$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}=\prod_{m=1}^{\infty}(1-q^m)^{-1}. \tag{2.2.9}$$

Remark. Equation (2.2.8) is due to Cauchy, and Eq. (2.2.9) is due to Euler.

Proof. First we note that (2.2.9) is obtained from (2.2.8) by setting $z=q$. In Corollary 2.4, set $a=\alpha^{-1}$, $b=\beta^{-1}$, $c=z$. Hence

$$1+\sum_{n=1}^{\infty}\frac{(\alpha-1)(\alpha-q)\cdots(\alpha-q^{n-1})(\beta-1)(\beta-q)\cdots(\beta-q^{n-1})z^n}{(q)_n(z)_n}$$

$$=\frac{(z\alpha)_\infty(z\beta)_\infty}{(z)_\infty(z\alpha\beta)_\infty},$$

and if we set $\alpha=\beta=0$ in this identity, we obtain (2.2.8). ■

COROLLARY 2.7. *If* $|q|<1$,

$$1+\sum_{n=1}^{\infty}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{m=1}^{\infty}(1-aq^{2m-1})(1+q^m).$$

Proof. Set $b=\beta^{-1}$ in Corollary 2.5. Hence

$$\sum_{n=0}^{\infty}\frac{(a)_n(\beta-1)(\beta-q)\cdots(\beta-q^{n-1})(-q)^n}{(q)_n(aq\beta)_n}=\frac{(aq;q^2)_\infty(-q)_\infty(aq^2\beta^2;q^2)_\infty}{(aq\beta)_\infty(-q\beta)_\infty}.$$

Now set $\beta=0$ in this identity and we obtain the desired result. ■

The next result, Jacobi's triple product identity, may be viewed as a corollary of Corollary 2.2; however, it is so important that we label it a theorem.

THEOREM 2.8. *For* $z\neq 0$, $|q|<1$,

$$\sum_{n=-\infty}^{\infty}z^nq^{n^2}=\prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1}). \tag{2.2.10}$$

Proof. For $|z|>|q|$, $|q|<1$,

$$\prod_{n=0}^{\infty}(1+zq^{2n+1})=\sum_{m=0}^{\infty}\frac{z^mq^{m^2}}{(q^2;q^2)_m}\qquad\text{(by (2.2.6))}$$

$$=\frac{1}{(q^2;q^2)_\infty}\sum_{m=0}^{\infty}z^mq^{m^2}(q^{2m+2};q^2)_\infty$$

$$=\frac{1}{(q^2;q^2)_\infty}\sum_{m=-\infty}^{\infty}z^mq^{m^2}(q^{2m+2};q^2)_\infty$$

(since $(q^{2m}\quad q^2)_\infty$ vanishes for m negative)

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m^2} \sum_{r=0}^{\infty} \frac{(-1)^r q^{r^2+2mr+r}}{(q^2;q^2)_r}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r z^{-r} q^r}{(q^2;q^2)_r} \sum_{m=-\infty}^{\infty} q^{(m+r)^2} z^{m+r}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-q/z)^r}{(q^2;q^2)_r} \sum_{m=-\infty}^{\infty} q^{m^2} z^m$$

$$= \frac{1}{(q^2;q^2)_\infty(-q/z;q^2)_\infty} \sum_{m=-\infty}^{\infty} q^{m^2} z^m.$$

This is the desired result. Note that absolute convergence pertains everywhere only so long as $|z| > |q|$, $|q| < 1$. However, the full result of the theorem follows either by invoking analytic continuation, or by observing that the entire argument may be carried out again with z^{-1} replacing z. ■

COROLLARY 2.9. *For* $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2-in}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2-in}(1 - q^{(2n+1)i})$$

$$= \prod_{n=0}^{\infty} (1 - q^{(2k+1)(n+1)})(1 - q^{(2k+1)n+i})(1 - q^{(2k+1)(n+1)-i}). \qquad (2.2.11)$$

Proof. Replace q by $q^{k+\frac{1}{2}}$ and then set $z = -q^{k+\frac{1}{2}-i}$ in (2.2.10). This substitution immediately yields the equality of the extremes in (2.2.11). Now

$$\sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2-in}(1 - q^{(2n+1)i})$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2-in} + \sum_{n=1}^{\infty} (-1)^n q^{(2k+1)n(n-1)/2+in}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2-in} + \sum_{n=-1}^{-\infty} (-1)^n q^{(2k+1)n(n+1)/2-in}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2-in}. \qquad ■$$

