In every case, we find that theorems on partitions correspond to generating function identities. The partition-theoretic interpretations of (7.1.6) and (7.1.7) are given in Section 7.3. If the reader wishes he may proceed by accepting Lemmas 7.2–7.4 and then going directly to the partition theorems in Section 7.3.

7.2 The Generating Functions

We shall consider, for $|x| < |q|^{-1}$, $|q| < 1$,

$$H_{k,i}(a;x;q) = \sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+n-in}a^n(1-x^iq^{2ni})(axq^{n+1})_\infty(a^{-1})_n}{(q)_n(xq^n)_\infty}, \quad (7.2.1)$$

$$J_{k,i}(a;x;q) = H_{k,i}(a;xq;q) - xqaH_{k,i-1}(a;xq;q). \quad (7.2.2)$$

We note that any value for a is admissible, even $a = 0$, since $a^n(a^{-1})_n = (a-1)(a-q)\cdots(a-q^{n-1})$ is merely a polynomial in a whose value at 0 is $(-1)^nq^{n(n-1)/2}$.

Very little can be said in way of motivation for this section. Actually extensive work in the theory of basic hypergeometric series and partition identities shows that "well-poised" basic hypergeometric series provide the generating functions for numerous families of partition identities. The series in (7.2.1) (once the infinite products have been factored out: "$\sum \alpha_n(Aq^n)_\infty^{\pm 1} = (A)_\infty^{\pm 1} \sum \alpha_n(A)_n^{\mp 1}$") is an example of a well-poised series.

LEMMA 7.1

$$H_{k,i}(a;x;q) - H_{k,i-1}(a;x;q) = x^{i-1}J_{k,k-i+1}(a;x;q). \quad (7.2.3)$$

Proof. Noting that

$$q^{-in}(1-x^iq^{2ni}) - q^{-(i-1)n}(1-x^{i-1}q^{2n(i-1)})$$
$$= q^{-in}(1-q^n) + x^{i-1}q^{n(i-1)}(1-xq^n),$$

we see that

$$H_{k,i}(a;x;q) - H_{k,i-1}(a;x;q)$$
$$= \sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+n}a^n(axq^{n+1})_\infty(a^{-1})_n}{(q)_n(xq^n)_\infty} q^{-in}(1-q^n)$$
$$+ \sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+n}a^n(axq^{n+1})_\infty(a^{-1})_n x^{i-1}q^{n(i-1)}(1-xq^n)}{(q)_n(xq^n)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^{kn} q^{kn^2+n} a^n (axq^{n+1})_\infty (a^{-1})_n q^{-in}}{(q)_{n-1}(xq^n)_\infty}$$

$$+ \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+n} a^n (axq^{n+1})_\infty (a^{-1})_n x^{i-1} q^{n(i-1)}}{(q)_n (xq^{n+1})_\infty}$$

$$= \sum_{n=0}^{\infty} \frac{x^{kn+k} q^{kn^2+n+2kn+k+1} a^{n+1} (axq^{n+2})_\infty (a^{-1})_{n+1} q^{-in-i}}{(q)_n (xq^{n+1})_\infty}$$

$$+ \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+n} a^n (axq^{n+1})_\infty (a^{-1})_n x^{i-1} q^{n(i-1)}}{(q)_n (xq^{n+1})_\infty}$$

$$= x^{i-1} \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+in} a^n (axq^{n+2})_\infty (a^{-1})_n}{(q)_n (xq^{n+1})_\infty} \Big\{(1 - axq^{n+1})$$

$$+ ax^{k-i+1} q^{2n(k-i)+k-i+1+n} \left(1 - \frac{q^n}{a}\right)\Big\}$$

$$= x^{i-1} \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+in} a^n (axq^{n+2})(a^{-1})_n}{(q)_n (xq^{n+1})_\infty} [1 - (xq)^{k-i+1} q^{n[2(k-i+1)-1]}]$$

$$- x^{i-1} \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+in} a^n (axq^{n+2})(a^{-1})_n}{(q)_n (xq^{n+1})_\infty} \{axq^{n+1}[1-(xq)^{k-i} q^{n[2(k-i)-1]}]\}$$

$$= x^{i-1}[H_{k,k-i+1}(a; xq; q) - axqH_{k,k-i}(a; xq; q)]$$

$$= x^{i-1} J_{k,k-i+1}(a; x; q). \qquad \blacksquare$$

LEMMA 7.2

$$J_{k,i}(a; x; q) - J_{k,i-1}(a; x; q) = (xq)^{i-1}(J_{k,k-i+1}(a; xq; q) - aJ_{k,k-i+2}(a; xq; q)). \qquad (7.2.4)$$

