ON THE PARITY OF PARTITION FUNCTIONS

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Abstract. Let S denote a subset of the positive integers, and let $p_S(n)$ be the associated partition function, that is $p_S(n)$ denotes the number of partitions of the positive integer n into parts taken from S. Thus, if S is the set of positive integers, then $p_S(n)$ is the ordinary partition function $p(n)$. In this paper, working in the ring of formal power series in one variable over the field of two elements $\mathbf{Z}/2\mathbf{Z}$, we develop new methods for deriving lower bounds for both the number of even values and the number of odd values taken by $p_S(n)$, for $n \leq N$. New very general theorems are obtained, and applications are made to several partition functions, including $p(n)$.

1. Introduction

As usual, let $p(n)$ denote the ordinary partition function, i.e., the number of ways a positive integer n can be represented as a sum of positive integers. It has long been conjectured that $p(n)$ is even approximately half of the time, or, more precisely,

$$\#\{n \leq N : p(n) \, \text{is even}\} \sim \tfrac{1}{2}N, \tag{1.1}$$

as $N \to \infty$. T. R. Parkin and D. Shanks [16] undertook the first extensive computations, which indicated that indeed (1.1) is likely true. Despite the venerability of the problem, it was not even known that $p(n)$ assumes even values infinitely often or $p(n)$ assumes odd values infinitely often until 1959, when O. Kolberg [8] established these facts. Other proofs of Kolberg's theorem were later found by J. Fabrykowski and M. V. Subbarao [5] and by M. Newman [10]. In 1983, L. Mirsky [9] established the first quantitative result by showing that

$$\#\left\{n \leq N : p(n) \;\text{ is even (odd)}\right\} > \frac{\log\log N}{2\log 2}. \tag{1.2}$$

An improvement was made by J.–L. Nicolas and A. Sárközy [12], who proved that

$$\#\left\{n \leq N : p(n) \;\text{ is even (odd)}\right\} > (\log N)^c, \tag{1.3}$$

for some positive constant c.

In the most recent investigations, the methods for finding lower bounds for the number of occurrences of even values of $p(n)$ have been somewhat different from those for odd values of $p(n)$. Greatly improving on previous results, Nicolas, I. Ruzsa, and Sárközy [11] proved that

$$\#\left\{n \leq N : p(n) \;\text{ is even }\right\} \gg \sqrt{N} \tag{1.4}$$

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and

$$\#\{n \le N : p(n) \text{ is odd }\} \gg \sqrt{N} e^{-(\log 2+\epsilon)\frac{\log N}{\log\log N}}. \tag{1.5}$$

In an appendix to their paper [11], J.–P. Serre, using modular forms, proved that

$$\#\{n \le N : p(n) \text{ is even }\} > c\sqrt{N}, \tag{1.6}$$

for every positive constant c. At present, this is the best result for even values of $p(n)$. The lower bound (1.6) has been improved by S. Ahlgren, [1] who used modular forms to prove that

$$\#\{n \le N : p(n) \text{ is odd }\} \gg \frac{\sqrt{N}}{\log N}, \tag{1.7}$$

provided that there is at least one value of n for which $p(n)$ is odd. The lower bounds (1.6) and (1.7) are currently the best known results.

Subbarao [22] first conjectured that in every arithmetic progression $n \equiv r \pmod{t}$ there are infinitely many values of n such that $p(n)$ is even and that there are infinitely many values of n for which $p(n)$ is odd. Several authors proved special cases of this conjecture, and a summary of these results can be found in K. Ono's paper [14]. For the case that $p(n)$ is even, Ono [13] proved that there are infinitely many n in every arithmetic progression $n \equiv r \pmod{t}$ such that $p(n)$ is even. He established an analogous result for odd values of $p(n)$, provided that there exists at least one such n for which $p(n)$ is odd. He then verified that indeed this is the case for all $t \le 10^5$. Two years later, Ono [14] proved that the density of primes t exceeds $1 - 1/10^{1500}$, for which the arithmetic progressions $n \equiv r \pmod{t}$, with $r \not\equiv 24^{-1} \pmod{t}$, have infinitely many values of n for which $p(n)$ is odd. The best quantitative results are due to Ahlgren [1] who has proved both (1.4) and (1.7) for all arithmetic progressions, with the same provision as above for odd values of $p(n)$. The theory of modular forms was the primary tool in proving all the results of Ahlgren, Ono, and Serre.

In this paper we develop new methods to examine the parity of $p(n)$. In particular, nothing from the theory of modular forms is used. Our methods are very simple and general, and, as we demonstrate, apply to a large variety of partition functions. All our work is effected in the ring of formal power series in one variable over the field of two elements $\mathbf{Z}/2\mathbf{Z}$.

In Section 3, using our first approach, we give easy proofs of analogues of (1.4) and (1.5) for a large class of partition functions, including $p(n)$; more generally, $p(r, s; n)$, the number of partitions of the positive integer n into parts congruent to r, s, or $r+s$ modulo $r+s$; and $c_3(n)$, the number of partitions of n into three colors. Note that $p(2, 1; n) = p(n)$.

Our second approach is given in Section 4. Here we use elementary differential equations over $\mathbf{Z}/2\mathbf{Z}$ and some basic ideas in elementary algebraic number theory to prove analogues of (1.4) and (1.5).

Using differential equations over $\mathbf{Z}/2\mathbf{Z}$, in Section 5, we generalize our ideas from Section 4 to a very general class of partition functions. Let S denote the set of positive integers coprime to a given fixed integer b, and let $p_S(n)$denote the number of partitions of a positive integer n into parts which belong to S. In Section 6, we apply the results of Section 5 to obtain lower bounds for both the number of even values and the number of odd values of the partition function $p_S(n)$. In Section 7, we give another application of the ideas in Section 5. Now let S denote the set of square-free integers which are relatively prime to a fixed positive integer b, and let $p_S(n)$ denote the number of partitions of the positive integer n into parts from S. We obtain lower bounds for both the number of even values and the number of odd values of $p_S(n)$. Zaharescu [24] had previously obtained results for the special case $b = 1$, i.e., when S is the set of square-free integers.

2. The ring A

Let $A := \mathbf{F}_2[[X]]$ be the ring of formal power series in one variable X over the field with two elements $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$, i.e.,

$$A = \{f(X) = \sum_{n=0}^{\infty} a_n X^n : a_n \in \mathbf{F}_2 \text{ for all } n\}. \tag{2.1}$$

The ring A is an integral domain. It is also a local ring, with maximal ideal generated by X. An element $f(X) = \sum_{n=0}^{\infty} a_n X^n \in A$ is invertible if and only if $a_0 = 1$. Since 0 and 1 are the only elements of $\mathbf{F}_2$, we may write any element $f(X) \in A$ in the form

$$f(X) = X^{n_1} + X^{n_2} + \cdots , \tag{2.2}$$

where the sum may be finite or infinite and $0 \le n_1 < n_2 < \cdots$. For any $f(X) \in A$, one has

$$f^2(X) = f(X^2). \tag{2.3}$$

In other words, if $f(X)$ is given by (2.2) then

$$f^2(X) = X^{2n_1} + X^{2n_2} + \cdots . \tag{2.4}$$

We need to know the shape of the division by $1 - X$ of a given series $f(X)$ as in (2.2). For any integers $0 \le a < b$ we have in A

$$\frac{X^a + X^b}{1 - X} = \frac{X^a(1 - X^{b-a})}{1 - X} = X^a + X^{a+1} + \cdots + X^{b-1}. \tag{2.5}$$

We put together pairs of consecutive terms $X^{n_{2k+1}} + X^{n_{2k+2}}$, and obtain

$$\begin{aligned} \frac{f(X)}{1-X} &= \frac{X^{n_1} + X^{n_2}}{1-X} + \frac{X^{n_3} + X^{n_4}}{1-X} + \cdots + \frac{X^{n_{2k+1}} + X^{n_{2k+2}}}{1-X} + \cdots \\ &= \left(X^{n_1} + X^{n_1+1} + \cdots + X^{n_2-1}\right) + \left(X^{n_3} + \cdots + X^{n_4-1}\right) + \cdots \\ &\qquad + \left(X^{n_{2k+1}} + \cdots + X^{n_{2k+2}-1}\right) + \cdots . \end{aligned} \tag{2.6}$$