We remark that Corollary 2.9 reduces to Corollary 1.7 when $k = i = 1$ once we observe that

$$\prod_{n=0}^{\infty}(1 - q^{3n+3})(1 - q^{3n+1})(1 - q^{3n+2}) = \prod_{n=1}^{\infty}(1 - q^n).$$

COROLLARY 2.10 (Gauss)

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \prod_{m=1}^{\infty}\frac{(1 - q^m)}{(1 + q^m)}, \tag{2.2.12}$$

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{m=1}^{\infty}\frac{(1 - q^{2m})}{(1 - q^{2m-1})}. \tag{2.2.13}$$

Proof. By (2.2.10) with $z = -1$,

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = (q^2; q^2)_\infty (q; q^2)_\infty (q; q^2)_\infty$$

$$= (q)_\infty (q; q^2)_\infty = (q)_\infty/(-q)_\infty$$

where the final equation follows from (1.2.5). Next

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \tfrac{1}{2}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2}$$

$$= \tfrac{1}{2}(q)_\infty(-q)_\infty(-1)_\infty$$

$$= (q)_\infty(-q)_\infty(-q)_\infty = (q^2; q^2)_\infty(-q)_\infty = (q^2; q^2)_\infty/(q; q^2)_\infty$$

where again the final equation follows from (1.2.5). ■

So far this section seems filled with much mathematics and little commentary. It has been the hope that the power of Theorem 2.1 and simple series manipulation would be fully appreciated if numerous significant results followed in rapid-fire order. The reader will have a chance to practice the techniques involved in the many examples at the end of this chapter.

2.3 Applications to Partitions

We shall prove four theorems on partitions utilizing either the actual results or the methods of Section 2.2. We conclude with an examination of "Durfee squares," which allows us to obtain (2.2.9) from purely combinatorial considerations. We begin with an interpretation of Corollary 2.9.

THEOREM 2.11. *Let $\mathscr{D}(k, i)$ denote all those partitions with distinct parts in which each part is congruent to* 0, $\pm i$ (modulo $2k + 1$). *Let* $p_e(\mathscr{D}(k, i), n)$ (resp. $p_o(\mathscr{D}(k, i), n)$) *denote the number of partitions of n taken from* $\mathscr{D}(k, i)$ *with an even* (resp. *odd*) *number of parts. Then*

NOTES ON THE JACOBI TRIPLE PRODUCT IDENTITY

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1. The Jacobi Triple Product Identity

These are notes on the *Jacobi Triple Product Identity* and its use in proving the *Euler Pentagonal Number Theorem* and the mod 5 and 7 *congruences for the partition number.* I have included in a few more of the details than I included in the lectures.

Let $\mathcal{P}$ be the set of all partitions and let $\mathcal{D}_n$ be the set of all partitions of n into distinct parts only. Let T_k denote the partition $(k, k-1, \ldots, 1)$.

Lemma 1.1. *[Sylvester's Decomposition]*

$$\mathcal{P} \times \{T_k\} \xrightarrow{\sim} \bigcup_{j \geq 0} \mathcal{D}_{k+j} \times (\mathcal{D}_j \cup \mathcal{D}_{j-1})$$

where $\mathcal{D}_0 \cup \mathcal{D}_{-1} = \mathcal{D}_0$.

Proof. Append the reverse $(1, 2, \ldots, k)$ of T_k to the top of the Ferrers diagram for $\pi \in P$, and consider the staircase that continues the profile of the Ferrers diagram for π. The length of the staircase is $k + j$. The staircase partitions the diagram into a partition α obtained by summing the columns of $\star$'s below the staircase, and a partition β obtained by summing the $\star$'s in rows above the staircase. The number of rows in β is j or $j-1$. The partitions α and β necessarily have distinct parts, induced by the staircase. The construction is clearly reversible. □

Theorem 1.2. *[Jacobi Triple Product Identity]*

$$\prod_{m \geq 1} \left(1 - q^{2m}\right)\left(1 + yq^{2m-1}\right)\left(1 + y^{-1}q^{2m-1}\right) = \sum_{k=-\infty}^{\infty} y^k q^{k^2}.$$