Proof.

$$J_{k,i}(a; x; q) - J_{k,i-1}(a; x; q)$$

$$= (H_{k,i}(a; xq; q) - H_{k,i-1}(a; xq; q))$$

$$- axq(H_{k,i-1}(a; xq; q) - H_{k,i-2}(a; xq; q))$$

$$= (xq)^{i-1} J_{k,k-i+1}(a; xq; q) - a(xq)^{i-1} J_{k-i+2}(a; xq; q). \qquad \blacksquare$$

In treating problems, there are numerous instances in which we require an infinite product representation of a generating function. Such representations are given in the following lemmas.

LEMMA 7.3. *For* $1 \leqslant i \leqslant k$, $|q| < 1$,

$$J_{k,i}(0; 1; q) = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i (\mathrm{mod}\ 2k+1)}}^{\infty} (1 - q^n)^{-1}. \tag{7.2.5}$$

Proof. By (7.2.2)

$$\begin{aligned} J_{k,i}(0; 1; q) &= H_{k,i}(0; q; q) \\ &= (q)_\infty^{-1} \sum_{n=0}^{\infty} q^{kn^2 + (k-i+1)n} (-1)^n q^{n(n-1)/2} (1 - q^{(2n+1)i}) \\ &= (q)_\infty^{-1} \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2 - in} (1 - q^{(2n+1)i}) \\ &= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i (\mathrm{mod}\ 2k+1)}}^{\infty} (1 - q^n)^{-1} \quad \text{(by Corollary 2.9).} \quad \blacksquare \end{aligned}$$

LEMMA 7.4. *For* $1 \leqslant i \leqslant k$, $|q| < 1$,

$$J_{k,i}(-q^{-1}; 1; q^2) = \prod_{\substack{n=1 \\ n \not\equiv 2 (\mathrm{mod} 4) \\ n \not\equiv 0, \pm (2i-1)(\mathrm{mod}\ 4k)}}^{\infty} (1 - q^n)^{-1}. \tag{7.2.6}$$

Proof.

$$\begin{aligned} {}_{,i}(-q^{-1}, 1; q^2) &= H_{k,i}(-q^{-1}; q^2; q^2) + qH_{k,i-1}(-q^{-1}; q^2; q^2) \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left\{ \sum_{n=0}^{\infty} (-1)^n q^{2kn + 2kn^2 - (2i-1)n} \right. \\ &\quad \times \frac{(1 - q^{2i+4ni})}{(1 + q^{2n+1})} + q \sum_{n=0}^{\infty} (-1)^n q^{2kn + 2kn^2 - (2i-3)n} \\ &\quad \left. \times \frac{(1 - q^{2i-2+4ni-4n})}{(1 + q^{2n+1})} \right\} \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{2kn + 2kn^2 + n} \\ &\quad \times \frac{(q^{-2in} - q^{2i+2ni} + q^{1-2in+2n} - q^{-1+2i+2ni-2n})}{(1 + q^{2n+1})} \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{2kn^2 + (2k+1-2i)n} (1 - q^{(2n+1)(2i-1)}) \end{aligned}$$

ISBN 0-201-13501-9

$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{2kn^2+(2k-2i+1)n}$$

$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} (1-q^{4kn+4k})(1-q^{4kn+2i-1})$$
$$\times (1-q^{4kn+4k-2i+1})$$

$$= \frac{1}{(q;q^2)_\infty (q^4;q^4)_\infty} \prod_{n=0}^{\infty} (1-q^{4k(n+1)})(1-q^{4kn+2i-1})$$
$$\times (1-q^{4k(n+1)-2i+1})$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 2 \pmod 4 \\ n \not\equiv 0, \pm(2i-1) \pmod{4k}}}^{\infty} (1-q^n)^{-1}. \qquad \blacksquare$$

7.3 The Rogers-Ramanujan Identities and Gordon's Generalization

In this section we shall utilize the analytic work in Section 7.2 to prove the following theorem, which is due to B. Gordon.

THEOREM 7.5. *Let $B_{k,i}(n)$ denote the number of partitions of n of the form $(b_1 b_2 \cdots b_s)$, where $b_j - b_{j+k-1} \geqslant 2$, and at most $i-1$ of the b_j equal 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Then $A_{k,i}(n) = B_{k,i}(n)$ for all n.*

Before we prove Theorem 7.5, it is appropriate to give center stage to its two most celebrated corollaries, the Rogers–Ramanujan identities (stated in terms of partitions).