If the sum on the right side of (2.2) which defines $f(X)$ is finite, say $f(X) = X^{n_1} + X^{n_2} + \cdots + X^{n_s}$, then

$$\frac{f(X)}{1-X} = \left(X^{n_1} + X^{n_1+1} + \cdots + X^{n_2-1}\right) + \cdots + \left(X^{n_{s-1}} + X^{n_{s-1}+1} + \cdots + X^{n_s-1}\right) \tag{2.7}$$

if s is even, and

$$\frac{f(X)}{1-X} = \left(X^{n_1} + \cdots + X^{n_2-1}\right) + \cdots + \left(X^{n_{s-2}} + \cdots + X^{n_{s-1}-1}\right) + \sum_{n=n_s}^{\infty} X^n \tag{2.8}$$

if s is odd. On A we have a natural derivation which sends an $f(X) \in A$ to $f'(X) = \frac{df}{dX} \in A$,

$$f(X) = \sum_{n=0}^{\infty} a_n X^n \to f'(X) = \sum_{n=1}^{\infty} n a_n X^{n-1}. \tag{2.9}$$

Note that for any $f(X) \in A$ one has

$$f''(X) = 0. \tag{2.10}$$

Let us also remark that for any $f(X)$ given in the form (2.2), the condition

$$f'(X) = 0 \tag{2.11}$$

is equivalent to the condition that all the exponents n_j are even numbers.

3. The parity problem for partition functions. First approach

Let

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

By the Jacobi triple product identity [2, p. 21, Thm. 2.8], [4, p. 35, Entry 19],

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{3.1}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q| < 1.$$

In particular, if $a = -q^r$ and $b = -q^s$, where r and s are positive integers and $|q| < 1$, then, by (3.1),

$$f(-q^r,-q^s) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\{(r+s)n^2+(r-s)n\}/2} = (q^r;q^{r+s})_\infty(q^s;q^{r+s})_\infty(q^{r+s};q^{r+s})_\infty. \tag{3.2}$$

Setting $r + s = t$, define the partition function $p(r,s;n)$ by

$$\sum_{n=0}^{\infty} p(r,s;n)q^n = \frac{1}{f(-q^r,-q^s)} = \frac{1}{(q^r;q^t)_\infty(q^s;q^t)_\infty(q^t;q^t)_\infty}, \tag{3.3}$$

where $|q| < 1$. Thus, $p(r,s;n)$ denotes the number of partitions of the positive integer n into parts congruent to either r, s, or t modulo t. In particular, if $r = 2$ and $s = 1$, then

$$\sum_{n=0}^{\infty} p(2,1;n)q^n = \frac{1}{f(-q^2,-q)} = \frac{1}{(q^2;q^3)_\infty(q;q^3)_\infty(q^3;q^3)_\infty} = \frac{1}{(q;q)_\infty}, \tag{3.4}$$

i.e., $p(2,1;n) = p(n)$, the ordinary partition function; moreover, in this case (3.2) reduces to Euler's pentagonal number theorem [2, p. 11, Cor. 1.7]. The partition function $p(r,s;n)$ appears in a famous theorem of I. Schur [20, Satz V], [21, pp. 453–50], but with additional retrictions on successive summands in the partition of n.

By reducing the coefficients modulo 2 and replacing q by X in (3.2), we find that, if $1/F_{r,s}(X)$ is the image of the infinite series of (3.2) in A, then

$$\frac{1}{F_{r,s}(X)} := \sum_{n=-\infty}^{\infty} X^{(tn^2+(r-s)n)/2}, \tag{3.5}$$

which we write in the form

$$1 = F_{r,s}(X)\left(1+\sum_{n=1}^{\infty}\left(X^{(tn^2+(r-s)n)/2}+X^{(tn^2-(r-s)n)/2}\right)\right). \tag{3.6}$$

Here, $F_{r,s}(X)$ has the form

$$F_{r,s}(X) = 1 + X^{n_1} + X^{n_2} + \cdots + X^{n_j} + \cdots. \tag{3.7}$$

where, of course, the positive integers $n_1 < n_2 < \cdots$ depend on r and s. Clearly, from (3.3) and (3.7)

$$\#\{1 \le n \le N : p(r,s;n) \text{ is odd }\} = \#\{n_j \le N\} \tag{3.8}$$

and

$$\#\{1 \le n \le N : p(r,s;n) \text{ is even }\} = N - \#\{n_j \le N\}. \tag{3.9}$$

We first establish a lower bound for $\#\{n_j \le N\}$. Using (3.7), write (3.6) in the form

$$\left(\sum_{j\ge 1} X^{n_j}\right)\left(1+\sum_{n=1}^{\infty}\left(X^{(tn^2+(r-s)n)/2}+X^{(tn^2-(r-s)n)/2}\right)\right)$$
$$=\sum_{n=1}^{\infty}\left(X^{(tn^2+(r-s)n)/2}+X^{(tn^2-(r-s)n)/2}\right). \tag{3.10}$$

Asymptotically, there are $\sqrt{2N/t}$ terms of the form $X^{(tm^2+(r-s)m)/2}$ less than X^N on the right side of (3.10). For a fixed positive integer n_j, we determine how many of these terms appear in a series of the form

$$X^{n_j}\left(1+\sum_{n=1}^{\infty}\left(X^{(tn^2+(r-s)n)/2}+X^{(tn^2-(r-s)n)/2}\right)\right), \tag{3.11}$$

arising from the left side of (3.10). Thus, for a fixed $n_j < N$,we estimate the number of integral pairs (m,n) of solutions of the equation

$$n_j + \frac{tn^2}{2} + \frac{(r-s)n}{2} = \frac{tm^2}{2} + \frac{(r-s)m}{2}, \tag{3.12}$$

which we put in the form

$$2n_j = tm^2 + (r-s)m - tn^2 - (r-s)n = (m-n)(tm+tn+r-s). \tag{3.13}$$

The number of divisors of $2n_j$ is $O_c\left(N^{\frac{c}{\log\log N}}\right)$ for any fixed $c > \log 2$, by a result of S. Wigert [23] and S. Ramanujan [17], [18, p. 80]. Thus each of the numbers $m-n$ and $tm+tn+r-s$ can assume at most $N^{\frac{c}{\log\log N}}$ values. Since the pair $(m-n, tm+tn+r-s)$ uniquely determines the pair (m,n), it follows that the number of solutions to (3.13) is $O_c\left(N^{\frac{c}{\log\log N}}\right)$, where c is any constant such that $c > 2\log 2$. A similar argument can be made for the terms in (3.10) of the form $X^{(tm^2-(r-s)m)/2}$.

Returning to (3.10) and (3.11), we see that each series of the form (3.11) has at most $N^{\frac{c}{\log\log N}}$ terms $X^{(tm^2+(r-s)m)/2}$ up to X^N that appear on the right side of (3.10). It follows that there are at least $N^{\frac{1}{2}-\frac{c}{\log\log N}}$ numbers $n_j \le N$ that are needed to match all the $\left[\sqrt{2N/t}\right]$ terms $X^{(tm^2+(r-s)m)/2}$ up to X^N on the right side of (3.10). Again, an analogous argument holds for terms of the form $X^{(tm^2-(r-s)m)/2}$. We have therefore proved the following theorem.