Proof. From Lemma 1.1, by counting partitions with respect to the sum of their parts, marked by q, we have

$$\begin{aligned} q^{\binom{k+1}{2}} \prod_{m\geq 1} (1-q^m)^{-1} &= \sum_{j\geq 0} \left[s^{k+j}\right] \prod_{a\geq 1} (1+sq^a) \\ &\quad \cdot \left(\left[t^j\right] \prod_{b\geq 1} \left(1+tq^b\right) + \left[t^{j-1}\right] \prod_{b\geq 1} \left(1+tq^b\right) \right) \\ &= \sum_{j\geq 0} \left[s^{k+j}t^j\right] (1+t) \prod_{m\geq 1} (1+sq^m)(1+tq^m) \\ &= \sum_{j\geq 0} \left[s^{k+j}t^j\right] \prod_{m\geq 1} (1+sq^m)\left(1+tq^{m-1}\right). \end{aligned}$$

We now change variables from s and t to s and u through $st=u$. Then

$$q^{\binom{k+1}{2}} \prod_{m\geq 1} (1-q^m)^{-1} = \left[s^k\right] \sum_{j\geq 0} \left[u^j\right] \prod_{m\geq 1} (1+sq^m)\left(1+us^{-1}q^{m-1}\right)$$

so

$$q^{\binom{k+1}{2}} \prod_{m\geq 1} (1-q^m)^{-1} = \left[s^k\right] \prod_{m\geq 1} (1+sq^m)\left(1+s^{-1}q^{m-1}\right). \tag{1}$$

We next sum over k from $-\infty$ to $+\infty$ by making use of the following symmetry in k. Replacing s by s^{-1}, we have

$$q^{\binom{k+1}{2}} \prod_{m\geq 1} (1-q^m)^{-1} = \left[s^{-k}\right] \prod_{m\geq 1} \left(1+s^{-1}q^m\right)\left(1+sq^{m-1}\right).$$

Now replace s by qS, noting that $\left[s^{-k}\right] = q^k\left[S^{-k}\right]$. Then

$$q^{\binom{k+1}{2}} \prod_{m\geq 1} (1-q^m)^{-1} = q^k\left[S^{-k}\right] \prod_{m\geq 1} \left(1+S^{-1}q^{m-1}\right)(1+Sq^m)$$

so, replacing S by s,

$$q^{\binom{-k+1}{2}} \prod_{m\geq 1} (1-q^m)^{-1} = \left[s^{-k}\right] \prod_{m\geq 1} (1+sq^m)\left(1+s^{-1}q^{m-1}\right)$$

since $\binom{k+1}{2} - k = \binom{-k+1}{2}$. Thus (1) holds with k replaced by $-k$. Thus summing (1) over k from $-\infty$ to $+\infty$ we have

$$\begin{aligned} \sum_{k=-\infty}^{\infty} s^k q^{\binom{k+1}{2}} \prod_{m\geq 1} (1-q^m)^{-1} &= \sum_{k=-\infty}^{\infty} s^k\left[s^k\right] \prod_{m\geq 1} (1+sq^m)\left(1+s^{-1}q^{m-1}\right) \\ &= \prod_{m\geq 1} (1+sq^m)\left(1+s^{-1}q^{m-1}\right) \end{aligned}$$

so

$$\sum_{k=-\infty}^{\infty} s^k q^{\binom{k+1}{2}} = \prod_{m\geq 1} (1-q^m)(1+sq^m)\left(1+s^{-1}q^{m-1}\right).$$

Replacing q by q^2,

$$\sum_{k=-\infty}^{\infty} s^k q^{k(k+1)} = \prod_{m\geq 1} \left(1-q^{2m}\right)\left(1+sq^{2m}\right)\left(1+s^{-1}q^{2m-2}\right).$$

Let $sq = y$. Then

$$\sum_{k=-\infty}^{\infty} y^k q^{k^2} = \prod_{m\geq 1} \left(1-q^{2m}\right)\left(1+yq^{2m-1}\right)\left(1+y^{-1}q^{2m-1}\right),$$

which completes the proof. □

Note that $\sum_{k=-\infty}^{\infty} y^k q^{k^2} \in Q\left[y, y^{-1}\right][[q]]$, the ring of formal power series in q with a coefficient ring that is *polynomial* in y and y^{-1}.

