A new proof of the inversion formula for the Dedekind Eta function

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1 Introduction and a Reformulation

The Dedekind eta function is defined for τ in the upper half complex plane $\mathcal{H}$ by

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{i2\pi n\tau}).$$

The inversion formula states that for all $\tau \in \mathcal{H}$,

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau), \tag{1}$$

where the branch of the square root function is taken which is positive for positive real numbers.

A number of proofs (1) exist, commonly making use of the residue theorem. The purpose of this brief note is to give a proof that does not. The proof is not elementary in the sense that it does use analytic continuation, but even this is fairly superficial. In fact, it likely that even the proof's use of integration could be eliminated by a detailed combinatoric argument. (I believe the proof is original, but I have't scoured the literature, so I make no guarantees.)

Our use of complex analysis goes no further than this; we note that (1) is equivalent by analytic continuation to the statement $\eta(i/y) = \sqrt{y} \cdot \eta(iy)$ for all $y > 0$. Upon taking logarithms, this becomes $\frac{1}{2}\log y = \log \eta(i/y) - \log \eta(iy)$. This is the statement we will prove.

2 A Proof

Note that

$$\begin{aligned}
\log \eta(iy) &= -\frac{\pi y}{12} + \sum_{n=1}^{\infty} \log(1 - e^{-2\pi n y}) \\
&= -\frac{\pi y}{12} - \sum_{n,m=1}^{\infty} \frac{e^{-2\pi m n y}}{m} \\
&= -\frac{\pi y}{12} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{e^{2\pi m y} - 1}.
\end{aligned}$$

Further, we know by logarithmic differentiation of the sine product, for instance, that

$$\frac{2\pi}{m(e^{2\pi m y} - 1)} = -\frac{\pi}{m} + \sum_{n=-\infty}^{\infty} \frac{1}{n^2 y^{-1} + m^2 y}.$$

(For completeness sake, I note that the sine product may be proven without recourse to the residue theorem, or, indeed, to Poisson summation.) This implies that

$$\begin{aligned}
\pi\log\eta(iy) &= -\frac{\pi^2}{12}y+\sum_{m=1}^{\infty}\left(\frac{\pi}{2m}-\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2y^{-1}+m^2y}\right)\\
&= -\frac{\pi^2}{12}y+\sum_{m=1}^{\infty}\left(\frac{\pi}{2m}-\sum_{n=1}^{\infty}\frac{1}{n^2y^{-1}+m^2y}-\frac{1}{2m^2y}\right)\\
&= -\frac{\pi^2}{12}(y+y^{-1})+\sum_{m=1}^{\infty}\left(\frac{\pi}{2m}-\sum_{n=1}^{\infty}\frac{1}{n^2y^-1+m^2y}\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\pi\log\eta(i/y)-\pi\log\eta(iy) &= \sum_{m=1}^{\infty}\left(\frac{\pi}{2m}-\sum_{n=1}^{\infty}\frac{1}{n^2y+m^2y^{-1}}\right)-\sum_{m'=1}^{\infty}\left(\frac{\pi}{2m'}-\sum_{n'=1}^{\infty}\frac{1}{n'^2y^{-1}+m'^2y}\right)\\
&= \lim_{N\to\infty}\sum_{m\le yN}\left(\frac{\pi}{2m}-\sum_{n=1}^{\infty}\frac{1}{n^2y+m^2y^{-1}}\right)-\sum_{m' leqN}\left(\frac{\pi}{2m'}-\sum_{n'=1}^{\infty}\frac{1}{n'^2y^{-1}+m'^2y}\right)\\
&= \lim_{N\to\infty}\frac{\pi}{2}(\log(yN)+\gamma+o(1))-\frac{\pi}{2}(\log N+\gamma+o(1))\\
&\qquad +\sum_{m'\le N}\sum_{n'=1}^{\infty}\frac{1}{n'^2y^{-1}+m'^2y}-\sum_{m\le yN}\sum_{n=1}^{\infty}\frac{1}{n^2y+m^2y^{-1}}\\
&= \frac{\pi}{2}\log y+\lim_{N\to\infty}\sum_{m\le N}\sum_{n=1}^{\infty}\frac{y}{n^2+m^2y^2}-\sum_{m\le yN}\sum_{n=1}^{\infty}\frac{y}{n^2y^2+m^2}\\
&= \frac{\pi}{2}\log y+\lim_{N\to\infty}y\left(\sum_{k\le N}\sum_{j>yN}\frac{1}{j^2+k^2y^2}-\sum_{j\le yN}\sum_{k>N}\frac{1}{j^2+k^2y^2}\right), \qquad (2)
\end{aligned}$$

where this last simplification is obtained by canceling out like terms and reindexing.