Theorem 3.1. *For each fixed c with $c > 2\log 2$ and N sufficiently large,*

$$\#\{n \le N : p(r,s;n) \text{ is odd }\} \ge N^{\frac{1}{2}-\frac{c}{\log\log N}}. \tag{3.14}$$

Corollary 3.2. *For each fixed c with $c > 2\log 2$ and N sufficiently large,*

$$\#\{n \le N : p(n) \text{ is odd }\} \ge N^{\frac{1}{2}-\frac{c}{\log\log N}}. \tag{3.15}$$

Next, we provide a lower bound for $\#\{n \le N : p(r,s;n)\ \text{ is even}\}$. Let $\{m_1, m_2, \dots\}$ be the complement of the set $\{0, n_1, n_2, \dots\}$ in the set of natural numbers $\{0, 1, 2, \dots\}$, and define

$$G_{r,s}(X) = X^{m_1} + X^{m_2} + \cdots \in A. \tag{3.16}$$

Then

$$G_{r,s}(X) + F_{r,s}(X) = 1 + X + X^2 + \cdots + X^k + \cdots = \frac{1}{1-X}. \tag{3.17}$$

Since, by (3.9),

$$\#\{m_j \le N\} = N - \#\{n_j \le N\} = \{n \le N : p(r,s;n) \text{ is even }\}, \tag{3.18}$$

we need a lower bound for $\#\{m_j \le N\}$. Using (3.17) in (3.6), we find that

$$\begin{aligned}
&1 + G_{r,s}(X)\left(1 + \sum_{n=1}^{\infty}\left(X^{(tn^2-(r-s)n)/2} + X^{(tn^2+(r-s)n)/2}\right)\right)\\
=&\frac{1}{1-X}\left(1 + \sum_{n=1}^{\infty}\left(X^{(tn^2-(r-s)n)/2} + X^{(tn^2+(r-s)n)/2}\right)\right)\\
=&\frac{1}{1-X}\big(1 + X^{(t-(r-s))/2} + X^{(t+(r-s))/2} + X^{(4t-2(r-s))/2} + X^{(4t+2(r-s))/2} + \cdots\\
&+X^{(tn^2-(r-s)n)/2} + X^{(tn^2+(r-s)n)/2} + X^{(t(n+1)^2-(r-s)(n+1))/2} + \cdots\big).
\end{aligned} \tag{3.19}$$

Here we have assumed without loss of generality (since $f(a,b) = f(b,a)$) that $r \ge s$, and have put the terms of the sum on the right side of (3.19) in increasing order of their exponents. By (2.6), we see that the right side of (3.19) equals

$$\begin{aligned}
&1+\cdots+X^{(t-r+s)/2-1}+\left(X^{(t+r-s)/2}+\cdots+X^{(4t^2-2(r-s))/2-1}\right)\\
&+\left(X^{(4t^2+2(r-s))/2}+\cdots+X^{(9t^2-3(r-s))/2-1}\right)+\cdots\\
&+\left(X^{(t(n-1)^2+(r-s)(n-1))/2}+\cdots+X^{(tn^2-(r-s)n)/2-1}\right)\\
&+\left(X^{(tn^2+(r-s)n)/2}+\cdots+X^{(t(n+1)^2-(r-s)(n+1))/2-1}\right)+\cdots. \qquad (3.20)
\end{aligned}$$

Note that if $(t-r+s)/2-1=0$, i.e., $s=1$, then the term $X^{(t-r+s)/2-1}$ is to be deleted from (3.20). Since the gap between $X^{(tn^2-(r-s)n)/2}$ and $X^{(tn^2+(r-s)n)/2}$ contains $(r-s)n$ terms that are missing from the series (3.20), and this comes after a segment of

$$\frac{tn^2}{2}-\frac{(r-s)n}{2}-\left(\frac{t(n-1)^2}{2}+\frac{(r-s)(n-1)}{2}\right)=2sn-s$$

terms that do appear in (3.20), we see that (3.20) contains asymptotically

$$\frac{2s}{2s+(r-s)}N=\frac{2s}{t}N$$

terms up to X^N. Now the sum in parentheses on the left side of (3.19) has asymptotically $2\sqrt{2N/t}$ nonzero terms up to X^N. Thus $G_{r,s}(X)$ must have at least $s\sqrt{N/(2t)}$ nonzero terms up to X^N in order for the left side of (3.19) to have at least $2sN/t$ terms up to X^N to match those on the right side of (3.19). We have therefore proved the following result.

Theorem 3.3. *For any fixed constant c such that $c<s/\sqrt{2t}$, and for N sufficiently large,*

$$\#\{n\le N : p(r,s;n) \text{ is even }\}\ge c\sqrt{N}. \qquad (3.21)$$

Corollary 3.4. *For each fixed constant c with $c<1/\sqrt{6}$, and for N sufficiently large,*

$$\#\{n\le N : p(n) \text{ is even }\}\ge c\sqrt{N}. \qquad (3.22)$$

We now give further applications of Theorems 3.1 and 3.3.

Recall Jacobi's identity [4, p. 39, Entry 24(ii)]

$$(q;q)_\infty^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}, \qquad |q|<1. \qquad (3.23)$$

Observe that

$$\frac{1}{(q;q)_\infty^3}=\sum_{n=0}^{\infty}c_3(n)q^n,$$

where $c_3(n)$ denotes the number of partitions of the positive integer n into parts with three distinct colors. The image of the right side of (3.23) in A is

$$\sum_{n=0}^{\infty}X^{n(n+1)/2}.$$

Hence, by the same arguments that we used to prove Theorem 3.1, we can derive the following theorem.

Theorem 3.5. *For each fixed c, with* $c > 2\log 2$*, and for* N *sufficiently large,*

$$\#\{n \leq N : c_3(n) \text{ is odd}\} \geq N^{\frac{1}{2}-\frac{c}{\log\log N}}.$$

In order to obtain a result for $c_3(n)$ like that in Theorem 3.3, we proceed as we did with (3.19) to find that

$$1 + G(X)\left(1+\sum_{n=1}^{\infty} X^{n(n+1)/2}\right) = \frac{1}{1-X}\left(1+\sum_{n=1}^{\infty} X^{n(n+1)/2}\right), \tag{3.24}$$

where $G(X)$ is the obvious analogue of $G_{r,s}(X)$ which we defined in (3.16). Next, observe that, by (2.6), the right hand side of (3.24) can be written in the form

$$\begin{aligned} &1 + \left(X^3 + X^4 + X^5\right) + \left(X^{10} + X^{11} + X^{12} + X^{13} + X^{14}\right) \\ &\quad + \cdots + \left(X^{n(2n+1)} + \cdots + X^{(2n+1)(n+1)-1}\right) + \cdots . \end{aligned} \tag{3.25}$$

The gap of missing terms between $X^{(2n-1)n}$ and $X^{n(2n+1)}$ consists of $2n$ terms, and this comes after a segment containing $2n-1$ terms that do appear. Thus, the series (3.25) has asymptotically $N/2$ terms up to X^N. The sum on the left side of (3.24) has asymptotically $\sqrt{N/2}$ terms up to X^N. Thus, $G(X)$ must have at least $\sqrt{N/2}$ terms up to X^N in order for the left side of (3.24) to have at least $N/2$ terms. Hence, we have proved the following theorem.

Theorem 3.6. *For each fixed c with* $c < 1/\sqrt{2}$*, and for* N *sufficiently large,*

$$\#\{n \leq N : c_3(n) \text{ is even}\} \geq c\sqrt{N}.$$

The ideas in this section can be generalized to any series that is *superlacunary.*

Definition 3.7. *A power series is superlacunary if it has the form*

$$\sum_{n=-\infty}^{\infty} d_n(a,b,c)q^{an^2+bn+c},$$

where a, b, *and* c *are integers, with* $a > 0$*, and the numbers* $d_n(a,b,c)$ *are constants.*

See a paper by K. Ono and S. Robins [15, Thm. 2] wherein they characterize superlacunary series that are certain kinds of cusp forms. The most interesting superlacunary series are those that can be represented as eta-products, i.e., as products of the form $(q^a;q^a)_\infty$, where a is a positive integer. We describe two such examples.