Example 1.1. *Find the number of integer points on the d-sphere of radius r.*

The d-sphere of radius r is given by

$$\left\{(z_1,\ldots,z_d)\in\mathbb{Z}^d \colon z_1^2+\cdots+z_d^2=r^2\right\}.$$

Then the number $c_{r,d}$ of such points is

$$c_{r,d} = \left|\left\{(z_1,\ldots,z_d)\in\mathbb{Z}^d \colon z_1^2+\cdots+z_d^2=r^2\right\}\right| = \left[x^{r^2}\right]\left(\sum_{i=-\infty}^{\infty} x^{i^2}\right)^d$$

so, by the Jacobi Triple Product Theorem, with $y=1$, we have

$$c_{r,d} = \left[x^{r^2}\right]\prod_{m\geq 1}\left(1-x^{2m}\right)^d\left(1+x^{2m-1}\right)^{2d}.$$

This has reduced the original question from a multivariate one to a univariate one.

The following result is an immediate consequence of the Jacobi Triple Product Identity.

Theorem 1.3. *[Euler Pentagonal Number Theorem]*

$$\prod_{m\geq 1}(1-q^m) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2}.$$

Proof. From the Jacobi Triple Product Identity,

$$\prod_{m\geq 1}\left(1-q^{2m}\right)\left(1+yq^{2m-1}\right)\left(1+y^{-1}q^{2m-1}\right) = \sum_{k=-\infty}^{\infty} y^k q^{k^2}.$$

First, replacing q by $q^{3/2}$ gives

$$\prod_{m\geq 1}\left(1-q^{3m}\right)\left(1+yq^{3m-3/2}\right)\left(1+y^{-1}q^{3m-3/2}\right) = \sum_{k=-\infty}^{\infty} y^k q^{3k^2/2}.$$

and then replacing y by $-q^{-1/2}$ gives

$$\prod_{m\geq 1}\left(1-q^{3m}\right)\left(1-q^{3m-2}\right)\left(1+q^{3m-1}\right) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2}.$$

The result follows immediately since the exponents one the right hand side give a complete set of residues modulo 3. □

The Euler Pentagonal Number Theorem has a combinatorial interpretation in terms of partitions.

Corollary 1.4. *The number of partitions in $\mathcal{D}_n$ with an even number of parts minus the number of partitions in $\mathcal{D}_n$ with an odd number of parts is equal to $(-1)^k$ if there is an integer k such that $n = k(3k-1)/2$ and is 0 otherwise.*

Proof. Let $d_k(n)$ be the number of partitions in $\mathcal{D}_n$ with k parts. Then

$$\sum_{k,n\geq 0} d_k(n)\, x^k q^n = \prod_{m\geq 1}(1+xq^m).$$

Let $e(n)$ be the number of partitions in $\mathcal{D}_n$ with an even number of parts minus the number of partitions in $\mathcal{D}_n$ with an odd number of parts. Then

$$\begin{aligned} e(n) &= \sum_{k\geq 0}(-1)^k d_k(n) = \sum_{k\geq 0}(-1)^k \left[x^k q^n\right]\prod_{m\geq 1}(1+xq^m) \\ &= [q^n]\sum_{k\geq 0}(-1)^k\left[x^k\right]\prod_{m\geq 1}(1+xq^m) \\ &= [q^n]\prod_{m\geq 1}(1-q^m) = [q^n]\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2}, \end{aligned}$$

by the Euler Pentagonal Number Theorem. Thus

$$e(n) = \begin{cases} (-1)^k & \text{if } n = k(3k-1)/2 \text{ for some integer } k, \\ 0 & \text{otherwise,} \end{cases}$$

which concludes the proof. □

2. Congruences for the partition number

We begin by proving an expansion theorem.

Theorem 2.1.

$$\prod_{m\geq 1}(1-q^m)^3 = \sum_{k\geq 0}(-1)^k(2k+1)\, q^{\binom{k+1}{2}}.$$

Proof. In the Jacobi Triple product Identity replace y by $-y$ to obtain

$$\prod_{m\geq 1}\left(1-q^{2m}\right)\left(1-yq^{2m-1}\right)\left(1-y^{-1}q^{2m-1}\right) = \sum_{k=-\infty}^{\infty}(-y)^k q^{k^2}.$$

But $\prod_{m\geq 1}\left(1-yq^{2m-1}\right) = (1-qy)\prod_{m\geq 1}\left(1-yq^{2m+1}\right)$ so

$$\prod_{m\geq 1}\left(1-yq^{2m+1}\right)\left(1-y^{-1}q^{2m-1}\right)\left(1-q^{2m}\right) = (1-qy)^{-1}\sum_{k=-\infty}^{\infty}(-y)^k q^{k^2}. \tag{2}$$