COROLLARY 7.6 (The first Rogers–Ramanujan identity). *The partitions of an integer n in which the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts $\equiv 1$ or 4 (modulo 5).*

Proof. Take $k = i = 2$ in Theorem 7.5. ■

COROLLARY 7.7 (The second Rogers–Ramanujan identity). *The partitions of an integer n in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts $\equiv 2$ or 3 (modulo 5).*

Proof. Take $k = i + 1 = 2$ in Theorem 7.5. ■

Proof of Theorem 7.5. Let $b_{k,i}(m, n)$ denote the number of partitions $(b_1 b_2 \cdots b_m)$ of n with exactly m parts such that $b_j \geqslant b_{j+1}$, $b_j - b_{j+k-1} \geqslant 2$, and at most $i-1$ of the b_j equal 1. Then for $1 \leqslant i \leqslant k$

$$b_{k,i}(m, n) = \begin{cases} 1 & \text{if} \quad m = n = 0 \\ 0 & \text{if} \quad m \leqslant 0 \quad \text{or} \quad n \leqslant 0 \quad \text{but} \quad (m, n) \neq (0, 0); \end{cases} \tag{7.3.1}$$

$$b_{k,0}(m, n) = 0; \tag{7.3.2}$$

for $1 \leqslant i \leqslant k$

$$b_{k,i}(m, n) - b_{k,i-1}(m, n) = b_{k,k-i+1}(m - i + 1, n - m). \tag{7.3.3}$$

Equations (7.3.1) and (7.3.2) are obvious once we recall that the only partition that is either of a nonpositive number or has a nonpositive number of parts is the empty partition of 0.

Equation (7.3.3) requires careful attention: $b_{k,i}(m, n) - b_{k,i-1}(m, n)$ enumerates the number of partitions among those enumerated by $b_{k,i}(m, n)$ that have exactly $i - 1$ appearances of 1. Let us transform this set of partitions by deleting the $i - 1$ ones, and then subtracting 1 from each of the remaining parts. The resulting partitions $(b_1' \cdots b_{m-i+1}')$ have $m - i + 1$ parts; they partition $n - m$, and the parts satisfy $b_j' - b_{j+k-1}' \geqslant 2$. Since originally 1 appeared $i - 1$ times and the total number appearances of ones and twos could not exceed $k - 1$ (due to the difference condition), we see that originally 2 appeared at most $(k - i + 1) - 1$ times, and thus after the transformation 1 appears at most $k - i + 1$ times. The transformation described above establishes a one-to-one correspondence between the partitions enumerated by $b_{k,i}(m, n) - b_{k,i-1}(m, n)$ and those enumerated by $b_{k,k-i+1}(m - i + 1, n - m)$. Hence (7.3.3) is established.

We now make a simple yet essential observation: the $b_{k,i}(m, n)$ $(0 \leqslant i \leqslant k)$ are *uniquely* determined by (7.3.1), (7.3.2), and (7.3.3). To see this, proceed by a double mathematical induction first on n and then on i. Equation (7.3.1) takes care of $n \leqslant 0$, $m \leqslant 0$, $i > 0$. Equation (7.3.2) handles all n when $i = 0$. Equation (7.3.3) represents $b_{k,i}(m, n)$ as a two-term sum in which the first term has a lower i index and the second a lower n index (since we can assume $m > 0$).

Now let us consider

$$J_{k,i}(0; x; q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{k,i}(m, n) x^m q^n.$$

From the fact that for $1 \leqslant i \leqslant k$

$$J_{k,i}(0; 0; q) = J_{k,i}(0; x; 0) = 1,$$

we see that for $1 \leqslant i \leqslant k$

$$c_{k,i}(m,n) = \begin{cases} 1 & \text{if} \quad m = n = 0 \\ 0 & \text{if} \quad m \leqslant 0 \quad \text{or} \quad n \leqslant 0 \quad \text{but} \quad (m,n) \neq (0,0). \end{cases} \tag{7.3.4}$$

From the fact that

$$J_{k,0}(0; k; q) = H_{k,0}(0; xq; q) = 0,$$

we see that

$$c_{k,0}(m,n) = 0. \tag{7.3.5}$$

Finally, by comparing coefficients of $x^m q^n$ on both sides of (7.2.4) with $a = 0$, we see that

$$c_{k,i}(m,n) - c_{k,i-1}(m,n) = c_{k,k-i+1}(m-i+1, n-m). \tag{7.3.6}$$

So we see that the $c_{k,i}(m,n)$ also satisfy the system of equations (7.3.1)–(7.3.3) that uniquely defines the $b_{k,i}(m,n)$. Therefore, $b_{k,i}(m,n) = c_{k,i}(m,n)$ for all m and n with $0 \leqslant i \leqslant k$.

