cauchy; as we shall see, this result provides the tool for doing everything else in this section.

THEOREM 2.1. *If* $|q| < 1$, $|t| < 1$, *then*

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-atq^n)}{(1-tq^n)}. \qquad (2.2.1)$$

Remark. We shall try always to state our theorems with as little notational disguise as possible. However, for the proofs, it seems only sensible to use the following standard abbreviations

$$(a)_n = (a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

$$(a)_\infty = (a; q)_\infty = \lim_{n\to\infty}(a; q)_n,$$

$$(a)_0 = 1.$$

We may define $(a)_n$ for all real numbers n by

$$(a)_n = (a)_\infty/(aq^n)_\infty.$$

The series in (2.2.1) is an example of a basic hypergeometric series. The study of basic series (or q-series, or Eulerian series) is an extensive branch of analysis and we shall only touch upon it in this book. Most of the theorems of this section may be viewed as elementary results in the theory of basic hypergeometric series. Theorem 2.1 has become known as the "q-analog of the binon series," for if we write $a = q^\alpha$ where α is a nonnegative integer, then (2.2.1) formally tends to

$$1 + \sum_{n=1}^{\infty} \binom{\alpha + n - 1}{n} t^n = (1 - t)^{-\alpha}, \quad \text{as} \quad q \to 1^-.$$

Proof. Let us consider

$$F(t) = \prod_{n=0}^{\infty} \frac{(1 - atq^n)}{(1 - tq^n)} = \sum_{n=0}^{\infty} A_n t^n \tag{2.2.2}$$

where $A_n = A_n(a, q)$. We note that the A_n exist since the infinite product is uniformly convergent for fixed a and q inside $|t| \leqslant 1 - \varepsilon$, and therefore it defines a function of t analytic inside $|t| < 1$.

Now

$$(1 - t)F(t) = (1 - at) \prod_{n=1}^{\infty} \frac{(1 - atq^n)}{(1 - tq^n)}$$

$$= (1 - at) \prod_{n=0}^{\infty} \frac{(1 - atq^{n+1})}{(1 - tq^{n+1})} = (1 - at)F(tq). \tag{2.2.3}$$

Clearly $A_0 = F(0) = 1$, and by comparing coefficients of t^n in the extremes of (2.2.3) we see that

$$A_n - A_{n-1} = q^n A_n - aq^{n-1} A_{n-1},$$

or

$$A_n = \frac{(1 - aq^{n-1})}{(1 - q^n)} A_{n-1}. \tag{2.2.4}$$

Iterating (2.2.4) we see that

$$A_n = \frac{(1 - aq^{n-1})(1 - aq^{n-2}) \cdots (1 - a)A_0}{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q)}$$

$$= \frac{(a)_n}{(q)_n}.$$

Substituting this value for A_n into (2.2.2), we obtain the theorem. ■

Euler found the two following special cases of Theorem 2.1. Each of these identities is directly related to partitions in Example 17 at the end of this chapter.

COROLLARY 2.2 (Euler). *For* $|t| < 1$, $|q| < 1$,

$$1 + \sum_{n=1}^{\infty} \frac{t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} (1 - tq^n)^{-1}, \tag{2.2.5}$$

$$1 + \sum_{n=1}^{\infty} \frac{t^n q^{\frac{1}{2}n(n-1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} (1 + tq^n). \tag{2.2.6}$$

Proof. Equation (2.2.5) follows immediately by setting $a = 0$ in (2.2.1). To obtain (2.2.6) we replace a by a/b and t by bz in (2.2.1); hence for $|bz| < 1$

$$1 + \sum_{n=1}^{\infty} \frac{(b-a)(b-aq)\cdots(b-aq^{n-1})z^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-azq^n)}{(1-bzq^n)}. \tag{2.2.7}$$

Now set $b = 0$, $a = -1$ in (2.2.7) and we derive (2.2.6) directly. ■

The following result is Heine's fundamental transformation, and it is instrumental in proving each of the succeeding four corollaries.

COROLLARY 2.3 (Heine). *For* $|q| < 1$, $|t| < 1$, $|b| < 1$

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})(1-b)(1-bq)\cdots(1-bq^{n-1})t^n}{(1-q)(1-q^2)\cdots(1-q^n)(1-c)(1-cq)\cdots(1-cq^{n-1})}$$

$$= \prod_{m=0}^{\infty} \frac{(1-bq^m)(1-atq^m)}{(1-cq^m)(1-tq^m)}$$

$$\times \left\{1 + \sum_{n=1}^{\infty} \frac{(1-c/b)(1-cq/b)\cdots(1-cq^{n-1}/b) \times}{(1-q)(1-q^2)\cdots(1-q^{n-1}) \times}\right.$$

$$\left.\frac{\times (1-t)(1-tq)\cdots(1-tq^{n-1})b^n}{\times (1-at)(1-atq)\cdots(1-atq^{n-1})}\right\}.$$

Proof.