Now

$$\begin{aligned} (1-qy)^{-1}\sum_{k=-\infty}^{\infty}(-y)^k q^{k^2} &= (1-qy)^{-1}\left(1+\sum_{k=1}^{\infty}\left((-y)^k+(-y)^{-k}\right)q^{k^2}\right) \\ &= 1+\sum_{m\geq 1}y^m q^m + \sum_{m\geq 0}y^m\sum_{k=1}^{\infty}\left((-y)^k+(-y)^{-k}\right)q^{k^2+m}. \end{aligned}$$

Let $k^2 + m = m'$ and eliminate m from the summation. Then $m = m' - k^2 \geq 0$ so $k^2 \leq m'$ so $k \leq \mu_{m'}$ where $\mu_{m'} = \lfloor\sqrt{m'}\rfloor$. Also $k \geq 1$ and $m \geq 0$ so $m' \geq 1$ whence the right hand side of the above expression is equal to $1 + \sum_m y^m q^m + \sum_{m'} q^{m'} \sum_{k=1}^{\mu_{m'}} \left((-y)^k + (-y)^{-k}\right) y^{m'-k^2}$ so

$$(1-qy)^{-1} \sum_{k=-\infty}^{\infty} (-y)^k q^{k^2} = 1 + \sum_{m\geq 1} qR_m$$

where $R_m = \sum_{k=1}^{\mu_m} \left((-y)^k + (-y)^{-k}\right) y^{m-k^2} + y^m$. But

$$\begin{aligned} R_m &= \sum_{k=2}^{\mu_m} (-1)^k y^{m-k(k-1)} + \sum_{k=1}^{\mu_m} (-1)^k y^{m-k(k+1)} \\ &= \sum_{k=1}^{\mu_m - 1} (-1)^{k+1} y^{m-k(k+1)} + \sum_{k=1}^{\mu_m} (-1)^k y^{m-k(k+1)} \\ &= (-1)^{\mu_m} y^{m-\mu_m^2-\mu_m}, \end{aligned}$$

so

$$(1-qy)^{-1} \sum_{k=-\infty}^{\infty} (-y)^k q^{k^2} = 1 + \sum_{m\geq 1} (-1)^{\mu_m} y^{m-\mu_m^2-\mu_m} q^m.$$

We may therefore set $y = q^{-1}$ in this expression. This gives

$$\begin{aligned} (1-qy)^{-1} \sum_{k=-\infty}^{\infty} (-y)^k q^{k^2} \Big|_{y=q^{-1}} &= 1 + \sum_{m\geq 1} (-1)^{\mu_m} y^{\mu_m^2+\mu_m} \\ &= 1 + \sum_{m\geq 1} (-1)^m y^{m^2+m} \left|\left\{i \geq 1 \colon \lfloor\sqrt{i}\rfloor = m\right\}\right| \\ &= 1 + \sum_{m\geq 1} (-1)^m (2m+1) y^{m^2+m}, \end{aligned}$$

since $\left|\{i \geq 1 \colon \lfloor\sqrt{i}\rfloor = m\}\right| = \left|\left\{i \geq 1 \colon m \leq i \leq (m+1)^2 - 1\right\}\right| = 2m+1$. But

$$\prod_{m\geq 1} \left(1 - yq^{2m+1}\right)\left(1 - y^{-1}q^{2m-1}\right)\left(1 - q^{2m}\right)_{y=q^{-1}} = \prod_{m\geq 1} \left(1 - q^{2m}\right)^3$$

so, from (2),

$$\prod_{m\geq 1} \left(1 - q^{2m}\right)^3 = 1 + \sum_{m\geq 1} (-1)^m (2m+1) q^{m^2+m},$$

and the result follows by replacing q by $q^{1/2}$. □

With these results, we may now prove a remarkable congruence for the partition number. The following lemma is needed.