In his notebooks [19, Chap. 17, Entries 8 (ix), (x)], [4, pp. 114–115], Ramanujan recorded the elegant identities

$$\sum_{n=-\infty}^{\infty} (6n+1)q^{3n^2+n} = \frac{(q;q)_\infty^5}{(q^2;q^2)_\infty^2} \tag{3.26}$$

and

$$\sum_{n=-\infty}^{\infty} (3n+1)q^{3n^2+2n} = \frac{(q;q)_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}. \tag{3.27}$$

These identities can be easily proved by employing the quintuple product identity; see [3, pp. 19–20]. The identity (3.26) was perhaps first rediscovered by N. J. Fine [6, p. 83], although he did not publish his proof for several years. The first published proof of (3.27) is due to B. Gordon [7].

Define the partition functions $P_1(n)$ and $P_2(n)$ by

$$\sum_{n=0}^{\infty} P_1(n)q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^5} = \frac{1}{(q;q)_\infty^3(q;q^2)_\infty^2}$$

and

$$\sum_{n=0}^{\infty} P_2(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2(q^4;q^4)_\infty^2} = \frac{1}{(q;q)_\infty(q;q^2)_\infty(q^4;q^4)_\infty^2},$$

respectively. Observe that $P_1(n)$ is the number of partitions of the positive integer n into parts with three distinct colors and into odd parts with two distinct colors, with the summands therefore having a total of five distinct colors. Also, $P_2(n)$ is the number of partitions of n into unrestricted parts of one color, odd parts of one color, and parts which are multiples of four having two distinct colors, for a total of four colors.

The images of the left sides of (3.26) and (3.27) in A are

$$\sum_{n=-\infty}^{\infty} X^{3n^2+n} \qquad \text{and} \qquad \sum_{n=-\infty}^{\infty} X^{12n^2+4n}, \tag{3.28}$$

respectively. By the methods of this section, we can therefore prove analogues of Theorem 3.1, wherein $p(r,s;n)$ is replaced by either $P_1(n)$ or $P_2(n)$. By the pentagonal number theorem, the image of (3.26) in (3.28) is the same as that of $(q^2;q^2)_\infty$. Hence, an analogue of Corollary 3.4 can be stated in which $p(n)$ is replaced by $P_1(n)$ and N is replaced by $N/2$. Also, observe that the image of (3.27) in (3.28) is $f(-X^{16},-X^8)$. Thus, from Theorem 3.3, we obtain the next corollary.

Corollary 3.8. *For any fixed c with $c < 2/\sqrt{3}$, and for N sufficiently large,*

$$\#\{n \le N : P_2(n) \text{ is even}\} \ge c\sqrt{N}.$$

4. The parity problem for $p(n)$. Second approach

We start from the generating function identity (3.4). Let us compute

$$\log F(q) = -\sum_{n=1}^{\infty} \log(1-q^n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{q^{nm}}{m}. \tag{4.1}$$

By applying the operator $q\frac{d}{dq}$ we obtain

$$\frac{qF'(q)}{F(q)} = q\frac{d}{dq}(\log F(q)) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nq^{nm} = \sum_{k=1}^{\infty} q^k \sum_{n|k} n = \sum_{k=1}^{\infty} \sigma(k)q^k, \tag{4.2}$$

where $\sigma(k)$ denotes the sum of divisors of k. If we denote by $H(X)$ the image in A of the series $\sum_{k=1}^{\infty} \sigma(k)X^k$, then from (4.2) we derive an equality in A, namely,

$$XF'(X) = F(X)H(X). \tag{4.3}$$

It is easy to see that

$$H(X)=\sum_{n,r\geq 0} X^{2^r(2n+1)^2}=\sum_{n=1}^{\infty} X^{n^2}+\sum_{n=1}^{\infty} X^{2n^2}. \tag{4.4}$$

We write $F(X)$ in the form $F(X)=1+X^{n_1}+X^{n_2}+\cdots$, and choose a large positive integer N. We proceed to derive from (4.3) a lower bound for $\#\{n_j\leq N\}$. Let us write (4.3) in the form

$$XF'(X)+(X^{n_1}+X^{n_2}+\cdots)H(X)=H(X)=\sum_{n=1}^{\infty} X^{n^2}+\sum_{n=1}^{\infty} X^{2n^2}. \tag{4.5}$$

Each of the terms X^{n^2} from the right side of (4.5) must also appear on the left side of (4.5). There are $[\sqrt{N}]$ such terms with $n^2\leq N$. If at least half of them are canceled by terms from the series $XF'(X)$, then $F'(X)$ has at least $[\sqrt{N}]$ terms up to X^N, and hence $F(X)$ has at least $[\sqrt{N}]$ terms up to X^N. This gives us the desired lower bound for $\#\{n_j\leq N\}$ in this case.

Assume now that less than half of the terms X^{n^2} with $n^2\leq N$ are canceled by terms from $XF'(X)$. It follows that at least $\sqrt{N}/2$ such terms are left to be canceled by terms from the series $(X^{n_1}+X^{n_2}+\cdots)H(X)$. Let us see how many terms of the form X^{m^2} with $m^2\leq N$ may appear in a series of the form $X^{n_j}H(X)$ for a fixed n_j. Thus we look separately at two diophantine equations in positive integers n,m, namely,

$$n_j+n^2=m^2, \tag{4.6}$$

and

$$n_j+2n^2=m^2. \tag{4.7}$$

If n,m satisfy (4.6), then both $n-m$ and $n+m$ are divisors of n_j. Thus each of these numbers can take at most $N^{\frac{c}{\log\log N}}$ values for any fixed $c>\log 2$ and any N large enough. The pair (n,m) being determined by $n+m$ and $n-m$, it follows that the equation (4.6) has at most $N^{\frac{2c}{\log\log N}}$ solutions.

We now turn to the equation (4.7). Here of course we have to use the constraint that $m^2\leq N$; otherwise we may have infinitely many solutions. We count the number of solutions as follows. By (4.7) it follows that $m+n\sqrt{2}$ divides n_j in the ring $\mathbf{Z}[\sqrt{2}]$, and the ideal generated by $m+n\sqrt{2}$ has norm n_j. We first count the number of ideals in $\mathbf{Z}[\sqrt{2}]$ generated by $m+n\sqrt{2}$ as n,m run over the set of solutions of the equation (4.7). Let $n_j=p_1^{k_1}\cdots p_s^{k_s}$ be the decomposition into prime factors of n_j. We take each prime factor p_i and look at its decomposition in $\mathbf{Z}[\sqrt{2}]$. There are three alternatives. The first is that p_i ramifies in $\mathbf{Z}[\sqrt{2}]$, in which case the ideal (p_i) is the square of a prime ideal of $\mathbf{Z}[\sqrt{2}]$. This only happens for $p_i=2$. The corresponding prime ideal in $\mathbf{Z}[\sqrt{2}]$ will be generated by $\sqrt{2}$. The second case is when p_i is inert in $\mathbf{Q}(\sqrt{2})$, that is, when (p_i) is a prime ideal in $\mathbf{Z}[\sqrt{2}]$. In this case, if n,m satisfy (4.7), then in $\mathbf{Z}[\sqrt{2}]$, p_i divides one of the factors $n+m\sqrt{2}$ or $n-m\sqrt{2}$. Say it divides the first one. Then we have

$$m+n\sqrt{2}=p_i(a+b\sqrt{2})$$

for some $a, b \in \mathbf{Z}$, from which it follows that p_i divides both n and m. For such a prime p_i, the exponent k_i needs to be an even number in order for the equation (4.7) to have solutions. Moreover, if k_i is even, then for any n, m satisfying (4.7) the exponent of the prime ideal (p_i) in each of the factors $m + n\sqrt{2}$ and $m - n\sqrt{2}$ equals $k_i/2$. The third case is when p_i splits in $\mathbf{Z}[\sqrt{2}]$, in which case one has $(p_i) = P_i P_i'$ for some prime ideals P_i, P_i' of $\mathbf{Z}[\sqrt{2}]$. For such a prime p_i, if n, m satisfy (4.7) and P_i divides $m + n\sqrt{2}$ then P_i' divides $m - n\sqrt{2}$.