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n t^n}{(q)_n(c)_n} = \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(q)_n} \cdot \frac{(cq^n)_\infty}{(bq^n)_\infty}$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_n t^n}{(q)_n} \cdot \frac{(c/b)_m b^m q^{nm}}{(q)_m}$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m b^m}{(q)_m} \frac{(atq^m)_\infty}{(tq^m)_\infty}$$

$$= \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m (t)_m b^m}{(q)_m (at)_m}. \qquad ■$$

COROLLARY 2.4 (Heine). *If* $|c| < |ab|$, $|q| < 1$,

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})(1-b)(1-bq)\cdots(1-bq^{n-1})(c/ab)^n}{(1-q)(1-q^2)\cdots(1-q^n)(1-c)(1-cq)\cdots(1-cq^{n-1})}$$

$$= \prod_{m=0}^{\infty} \frac{(1-cq^m/a)(1-cq^m/b)}{(1-cq^m)(1-cq^m/ab)}.$$

Proof. By Corollary 2.3,

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c/ab)^n}{(q)_n(c)_n} = \frac{(b)_\infty(c/b)_\infty}{(c)_\infty(c/ab)_\infty} \sum_{n=0}^{\infty} \frac{(c/ab)_n b^n}{(q)_n}$$

$$= \frac{(b)_\infty(c/b)_\infty}{(c)_\infty(c/ab)_\infty} \cdot \frac{(c/a)_\infty}{(b)_\infty} = \frac{(c/a)_\infty(c/b)_\infty}{(c)_\infty(c/ab)_\infty}. \qquad ■$$

COROLLARY 2.5 (Bailey). *If* $|q| < \min(1, |b|)$, *then*

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})(1-b)(1-bq)\cdots(1-bq^{n-1})(-q/b)^n}{(1-q)(1-q^2)\cdots(1-q^n)(1-aq/b)(1-aq^2/b)\cdots(1-aq^n/b)}$$

$$= \prod_{m=0}^{\infty} \frac{(1-aq^{2m+1})(1+q^{m+1})(1-aq^{2m+2}/b^2)}{(1-aq^{m+1}/b)(1+q^{m+1}/b)}.$$

Proof. By Corollary 2.3 (interchanging a and b)

$$\sum_{n=0}^{\infty} \frac{(b)_n(a)_n(-q/b)^n}{(q)_n(aq/b)_n} = \frac{(a)_\infty(-q)_\infty}{(aq/b)_\infty(-q/b)_\infty} \sum_{m=0}^{\infty} \frac{(q/b)_m(-q/b)_m a^m}{(q)_m(-q)_m}$$

$$= \frac{(a)_\infty(-q)_\infty}{(aq/b)_\infty(-q/b)_\infty} \sum_{m=0}^{\infty} \frac{(q^2/b^2;q^2)_m a^m}{(q^2;q^2)_m}$$

$$= \frac{(a)_\infty(-q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b)_\infty(-q/b)_\infty(a;q^2)_\infty}$$

$$= \frac{(aq;q^2)_\infty(-q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b)_\infty(-q/b)_\infty}. \qquad ■$$

We remark that Corollary 2.4 is commonly referred to as the "q-analog of Gauss's theorem," while Corollary 2.5 is the "q-analog of Kummer's theorem."

COROLLARY 2.6. *If* $|q| < 1$,

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2-n}z^n}{(1-q)(1-q^2)\cdots(1-q^n)(1-z)(1-zq)\cdots(1-zq^{n-1})}$$

$$= \prod_{m=0}^{\infty} (1-zq^m)^{-1}, \tag{2.2.8}$$

Therefore if

$$f(x, y) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} p_{b,c}(a - c)x^b y^c q^a$$

$$= 1 + \sum_{n=1}^{\infty} \frac{xy^n q^n}{(xq)_n},$$

then we need only show $f(x, y) = f(y, x)$ to obtain the desired result:

$$f(x, y) - 1 + x = \sum_{n=0}^{\infty} \frac{xy^n q^n}{(xq)_n}$$

$$= x \sum_{n=0}^{\infty} \frac{(0)_n (q)_n (yq)^n}{(q)_n (xq)_n}$$

$$= x \frac{(q)_\infty}{(xq)_\infty (yq)_\infty} \sum_{n=0}^{\infty} \frac{(x)_n (yq)_n q^n}{(q)_n} \qquad \text{(by Corollary 2.3)}$$

$$= x \frac{(q)_\infty}{(xq)_\infty (yq)_\infty} \sum_{n=0}^{\infty} \frac{(yq)_n (x)_n q^n}{(q)_n (0)_n}$$

$$= x \frac{(q)_\infty}{(xq)_\infty (yq)_\infty} \frac{(x)_\infty (yq^2)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} \frac{(q)_n x^n}{(q)_n (yq^2)_n}$$

(by Corollary 2.3)

$$= (1 - x) \sum_{n=0}^{\infty} \frac{x^{n+1}}{(yq)_{n+1}}.$$

Therefore

$$f(x, y) = (1 - x) \sum_{n=0}^{\infty} \frac{x^n}{(yq)_n} = \sum_{n=0}^{\infty} \frac{x^n}{(yq)_n} - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(yq)_n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{x^n}{(yq)_n} - \sum_{n=1}^{\infty} \frac{x^n(1 - yq^n)}{(yq)_n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{yx^n q^n}{(yq)_n} = f(y, x).$$

■

As a third example we consider a second refinement of Euler's theorem (Corollary 1.2) due to N. J. Fine.

THEOREM 2.13. *The number of partitions of n into distinct parts with largest part k equals the number of partitions of n into odd parts such that* $2k + 1$ *equals the largest part plus twice the number of parts.*

Proof. Combinatorial reasoning of the type used in the previous three proofs shows that Fine's theorem is equivalent to the following assertion:

$$\sum_{j=0}^{\infty} \frac{t^{j+1}q^{2j+1}}{(tq;q^2)_{j+1}} = tq\sum_{j=0}^{\infty}(-q)_j q^j t^j.$$

Now

$$\begin{aligned}
&tq\sum_{j=0}^{\infty}(-q)_j q^j t^j\\
&= tq\sum_{j=0}^{\infty}\frac{(q^2;q^2)_j q^j t^j}{(q)_j}\\
&= tq(q^2;q^2)_\infty \sum_{j=0}^{\infty}\frac{t^j q^j}{(q)_j}\frac{1}{(q^{2j+2};q^2)_\infty}\\
&= tq(q^2;q^2)_\infty \sum_{j=0}^{\infty}\frac{t^j q^j}{(q)_j}\sum_{m=0}^{\infty}\frac{q^{2jm+2m}}{(q^2;q^2)_m} \qquad \text{(by (2.2.5))}\\
&= tq(q^2;q^2)_\infty \sum_{m=0}^{\infty}\frac{q^{2m}}{(q^2;q^2)_m}\sum_{j=0}^{\infty}\frac{t^j q^{j(2m+1)}}{(q)_j}\\
&= tq(q^2;q^2)_\infty \sum_{m=0}^{\infty}\frac{q^{2m}}{(q^2;q^2)_m(tq^{2m+1})_\infty} \qquad \text{(by (2.2.5))}\\
&= \frac{tq(q^2;q^2)_\infty}{(tq)_\infty}\sum_{m=0}^{\infty}\frac{(tq;q^2)_m(tq^2;q^2)_m q^{2m}}{(q^2;q^2)_m}\\
&= \frac{tq(q^2;q^2)_\infty}{(tq)_\infty}\frac{(tq^2;q^2)_\infty(tq^3;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(q^2;q^2)_m t^m q^{2m}}{(q^2;q^2)_m(tq^3;q^2)_m}\\
&\qquad\qquad \text{(by Corollary 2.3)}\\
&= \sum_{m=0}^{\infty}\frac{t^{m+1}q^{2m+1}}{(tq;q^2)_{m+1}}. \qquad \blacksquare
\end{aligned}$$

We conclude this chapter with a look at a property of partitions called the Durfee square, and we utilize it to provide a new proof of (2.2.9).

Combinatorial Proof of Eq. (2.2.9). To each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ we may assign a parameter $d(\lambda)$ as the number of λ_j such that $\lambda_j \geqslant j$. Let us see what $d(\lambda)$ measures in the graphical representation of λ. Suppose $\lambda = (124^2 57^2)$, then $d(\lambda) = 4$, the graphical representation is