MacMahon used this recursion formula to compute $p(n)$ up to $n = 200$. Here are some sample values from his table.

$$\begin{aligned}
p(1) &= 1 \\
p(5) &= 7 \\
p(10) &= 42 \\
p(15) &= 176 \\
p(20) &= 627 \\
p(25) &= 1{,}958 \\
p(30) &= 5{,}604 \\
p(40) &= 37{,}338 \\
p(50) &= 204{,}226 \\
p(100) &= 190{,}569{,}292 \\
p(200) &= 3{,}972{,}999{,}029{,}388
\end{aligned}$$

These examples indicate that $p(n)$ grows very rapidly with n. The largest value of $p(n)$ yet computed is $p(14{,}031)$, a number with 127 digits. D. H. Lehmer [42] computed this number to verify a conjecture of Ramanujan which asserted that $p(14{,}031) \equiv 0 \pmod{11^4}$. The assertion was correct. Obviously, the recursion formula in (8) was not used to calculate this value of $p(n)$. Instead, Lehmer used an asymptotic formula of Rademacher [54] which implies

$$p(n) \sim \frac{e^{K\sqrt{n}}}{4n\sqrt{3}} \quad \text{as } n \to \infty,$$

where $K = \pi(2/3)^{1/2}$. For $n = 200$ the quantity on the right is approximately 4×10^{12} which is remarkably close to the actual value of $p(200)$ given in MacMahon's table.

In the sequel to this volume we give a derivation of Rademacher's asymptotic formula for $p(n)$. The proof requires considerable preparation from the theory of elliptic modular functions. The next section gives a crude upper bound for $p(n)$ which involves the exponential $e^{K\sqrt{n}}$ and which can be obtained with relatively little effort.

14.7 An upper bound for $p(n)$

Theorem 14.5 *If $n \geq 1$ we have $p(n) < e^{K\sqrt{n}}$, where $K = \pi(2/3)^{1/2}$.*

PROOF. Let

$$F(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1} = 1 + \sum_{k=1}^{\infty} p(k)x^k,$$

and restrict x to the interval $0 < x < 1$. Then we have $p(n)x^n < F(x)$, from which we obtain $\log p(n) + n \log x < \log F(x)$, or

$$\log p(n) < \log F(x) + n \log \frac{1}{x}. \tag{9}$$

We estimate the terms $\log F(x)$ and $n \log(1/x)$ separately. First we write

$$\log F(x) = -\log \prod_{n=1}^{\infty}(1 - x^n) = -\sum_{n=1}^{\infty} \log(1 - x^n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{x^{mn}}{m}$$
$$= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} (x^m)^n = \sum_{m=1}^{\infty} \frac{1}{m} \frac{x^m}{1 - x^m}.$$

Since we have

$$\frac{1 - x^m}{1 - x} = 1 + x + x^2 + \cdots + x^{m-1},$$

and since $0 < x < 1$, we can write

$$mx^{m-1} < \frac{1 - x^m}{1 - x} < m,$$

and hence

$$\frac{m(1 - x)}{x} < \frac{1 - x^m}{x^m} < \frac{m(1 - x)}{x^m}.$$

Inverting and dividing by m we get

$$\frac{1}{m^2} \frac{x^m}{1 - x} \le \frac{1}{m} \frac{x^m}{1 - x^m} \le \frac{1}{m^2} \frac{x}{1 - x}.$$

Summing on m we obtain

$$\log F(x) = \sum_{m=1}^{\infty} \frac{1}{m} \frac{x^m}{1 - x^m} \le \frac{x}{1 - x} \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} \frac{x}{1 - x} = \frac{\pi^2}{6t},$$

where

$$t = \frac{1 - x}{x}.$$

Note that t varies from ∞ to 0 through positive values as x varies from 0 to 1.

Next we estimate the term $n \log(1/x)$. For $t > 0$ we have $\log(1 + t) < t$. But

$$1 + t = 1 + \frac{1 - x}{x} = \frac{1}{x}, \qquad \text{so } \log \frac{1}{x} < t.$$

Now

$$\text{(10)} \qquad \log p(n) < \log F(x) + n \log \frac{1}{x} < \frac{\pi^2}{6t} + nt.$$

The minimum of $(\pi^2/6t) + nt$ occurs when the two terms are equal, that is, when $\pi^2/(6t) = nt$, or $t = \pi/\sqrt{6n}$. For this value of t we have

$$\log p(n) < 2nt = 2n\pi/\sqrt{6n} = K\sqrt{n}$$

so $p(n) < e^{K\sqrt{n}}$, as asserted. □

Note. J. H. van Lint [48] has shown that with a little more effort we can obtain the improved inequality

$$\text{(11)} \qquad p(n) < \frac{\pi e^{K\sqrt{n}}}{\sqrt{6(n-1)}} \quad \text{for } n > 1.$$

Since $p(k) \geq p(n)$ if $k \geq n$, we have, for $n > 1$,

$$F(x) > \sum_{k=n}^{\infty} p(k)x^k \geq p(n) \sum_{k=n}^{\infty} x^k = \frac{p(n)x^n}{1-x}.$$

Taking logarithms we obtain, insteady of (9), the inequality

$$\log p(n) < \log F(x) + n \log \frac{1}{x} + \log(1 - x).$$

Since $1 - x = tx$ we have $\log(1 - x) = \log t - \log(1/x)$, hence (10) can be replaced by

$$\text{(12)} \qquad \log p(n) < \frac{\pi^2}{6t} + (n-1)t + \log t.$$

An easy calculation with derivatives shows that the function

$$f(t) = \frac{\pi^2}{6t} + (n-1)t + \log t$$

has its minimum at

$$t = \frac{-1 + \sqrt{1 + [4(n-1)\pi^2/6]}}{2(n-1)}.$$

Using this value of t in (12) and dropping insignificant terms we obtain (11).

14.8 Jacobi's triple product identity

This section describes a famous identity of Jacobi from the theory of theta functions. Euler's pentagonal number theorem and many other partition identities occur as special cases of Jacobi's formula.