Let us rewrite now the prime decomposition of n_j in the form

$$n_j = 2^k p_1^{k_1} \cdots p_r^{k_r} q_1^{2l_1} \cdots q_t^{2l_t}, \tag{4.8}$$

where $p_1, \ldots, p_r$ are the prime divisors of n_j which split in the field $\mathbf{Q}(\sqrt{2})$ and $q_1, \ldots, q_t$ are the prime divisors of n_j which are inert in $\mathbf{Q}(\sqrt{2})$. As remarked above, we may assume that the exponents of $q_1, \ldots, q_t$ in n_j are even numbers; otherwise there are no solutions to (4.7).

Next, the decomposition of n_j into prime ideals in $\mathbf{Z}[\sqrt{2}]$ will be

$$n_j = (\sqrt{2})^{2k} P_1^{k_1} P_1'^{k_1} \cdots P_r^{k_r} P_r'^{k_r} (q_1)^{2l_1} \cdots (q_t)^{2l_t}. \tag{4.9}$$

Now for any integers n, m satisfying (4.7), the exponent of $(\sqrt{2})$ in $m + n\sqrt{2}$ must equal k, and for each $i = 1, \ldots, t$, the exponent of (q_i) in $m + n\sqrt{2}$ must equal l_i. For $i = 1, \ldots, r$, if b_i denotes the exponent of P_i in $m+n\sqrt{2}$, then the exponent of P_i in the remaining factor $m-n\sqrt{2}$ will equal $k_i - b_i$, and so, the exponent of P_i' in $m+n\sqrt{2}$ will also equal $k_i - b_i$. That is to say, the exponents of $P_1', \ldots, P_r'$ in $m + n\sqrt{2}$ are uniquely determined by the exponents of $P_1, \ldots, P_r$, respectively. Since b_i may assume values from 0 to k_i for $i = 1, \ldots, r$, it follows that there are only $(1 + k_1) \cdots (1 + k_r)$ possible choices for the ideal $(m + n\sqrt{2})$. Note that $(1 + k_1) \cdots (1 + k_r)$ is not larger than the number of divisors of n_j, which in turn is not larger than $N^{\frac{c}{\log\log N}}$. Thus the number of ideals of the form $(m+n\sqrt{2})$, with n, m satisfying (4.7), is bounded by $N^{\frac{c}{\log\log N}}$. Next, let us see how many solutions n, m produce the same ideal $(n + m\sqrt{2})$. If n, m and, respectively, n', m' satisfy (4.7) and the ideals $(n + m\sqrt{2})$ and $(m' + n'\sqrt{2})$ coincide, then there exists a unit u in the ring $\mathbf{Z}[\sqrt{2}]$ such that

$$m' + n'\sqrt{2} = u(m + n\sqrt{2}). \tag{4.10}$$

Since n, m, n', m' are positive integers, u will be positive. If we denote by $\eta > 1$ the fundamental unit (actually $\eta = 1 + \sqrt{2}$), then we have $u = \eta^v$ for some integer v. Since both factors $m' + n'\sqrt{2}$ and $m + n\sqrt{2}$ are $O(\sqrt{N})$, from (4.10) it follows that $|v|$ is bounded by $\log N$. Thus the number of solutions n, m of (4.7) which produce the same ideal $(m + n\sqrt{2})$ is bounded by $\log N$. In conclusion, the number of solutions to (4.6) and (4.7) are both bounded by $N^{\frac{2c}{\log\log N}}$. As a consequence, at most $N^{\frac{2c}{\log\log N}}$ terms of the form X^{m^2} with $m^2 \le N$ may be canceled by each of the series of the form $x^{n_j} H(X)$. Hence one must have

$$\#\{n_j \le N\} \ge N^{\frac{1}{2} - \frac{2c}{\log\log N}} \tag{4.11}$$

in order for all the remaining $\sqrt{N}/2$ terms up to X^N from the right side of (4.5) to be canceled by the left side of (4.5). This completes the second proof of Theorem 3.1.

We now provide another proof for Theorem 3.3. We start by introducing the same series

$$G(X) = X^{m_1} + X^{m_2} + \cdots \in A, \tag{4.12}$$

where the set $\{m_1, m_2, \dots\}$ is the complement of the set $\{0, n_1, n_2, \dots\}$ in the set of natural numbers. We have as before

$$G(X) + F(X) = \frac{1}{1-X}\,. \tag{4.13}$$

Again we need a lower bound for $\#\{m_j \le N\}$, where N is a fixed large positive integer. This time, instead of (3.6) we are going to use the equality (4.3). Differentiating (4.13) we have

$$G'(X) + F'(X) = \frac{1}{(1-X)^2} \tag{4.14}$$

In (4.13) and (4.14) we solve for $F(X)$ and $F'(X)$ respectively, then introduce these expressions in (4.3) to obtain

$$\frac{X}{(1-X)^2} - XG'(X) = XF'(X) = F(X)H(X) = \left(\frac{1}{1-X} - G(X)\right)H(X). \tag{4.15}$$

Multiplying both sides of (4.15) by $1-X$, we see that

$$\frac{X}{1-X} - X(1-X)G'(X) = \left(1-(1-X)G(X)\right)H(X), \tag{4.16}$$

which we write in the form

$$X(1-X)G'(X) + \left(1-(1-X)G(X)\right)H(X) = X + X^2 + X^3 + \cdots + X^n + \cdots\,. \tag{4.17}$$

Denote $M = \#\{m_j \le N\}$. Then $(1-(1-X)G(X))$ has at most $1+2M$ nonzero terms up to X^N. Since $H(X)$ has at most $(1+1/\sqrt{2})\sqrt{N}$ nonzero terms up to X^N, we deduce that

$$\#\{\text{nonzero terms in}\,(1-(1-X)G(X))H(X) \ \text{ up to}\,X^N\,\} \le (1+2M)(1+1/\sqrt{2})\sqrt{N}. \tag{4.18}$$

Similarly, $G'(X)$ has at most M nonzero terms up to X^N, therefore

$$\#\{\text{nonzero terms in}\,X(1-X)G'(X) \ \text{ up to}\,X^N\,\} \le 2M. \tag{4.19}$$

Since the right side of (4.17) has N nonzero terms up to X^N, from (4.17), (4.18) and (4.19) we deduce that

$$2M + (1+2M)\left(1+\frac{1}{\sqrt{2}}\right)\sqrt{N} \ge N. \tag{4.20}$$

This completes a second proof of Theorem 3.3. More precisely, from (4.20) it follows that for any constant $c < 1 - 1/\sqrt{2}$ and any N large enough one has

$$\#\{n \le N : p(n) \text{ is even }\} \ge c\sqrt{N}. \tag{4.21}$$

5. Results for more general partition functions

In this section we use the method employed in the previous section to study the parity of more general partition functions. Denote $\mathbf{N} = \{0, 1, 2, \dots\}$ and $\mathbf{N}^* = \{1, 2, \dots\}$. For any subset S of $\mathbf{N}^*$ and any positive integer n we denote by $p_S(n)$ the number of partitions of n with all parts in S. The object of this section is to study the parity of $p_S(n)$ for a given S, by means of the associated differential equation obtained below, which is similar to (4.3).