We seek to estimate this last limit by integrals. It follows by routine considerations of monoticity that for $x>0$ and $A\ge 1$

$$\int_0^{\lfloor A\rfloor}\frac{d\alpha}{\alpha^2+x}\le\sum_{n\le A}\frac{1}{n^2+x}\le\int_0^{\lfloor A\rfloor+1}\frac{d\alpha}{\alpha^2+x},\ \text{ and }\ \int_{\lfloor A\rfloor+1}^{\infty}\frac{d\beta}{\beta^2+x}\le\sum_{n\le A}\frac{1}{n^2+x}\le\int_{\lfloor A\rfloor}^{\infty}\frac{d\beta}{\beta^2+x}.$$

Hence for A and x as above,

$$\left|\sum_{n\le A}\frac{1}{n^2+x}-\int_0^{\lfloor A\rfloor}\frac{d\alpha}{\alpha^2+x}\right|\le\int_{\lfloor A\rfloor}^{\lfloor A\rfloor+1}\frac{d\alpha}{\alpha^2+x}=O(\frac{1}{A^2+x}),$$

and likewise,

$$\sum_{n>A}\frac{1}{n^2+x}-\int_A^{\infty}\frac{d\beta}{\beta^2+x}=O(\frac{1}{A^2+x}),$$

where $O(\cdot)$ is used with respect to both A and x.

Hence, fixing positive y and using $O(\cdot)$ notation with respect to N, j, and α, we have

$$
\begin{aligned}
\sum_{k\le N}\sum_{j>yN}\frac{1}{j^2+k^2y^2} &= \sum_{j>yN}\left(\int_0^N \frac{d\alpha}{y^2\alpha^2+j^2}+O(\frac{1}{y^2N^2+j^2})\right)\\
&= \int_0^N \sum_{j>yN}\frac{d\alpha}{y^2\alpha^2+j^2}+O\left(\sum_{j>yN}\frac{1}{y^2N^2+j^2}\right)\\
&= \int_0^N \int_{yN}^\infty \frac{d\beta}{y^2\alpha^2+\beta^2}+O\left(\frac{1}{y^2N^2+y^2\alpha^2}\right)d\alpha+O\left(\sum_{j>yN}\frac{1}{j^2}\right)\\
&= \int_0^N \int_{yN}^\infty \frac{d\beta d\alpha}{y^2\alpha^2+\beta^2}+N\cdot O(\frac{1}{N^2})+O(\frac{1}{N})\\
&= \int_0^N \int_{yN}^\infty \frac{d\beta d\alpha}{y^2\alpha^2+\beta^2}+O(\frac{1}{N}), \qquad (3)
\end{aligned}
$$

where the interchange of sum and integral is justified by the monotone convergence theorem, for instance. Analogously,

$$
\sum_{j\le yN}\sum_{k>N}\frac{1}{j^2+k^2y^2} = \int_0^{yN}\int_N^\infty \frac{d\beta d\alpha}{\alpha^2+y^2\beta^2}+O(\frac{1}{N}). \qquad (4)
$$

Making a change of variables $\alpha' = y\alpha$ in (3), we have

$$
\sum_{k\le N}\sum_{j>yN}\frac{1}{j^2+k^2y^2} = \frac{1}{y}\int_0^{yN}\int_{yN}^\infty \frac{d\beta d\alpha}{\alpha'^2+\beta^2}+O(\frac{1}{N}).
$$

Similarly, making a change of variables $\beta' = y\beta$ in (4), we have

$$
\sum_{j\le yN}\sum_{k>N}\frac{1}{j^2+k^2y^2} = \frac{1}{y}\int_0^{yN}\int_{yN}^\infty \frac{d\beta' d\alpha}{\alpha^2+\beta'^2}+O(\frac{1}{N}).
$$

Therefore (2) becomes

$$
\pi\log\eta(i/y)-\pi\log\eta(iy) = \frac{\pi}{2}\log y+\lim_{N\to\infty}O(\frac{1}{N}) = \frac{\pi}{2}\log y,
$$

which, upon dividing by π, is what we wanted to prove.