Lemma 2.2. *Let $a_0, a_1, \ldots$ be integers, and let m be a non-negative integer not congruent to 0 modulo 5. Then*

$$[q^m]\left(a_0 + a_1 q + a_2 q^2 + \cdots\right)^5 \equiv 0 \bmod 5.$$

Proof. Now

$$\begin{aligned}[q^m]\left(a_0 + a_1 q + a_2 q^2 + \cdots\right)^5 &= [q^m]\left(a_0 + a_1 q + \cdots + a_m q^m\right)^5 \\ &= \sum_{i_0,\ldots,i_m \geq 0} \frac{5!}{i_0! \ldots i_m!} a_0^{i_0} \cdots a_m^{i_m},\end{aligned}$$

where the sum is over all $i_0, \ldots, i_m$ such that $i_0 + \cdots + i_m = 5$ and $i_1 + 2i_2 + \cdots + mi_m = m$. But m is not congruent to 0 modulo 5 so not all of $i_1, 2i_2, \cdots, mi_m$ are congruent to 0 modulo 5. Suppose that ji_j is not congruent to 0 modulo 5. Then, in particular, i_j is not congruent to 0 modulo 5. But $0 \leq i_j \leq 5$ so $i_j \neq 0$. Thus none of $i_0, \ldots, i_m$ is equal to 5 since their sum is 5. Then

$$\frac{5!}{i_0! \ldots i_m!} \equiv 0.$$

The result follows since $a_0^{i_0} \cdots a_m^{i_m}$ is an integer. □

The above lemma is in fact more general, since "5" may be replaced by an arbitrary prime throughout (primality is necessary since, for example, $4!/2!^2$ is not congruent to 0 modulo 4).

Theorem 2.3. $p(5n-1) \equiv 0$ *mod* 5.

Proof. Throughout this proof, I shall use $\equiv$ to denote congruence modulo 5. Let

$$F(q) = q \prod_{k \geq 1} \left(1 - q^k\right)^4.$$

Then

$$F(q) = q \prod_{i \geq 1} \left(1 - q^i\right) \prod_{k \geq 1} \left(1 - q^k\right)^3.$$

From the Euler Pentagonal Number Theorem and Theorem 2.1 we have

$$\begin{aligned}F(q) &= q \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2} \sum_{k \geq 0} (-1)^k (2k+1) q^{\binom{k+1}{2}} \\ &= \sum_{m=-\infty, k \geq 0}^{\infty} (-1)^{m+k} (2k+1) q^{1 + m(3m-1)/2 + \binom{k+1}{2}}.\end{aligned}$$

Now note that $q \prod_{i \geq 1} \left(1 - q^i\right)^{-1} = F(q) \prod_{k \geq 1} \left(1 - q^k\right)^{-5}$. Then

$$p(5j-1) = \left[q^{5j-1}\right] \prod_{i \geq 1} \left(1 - q^i\right)^{-1} = \left[q^{5j}\right] F(q) \prod_{k \geq 1} \left(1 - q^k\right)^{-5}$$

so

$$p(5j-1) = \sum_{n \geq 0} \left([q^n] F(q)\right) \left(\left[q^{5j-n}\right] \prod_{k \geq 1} \left(1 - q^k\right)^{-5}\right). \quad (3)$$

There are two cases.

Case 1: *Assume that* $n \equiv 0$. Then $[q^n] F(q)$ is non-zero if $1 + m(3m-1)/2 + \binom{k+1}{2} \equiv n$. Now consider $1 + m(3m-1)/2$. If m is odd, so $m = 2a+1$, then $1 + m(3m-1)/2 = 1 + m(3a+1) \equiv 1 + m(-2a+1) = 1 + m(-m+2) = -m^2 + 2m + 1$. If m is even, so $m = 2a$, then $1 + m(3m-1)/2 = 1 + a(6a-1) \equiv 1 - 4a(a-1) = 1 - 2m(a-1) = -m^2 + 2m + 1$. Thus, for any m, $1 + m(3m-1)/2 \equiv -m^2 + 2m + 1$. Then

$$1 + m(3m-1)/2 \equiv -A^2 + 2A + 1 \text{ if } m \equiv A.$$

Similarly, for $\binom{k+1}{2}$, if k odd then $k = 2b+1$, so $\binom{k+1}{2} = k(b+1) \equiv k(-4b+1) \equiv k(-2k+3) = -2k^2 + 3k \equiv 3k^2 - 2k$. If k is even, so $k = 2b$, then $\binom{k+1}{2} = b(k+1) \equiv -4b(k+1) = -2k(k+1) \equiv 3k^2 - 2k$. Thus, for any k, $\binom{k+1}{2} \equiv 3k^2 - 2k$. Then