As before, we start from the generating function identity

$$F_S(q) := 1 + \sum_{n=1}^{\infty} p_S(n) q^n = \prod_{n \in S} \frac{1}{1 - q^n}\,. \tag{5.1}$$

We again compute

$$\log F_S(q) = -\sum_{n \in S} \log(1 - q^n) = \sum_{n \in S} \sum_{m=1}^{\infty} \frac{q^{nm}}{m}\,. \tag{5.2}$$

By applying the operator $q\frac{d}{dq}$ we obtain

$$\frac{qF_S'(q)}{F_S(q)} = q\frac{d}{dq}(\log F_S(q)) = \sum_{n \in S} \sum_{m=1}^{\infty} nq^{nm} = \sum_{k=1}^{\infty} q^k \sum_{n \in S, n|k} n = \sum_{k=1}^{\infty} \sigma_S(k) q^k, \tag{5.3}$$

where we denote by $\sigma_S(k)$ the sum of those divisors of k which belong to S. Let $H_S(X)$ denote the image in A of the series $\sum_{k=1}^{\infty} \sigma_S(k) q^k$. Then from (5.3) we derive

$$XF_S'(X) = F_S(X) H_S(X). \tag{5.4}$$

In the previous section we relied heavily on the properties of the particular function $H(X)$ given by (4.4). We need to understand which properties the series $H_S(X)$ needs to satisfy in order to be able to obtain results on the parity of $p_S(n)$, on the same lines as in the proofs from the previous section. We have seen that a certain diophantine equation plays an important role in the proof of the lower bound for the number of $n \le N$ for which $p(n)$ is odd. Also, the fact that the series $H(X)$ from (4.4) does not have too may nonzero terms up to X^N played an essential role in the proof of the lower bound for the number of $n \le N$ for which $p(n)$ is even. With these in mind, we introduce two counting functions. For any positive integer N and any element

$$g(X) = X^{m_1} + X^{m_2} + \cdots + X^{m_j} + \cdots \in A, \tag{5.5}$$

we set

$$B(g(X), N) := \#\{m_j \le N\}. \tag{5.6}$$

Thus if the sequence of nonzero terms from the right side of (5.5) is sparse, $B(g(X), N)$ will be significantly smaller than N, as $N \to \infty$. In particular, we say that *the sequence of nonzero terms of $g(X)$ has zero density*, provided one has

$$\lim_{n \to \infty} \frac{B(g(X), N)}{N} = 0. \tag{5.7}$$

The second counting function is defined as follows. For any $g(X)$ as in (5.5), any positive integer N and any integer a with $1 \leq a \leq N$, denote by $D(g(X), N, a)$ the number of solutions of the equation

$$a = m_j - m_i \tag{5.8}$$

where the exponents m_i, m_j appear on the right side of (5.5), and $m_j \leq N$. Then set

$$D(g(X), N) = \max_{1 \leq a \leq N} D(g(X), N, a).$$

If there is at most one term on the right side of (5.5), that is, if $g(X) = 0$ or $g(X) = X^m$ for some integer $m \geq 0$, then evidently $D(g(X), N) = 0$ for any N. For any element $g(X)$ of A which has at least two nonzero terms on the right side of (5.5), one has $D(g(X), N) \geq 1$ for N large enough. Note that for any $g(X) \in A$ and any positive integer N one has

$$D(g(X), N) \leq B(g(X), N). \tag{5.9}$$

Let now S be a subset of $\mathbf{N}^*$. We would like to obtain information on the parity of the associated partition function $p_S(n)$ in terms of the behavior of the counting functions $B(H_S(X), N)$ and $D(H_S(X), N)$. We assume $H_S(X)$ has at least two terms so that $D(H_S(X), N) \geq 1$ for N large enough, and choose such an N.

Let us consider first the problem of providing a lower bound for the number of integers $n \leq N$ for which $p_S(n)$ is odd. We write

$$F_S(X) = 1 + X^{n_1} + X^{n_2} + \cdots + X^{n_j} + \cdots, \tag{5.10}$$

with $1 \leq n_1 < n_2 < \cdots < n_j < \ldots$, and

$$H_S(X) = X^{m_1} + X^{m_2} + \cdots + X^{m_j} + \cdots, \tag{5.11}$$

with $1 \leq m_1 < m_2 < \cdots < m_j < \ldots$. The sums on the right side of (5.10) and (5.11) may be finite or infinite. The problem is to provide a lower bound for $\#\{n_j \leq N\}$. We use the equality (5.4), which we write in the form

$$XF'_S(X) + (X^{n_1} + \cdots + X^{n_k} + \cdots)\, H_S(X) = H_S(X) = X^{m_1} + X^{m_2} + \cdots + X^{m_j} + \cdots. \tag{5.12}$$

The number of nonzero terms up to X^N on the right side of (5.12) equals $B(H_S(X), N)$. These terms must also appear on the left side of (5.12). If at least half of these terms do appear in $XF'_S(X)$, then, since $F(X)$ has at least as many nonzero terms up to X^N as $XF'(X)$ has, it follows that

$$\#\{n_j \leq N\} \geq \frac{B(H_S(X), N)}{2}. \tag{5.13}$$

Assume now that less than half of the nonzero terms from the right side of (5.12) appear in $XF'(X)$. Then at least half of these terms appear in the series

$$(X^{n_1} + \cdots + X^{n_k} + \cdots)(X^{m_1} + \cdots + X^{m_i} + \cdots). \tag{5.14}$$

For each fixed $n_k \leq N$, the number of solutions of the equation

$$n_k + m_i = m_j \tag{5.15}$$

is bounded by $D(H_S(X), N)$. Therefore each series of the form $X^{n_k} H(X)$ contains at most $D(H_S(X), N)$ terms from the right side of (5.12). We deduce that there must be at least $B(H_S(X),N)/\{2D(H_S(X),N)\}$ terms X^{n_k} with $n_k \le N$ in order for all the terms up to X^N from the right side of (5.12) to also appear on the left side of that equality. If we compare this with (5.13) we see that in all cases one has

$$\#\{n_j \le N\} \ge \frac{B(H_S(X),N)}{2D(H_S(X),N)}\,. \tag{5.16}$$

This gives the required lower bound for the number of integers $n \le N$ for which $p_S(n)$ is odd, assuming that $H_S(X)$ has at least two nonzero terms. We claim that if $H_S(X)$ is not zero, this series always has at least two nonzero terms.

Indeed, let us assume that there exists a subset S of $\mathbf{N}^*$ for which $H_S(X)$ consists of exactly one nonzero term, that is, $H_S(X) = X^m$ for some integer $m \ge 1$. Then $\sigma_S(m)$ is odd, and $\sigma_S(n)$ is even for any integer $n \ge 1$ with $n \ne m$. Since $\sigma_S(m)$ is odd, it follows that m has an odd number of odd divisors which belong to S. As a consequence, at least one element of S is odd. Let m_0 be the smallest odd integer which belongs to S. Then by the definition of σ_S it is clear that $\sigma_S(m_0)$ is odd, and so m_0 coincides with m. Now let us consider the parity of $\sigma_S(2m)$. Any odd divisor of $2m$ must also divide m. But m is the smallest odd element of S. It follows that m is the only odd divisor of $2m$ which belongs to S, and hence $\sigma_S(2m)$ is odd. The contradiction obtained shows that $H_S(X)$ always has at least two nonzero terms, provided $H_S(X) \ne 0$. In fact, the reasoning above shows that if S has at least one odd element, and if m_0 is the smallest odd integer which belongs to S, then $\sigma_S(2^k m_0)$ is odd for any integer $k \ge 0$. This implies that for any constant $c < 1/\log 2$ and any S which contains at least one odd element one has

$$B(H_S(X),N) \ge c \log N \tag{5.17}$$

for all N large enough.

On the other hand, if S consists only of even numbers, then $\sigma_S(n)$ will be even for any integer $n \ge 1$, and so $H_S(X)$ will be identically zero. In that case the equality (5.4) reduces to

$$XF_S'(X) = 0. \tag{5.18}$$

This implies that all the terms from $F_S(X)$ have even exponents, but no information on the number of nonzero terms of $F_S(X)$ up to X^N can be derived from (5.18). For this reason, in what follows we will only consider sets S which have at least one odd element. Actually this is not much of a restriction, in the sense that the general case can be reduced to this one. More precisely, let S be a subset of $\mathbf{N}^*$ which consists of even numbers only. Let 2^r denote the largest power of 2 which divides each of the elements of S. Say $S = \{2^r a_1, 2^r a_2, \dots\}$. Consider the set $S' = \{a_1, a_2, \dots\}$. Evidently $p_S(n) = 0$ unless n is a multiple of 2^r. If $n = 2^r k$, then clearly $p_S(n) = p_{S'}(k)$. In particular the parity problem for $p_S(n)$ reduces to the analogous problem for $p_{S'}(k)$, and the new set S' is of the type we want, in the sense that it contains at least one odd element. These being said, we now state our lower bound for the number of $n < N$ for which $p_S(n)$ is odd, where S is of the type above.

Theorem 5.1. *Let S be a set of positive integers containing at least one odd element. Then for all N large enough,*

$$\#\{n \leq N : p_S(n) \text{ is odd }\} \geq \frac{B(H_S(X), N)}{2D(H_S(X), N)}. \tag{5.19}$$

We now turn to the problem of providing lower bounds for the number of integers $n \leq N$ for which $p_S(n)$ is even, where S is a given set of positive integers. Let $F_S(X)$ and $H_S(X)$ be as in (5.10) and (5.11), respectively. Let $\{r_1, r_2, \dots\}$ denote the complement of the set $\{n_1, n_2, \dots\}$ in $\mathbf{N}$, and define

$$G_S(X) = X^{r_1} + X^{r_2} + \cdots \in A. \tag{5.20}$$

Fix a large integer N. We need a lower bound for $\#\{r_j \leq N\}$. As before, one has

$$G_S(X) + F_S(X) = \frac{1}{1-X}. \tag{5.21}$$

Differentiating (5.21) we obtain

$$G_S'(X) + F_S'(X) = \frac{1}{(1-X)^2}. \tag{5.22}$$

As in the previous section, at this point we solve for $F_S(X)$ and $F_S'(X)$ in (5.21) and (5.22), respectively, and then introduce these expressions in (5.4). We find as before that

$$\frac{X}{(1-X)^2} - XG_S'(X) = \left(\frac{1}{1-X} - G_S(X)\right) H_S(X). \tag{5.23}$$

Multiplying both sides by $1-X$, we find that

$$\frac{X}{1-X} - X(1-X)G_S'(X) = (1-(1-X)G_S(X))H_S(X), \tag{5.24}$$

which we write in the form

$$X(1-X)G_S'(X) + (1-(1-X)G_S(X))H_S(X) = X + X^2 + \cdots + X^n + \cdots. \tag{5.25}$$

The right side of (5.25) has N nonzero terms up to X^N. Let $M = \#\{r_j \leq N\}$. Then $G_S'(X)$ has at most M nonzero terms up to X^N, and hence the series $X(1-X)G_S'(X)$ has at most $2M$ nonzero terms up to X^N. Next, the series $1-(1-X)G_S(X)$ has at most $1+2M$ nonzero terms up to X^N, while the number of nonzero terms up to X^N in $H_S(X)$ equals $B(H_S(X), N)$. Therefore the product $(1-(1-X)G_S(X))H_S(X)$ will have at most $(1+2M)B(H_S(X), N)$ terms. Hence the total number of nonzero terms up to X^N on the left side of (5.25) is at most $2M + (1+2M)B(H_S(X), N)$. It follows that

$$2M + (1+2M)B(H_S(X), N) \geq N, \tag{5.26}$$

which implies that

$$\#\{r_j \leq N\} \geq \frac{N - B(H_S(X), N)}{2(1 + B(H_S(X), N))}. \tag{5.27}$$

We therefore have obtained the following result.

Theorem 5.2. *For any set S of positive integers and for any positive integer N,*

$$\#\{n \le N : p_S(n) \text{ is even } \} \ge \frac{N - B(H_S(X), N)}{2(1 + B(H_S(X), N))} \,. \tag{5.28}$$

Theorems 5.1 and 5.2 provide lower bounds for the number of integers $n \le N$ for which $p_S(n)$ is odd, respectively even, in terms of the counting functions $B(H_S(X), N)$ and $D(H_S(X), N)$. Some applications will be given in the next sections, where we study certain particular classes of partition functions. We end this section with the following result.

Theorem 5.3. *Let S be a set of positive integers containing at least one odd element, for which the sequence of nonzero terms of $H_S(X)$ has zero density. Then there are infinitely many positive integers n for which $p_S(n)$ is even, and there are infinitely many positive integers n for which $p_S(n)$ is odd.*

The fact that there are infinitely many positive integers N for which $p_S(N)$ is even, follows directly from Theorem 5.2, on combining (5.28) with (5.7). It remains to show that there are infinitely many integers n for which $p_S(n)$ is odd. Notations being as in (5.10) and (5.11), let us assume that there are only finitely many integers n for which $p_S(n)$ is odd. Then $F_S(X)$ is a polynomial in X, say

$$F_S(X) = 1 + X^{n_1} + X^{n_2} + \cdots + X^{n_k}. \tag{5.29}$$

Since the sequence of nonzero terms of $H_S(X)$ is assumed to have zero density, there must be gaps larger than n_k between consecutive terms in $H_S(X)$. Let us choose such a gap, say

$$H_S(X) = X^{m_1} + \cdots + X^{m_s} + X^{m_{s+1}} + \cdots \,, \tag{5.30}$$

where $m_{s+1} - m_s > n_k$. Consider now the product $F_S(X)H_S(X)$. The point here is that the term $X^{n_k+m_s}$, obtained by multiplying X^{n_k} from the right side of (5.29) with X^{m_s} from the right side of (5.30), cannot be canceled by any other term. Therefore $X^{n_k+m_s}$ appears as one of the nonzero terms on the right side of (5.4). Clearly no term from the left side of (5.4) can have such a high exponent. In conclusion $F_S(X)$ must have infinitely many nonzero terms, and this completes the proof of the theorem.

We close this section with an application of Theorem 5.3. Let S consist of those positive integers with all prime factors congruent to 3 (mod 4). Thus S is the monoid generated by the prime numbers congruent to 3(mod 4). We need an explicit description of $H_S(X)$. It is easy to see, for any given positive integer m, that X^m appears as a nonzero term in $H_S(X)$ if and only if m can be expressed as a sum of two squares. Then we may apply Theorem 5.3, since the sequence of numbers which are sums of two squares has zero density. We thus have the following result.

Theorem 5.4. *Let S consist of those positive integers with all prime factors congruent to* 3 *(mod* 4*). Then there are infinitely many positive integers n for which $p_S(n)$ is even, and there are infinitely many positive integers n for which $p_S(n)$ is odd.*

6. Partitions in parts coprime with a given number

In this section we study the parity problem for the number of partitions in parts which are relatively prime to a given number. Thus we fix a positive integer, b say, and then let S be the set of positive integers which are relatively prime to b. In order to apply the results from the previous section, one needs firstly to find $H_S(X)$ explicitly, secondly to estimate $B(H_S(X), N)$, and finally to provide an upper bound for $D(H_S(X), N)$. Then insert the results in the statements of Theorems 5.1 and 5.2.

Let us consider the prime decomposition of the number $2b$, say

$$2b = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k},$$

with $p_1 = 2$ and $p_2, \dots, p_k$ distinct odd primes, and $r_1, \dots, r_k \geq 1$. Denote $b_0 = p_1 p_2 \cdots p_k$.

A brief computation shows that

$$H_S(X) = \sum_{d|b_0} \sum_{n=1}^{\infty} X^{dn^2}. \tag{6.1}$$

If $b = 1$ we are in the case of the unrestricted partition function $p(n)$. In that case $b_0 = 2$ so $d = 1$ or $d = 2$, and the right side of (6.1) reduces to the right side of (4.4).