$$\binom{k+1}{2} \equiv 3B^2 - 2B \text{ if } k \equiv B.$$

By direct computation,

$$\left(-A^2 + 2A + 1 \bmod 5 \colon A = 0, \ldots, 4\right) = (1, 2, 1, 3, 3)$$

and

$$\left(3B^2 - 2B \bmod 5 \colon B = 0, \ldots, 4\right) = (0, 1, 3, 1, 0).$$

Then $1 + m(3m-1)/2 + \binom{k+1}{2} \equiv 0$ implies that $A = 1$ and $B = 2$, since the only two residue classes, one from each of the above two lists, that sum to 0 mod 5 are 2 and 3, which implies that $m \equiv 1$ and $k \equiv 2$. Thus $2k + 1 \equiv 0$. We conclude that $[q^n] F(q) \equiv 0$. Thus the contribution to the right hand side of (3) is 0 from this case.

Case 2: *Assume that* n *is not congruent to* 0 *modulo* 5. Then neither is $5j - n$, so, from Lemma 2.2, $\left[q^{5j-n}\right] \prod_{k \geq 1} \left(1 - q^k\right)^{-5} \equiv 0$ since $\left(1 - q^k\right)^{-5}$ is a series with integer coefficients. It follows that the contribution to the right hand side of (3) is 0 in this case.

We conclude from (3) that $p(5j - 1) \equiv 0$, establishing the result. □

The following mod 7 congruence may be obtained by a similar argument.

Theorem 2.4. $p(7n - 2) \equiv 0$ *mod* 7.

Proof. Throughout this proof, I shall use $\equiv$ to denote congruence modulo 7. Let

$$G(q) = q^2 \prod_{i \geq 1} \left(1 - q^i\right)^6.$$

Then

$$G(q) = q^2 \prod_{i \geq 1} \left(1 - q^i\right)^3 \prod_{i \geq k} \left(1 - q^k\right)^3.$$

From Theorem 2.1 we have

$$G(q) = \sum_{j,k\geq 0} (-1)^{k+j} (2k+1)(2k+1)\, q^{2+\binom{j+1}{2}+\binom{k+1}{2}}.$$

Now note that $q^2 \prod_{i\geq 1} \left(1-q^i\right)^{-1} = G(q) \prod_{k\geq 1} \left(1-q^k\right)^{-7}$. Then

$$\begin{aligned} p(7j-2) &= \left[q^{7j-2}\right] \left(1-q^i\right)^{-1} = \left[q^{7j}\right] G(q) \prod_{k\geq 1} \left(1-q^k\right)^{-7} \\ &= \sum_{n\geq 0} \left(\left[q^n\right] G(q)\right) \left(\left[q^{7j-n}\right] \prod_{k\geq 1} \left(1-q^k\right)^{-7} \right). \end{aligned}$$

so

$$p(7j-2) = \sum_{n\geq 0} \left(\left[q^n\right] G(q)\right) \left(\left[q^{7j-n}\right] \prod_{k\geq 1} \left(1-q^k\right)^{-7} \right) \tag{4}$$

There are two cases.

Case 1: *Assume that* $n \equiv 0$.We now proceed as before. It follows easily (the details are omitted) that

$$2 + \binom{j+1}{2} + \binom{k+1}{2} \equiv (2A+1)^2 + (2B+1)^2 \text{ if } j \equiv A \text{ and } k \equiv B.$$

But by direct computation

$$\left((2A+1) \mod 5 \colon A = 0, \ldots, 4\right) = (1, 2, 4, 0, 4, 2, 1)$$

so $2 + \binom{j+1}{2} + \binom{k+1}{2} \equiv 0$ implies that $A = B = 3$. Thus $j = 7J + 3$ and $k = 7K + 3$ for some J and K, so $2j+1, 2k+1 \equiv 0$. We conclude that $\left[q^n\right] G(q) \equiv 0$. Thus the contribution to the right hand side of (4) is 0 from this case.

Case 2: *Assume that* n *is not congruent to* 0 *modulo* 7. Then neither is $7j - n$, so, from the comment folowing Lemma 2.2, $\left[q^{7j-n}\right] \prod_{k\geq 1} \left(1-q^k\right)^{-7} \equiv 0$ since $\left(1-q^k\right)^{-7}$ is a series with integer coefficients. It follows that the contribution to the right hand side of (4) is 0 in this case.

We conclude from (4) that $p(7j-2) \equiv 0$, establishing the result. □