¿From (6.1) one easily obtains an asymptotic formula for $B(H_S(X), N)$. More precisely, one has

$$B(H_S(X), N) \sim \sqrt{N} \sum_{d|b_0} \frac{1}{\sqrt{d}} = \sqrt{N} \prod_{p|2b} \left(1 + \frac{1}{\sqrt{p}}\right), \tag{6.2}$$

where the product is over all prime factors of $2b$. In order to bound $D(H_S(X), N)$ one needs to take an integer a, with $1 \leq a \leq N$ and then bound the number of solutions of the equation

$$a = dm^2 - d'n^2, \tag{6.3}$$

with d, d' divisors of b_0 and n, m positive integers bounded, say, by $\sqrt{N}$.

If $d = d'$ we may proceed as with (4.6), since in this case $m - n$ and $m + n$ are divisors of a. In case $d \neq d'$, we first divide (6.3) by the greatest common divisor of d and d'. So we may assume in the following that d and d' are relatively prime, and not both of them equal to 1. Then $D := dd' > 1$ and D will be a divisor of b_0, Next, we multiply (6.3) by d, and obtain an equation of the form

$$ad = x^2 - Dy^2, \tag{6.4}$$

where $x = dm$ and $y = n$. Then we treat the equation (6.4) in the same way we treated (4.7), working this time in the ring of integers of the real quadratic field $\mathbf{Q}(\sqrt{D})$. One finds in this way that the number of solutions to (6.3) is $O_{b,c}\left(N^{\frac{c}{\log\log N}}\right)$ for any fixed $c > 2\log 2$. Therefore

$$D(H_S(X), N) = O_{b,c}\left(N^{\frac{c}{\log\log N}}\right). \tag{6.5}$$

Using (6.2) and (6.5) in Theorem 5.1 and Theorem 5.2 we obtain the following results.

Theorem 6.1. *Let b be a positive integer, and let S denote the set of positive integers which are relatively prime to b. Then, for each fixed c, with $c > 2\log 2$, and for N sufficiently large in terms of b and c,*

$$\#\{n \le N : p_S(n) \text{ is odd}\} \ge N^{\frac{1}{2} - \frac{c}{\log\log N}}. \tag{6.6}$$

Theorem 6.2. *Let b be a positive integer and let S denote the set of positive integers which are relatively prime to b. Then, for each fixed c with $c < \left(2\prod_{p|2b}(1+1/\sqrt{p})\right)^{-1}$, and for N sufficiently large in terms of b and c,*

$$\#\{n \le N : p_S(n) \text{ is even}\} \ge c\sqrt{N}. \tag{6.7}$$

7. Partitions in parts square free and coprime to a given number

In this section we take S to be the set of square free numbers which are relatively prime to a given positive integer b. In the case $b = 1$, we had investigated this problem in [24].

Is is easy to see that, for any positive integer b,

$$H_S(X) = \sum_{n_1,\dots,n_k=1}^{\infty} X^{p_1^{n_1}\cdots p_k^{n_k}}, \tag{7.1}$$

where $p_1, \dots, p_k$ are as in the previous section.

In order to provide an asymptotic result for $B(H_S(X), N)$, note that the inequality $p_1^{n_1}\cdots p_k^{n_k} \le N$ is equivalent to $n_1 \log p_1 + \cdots + n_k \log p_k \le \log N$. This means that we need to count lattice points $(n_1, \dots, n_k)$ inside a $k-$dimensional body which looks like a tetrahedron. The number of lattice points inside this body will equal the volume of the body, plus an error bounded in terms of the surface area of the body. The volume of the body is of size $\log^k N$ (times a constant c which depends on the numbers $\log p_1, \dots, \log p_k$). The surface area will be bounded by $\log^{k-1} N$. Therefore we can derive an asymptotic formula,

$$B(H_S(X), N) \sim c \log^k N. \tag{7.2}$$

Here the constant c equals the volume of the body Ω in $\mathbf{R}^k$ given by

$$\Omega = \{(x_1, \dots, x_k) \in \mathbf{R}^k : x_1, \dots, x_k \ge 0, x_1 \log p_1 + \cdots + x_k \log p_k \le 1\}.$$

A linear change of variables $y_j = x_j \log p_j, j = 1, \dots, k$, transforms our body to

$$\{(y_1, \dots, y_k) \in \mathbf{R}^k : y_1, \dots, y_k \ge 0, y_1 + \cdots + y_k \le 1\},$$

which has volume $\frac{1}{k!}$. Hence $c = 1/(k! \log p_1 \cdots \log p_k)$. If we denote by $\omega(n)$ the number of distinct prime factors of a positive integer n, then $k = \omega(2b)$, and (7.2) may be written in the form

$$B(H_S(X), N) \sim \frac{(\log N)^{\omega(2b)}}{\omega(2b)! \prod_{p|2b} \log p}. \tag{7.3}$$

In order to obtain via Theorem 5.1 a nontrivial lower bound for the number of $n \le N$ with $p_S(n)$ odd, we need for $D(H_S(X), N)$ an upper bound which is of smaller order of magnitude than the size of $B(H_S(X), N)$. Generalizing the reasoning employed in [24], we obtain a bound of the form

$$\#\{n \le N : p_S(n) \text{ odd } \} \gg (\log N)^{\left[\frac{k+2}{3}\right]}, \tag{7.4}$$

where $[x]$ denotes the greatest integer $\le x$. The reasoning is as follows. We need to bound the number of solutions $n_1, \dots, n_k, m_1, \dots, m_k$ of a diophantine equation of the form

$$a = p_1^{m_1} \cdots p_k^{m_k} - p_1^{n_1} \cdots p_k^{n_k}.$$

We write the set $\{1, \dots, k\}$ as a disjoint union of three sets:

$$M_1 = \{j : m_j > n_j\},$$

$$M_2 = \{j : m_j < n_j\},$$

and

$$M_3 = \{j : m_j = n_j\}.$$

For any $j \in M_1$, n_j is uniquely determined by a, more precisely $p_j^{n_j}$ is the largest power of p_j which divides a. Similarly, for any $j \in M_2$, m_j is uniquely determined by a. Dividing the above equation by the factors of the above two types, we are left with an equation of the form:

$$a' = \left(\prod_{j \in M_3} p_j^{t_j} \right) \left(\prod_{j \in M_1} p_j^{t_j} - \prod_{j \in M_2} p_j^{t_j} \right),$$

where $t_j = m_j - n_j$ if $j \in M_1$, $t_j = n_j - m_j$ if $j \in M_2$, and $t_j = n_j = m_j$ if $j \in M_2$. Here the point is that any of the three products $\prod_{j \in M_1} p_j^{t_j}$, $\prod_{j \in M_2} p_j^{t_j}$ and $\prod_{j \in M_3} p_j^{t_j}$, is uniquely determined by the other two (and by a' of course). We now choose that set among M_1, M_2 and M_3 which has the largest number of elements. That set will have at least $k/3$ elements. The other two sets together will have at most $[2k/3]$ elements. We allow the parameters t_j , with j in the union of those two sets, to move freely. Then the remaining t_j will be uniquely determined, as was observed above. Now each t_j is bounded by $\log N$. It follows that $D(H_S(X), N)$ will be bounded by $(\log N)^{[2k/3]}$. This in turn gives the lower bound stated above for the number of odd values of $p_S(n)$. We have obtained the following results.

Theorem 7.1. *Let b be a fixed positive integer and denote by S the set of square free positive integers which are relatively prime to b. Then for N large in terms of b,*

$$\#\{n \le N : p_S(n) \textit{ is odd}\} \gg (\log N)^{[(\omega(2b)+2)/3]}. \tag{7.5}$$

Theorem 7.2. *Let b be a fixed positive integer and denote by S the set of square free positive integers which are relatively prime to b. Then for each fixed c, with $c < (1/2)\omega(2b) \prod_{p|2b} \log p$, and for N large enough in terms of b and c,*

$$\#\{n \le N : p_S(n) \textit{ is even}\} \ge \frac{cN}{(\log N)^{\omega(2b)}}. \tag{7.6}$$

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