Hence, since $\sum_{m \geqslant 0} b_{k,i}(m,n) = B_{k,i}(n)$, we see that

$$\begin{aligned} \sum_{n \geqslant 0} B_{k,i}(n) q^n &= \sum_{m \geqslant 0} \sum_{n \geqslant 0} b_{k,i}(m,n) q^n \\ &= J_{k,i}(0; 1; q) \\ &= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i (\bmod 2k+1)}}^{\infty} (1 - q^n)^{-1} \qquad \text{(by (7.2.5))} \\ &= \sum_{n \geqslant 0} A_{k,i}(n) q^n \qquad \text{(by Theorem 1.1, Eq. (1.2.3))}. \end{aligned}$$

Comparing coefficients of q^n in the extremes of the string of equations above, we see that $A_{k,i}(n) = B_{k,i}(n)$. ■

Theorem 7.5 has an analytic counterpart, which we shall prove before proceeding.

THEOREM 7.8. *For* $1 \leqslant i \leqslant k$, $k \geqslant 2$, $|q| < 1$

$$\sum_{n_1, n_2, \ldots, n_{k-1} \geqslant 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_i + N_{i+1} + \cdots + N_{k-1}}}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i (\bmod 2k+1)}}^{\infty} (1 - q^n)^{-1} \tag{7.3.7}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

Proof. We shall prove that

$$J_{k,i}(0;x;q)=\sum_{n_1,n_2,\ldots,n_{k-1}\geqslant 0}\frac{x^{N_1+N_2+\cdots+N_{k-1}}q^{N_1{}^2+N_2{}^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}. \tag{7.3.8}$$

We obtain Eq. (7.3.7) from (7.3.8) by setting $x = 1$ and invoking (7.2.5).

Equation (7.3.8) itself follows from

$$J_{k,i}(0;x;q) = \sum_{n\geqslant 0}\frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}J_{k-1,i}(0;xq^{2n};q), \tag{7.3.9}$$

which may be seen immediately by induction on k once we observe that $J_{k,k+1}(0;x;q) = J_{k,k}(0;x;q)$ (set $i = k+1$ in (7.2.4) and recall that $J_{k,0}(0;xq;q) = H_{k,0}(0,xq^2;q) = 0$) and that $J_{1,1}(0;x;q) = 1$ (since by (7.2.4) $J_{1,1}(0;x;q) = J_{1,1}(0;xq;q) = J_{1,1}(0,xq^2;q) = \cdots = J_{1,1}(0;xq^n;q) \to J_{1,1}(0;0;q) = 1$).

To prove (7.3.9), we define

$$R_{k,i}(x;q) = \sum_{n\geqslant 0}\frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}J_{k-1,i}(0;xq^{2n};q).$$

Then for $1 \leqslant i \leqslant k$,

$$R_{k,i}(0;q) = R_{k,i}(x;0) = 1 \tag{7.3.10}$$

and

$$R_{k,0}(x;q) = 0. \tag{7.3.11}$$

Finally

$$\begin{aligned}
&R_{k,i}(x;q) - R_{k,i-1}(x;q)\\
&\quad= \sum_{n\geqslant 0}\frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(J_{k-1,i}(0;xq^{2n};q) - q^nJ_{k-1,i-1}(0;xq^{2n};q))\\
&\quad= \sum_{n\geqslant 0}\frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(J_{k-1,i-1}(0;xq^{2n};q)\\
&\qquad + (xq^{2n+1})^{i-1}J_{k-1,k-i}(0;xq^{2n+1};q) - q^nJ_{k-1,i-1}(0;xq^{2n};q))\\
&\quad= \sum_{n\geqslant 1}\frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(1-q^n)J_{k-1,i-1}(0;xq^{2n};q)\\
&\qquad + (xq)^{i-1}\sum_{n\geqslant 0}\frac{x^{(k-1)n}q^{(k-1)n^2+(k+i-2)n}}{(q)_n}J_{k-1,k-i}(0;xq^{2n+1};q)
\end{aligned}$$

$$= x^{k-1}q^{2k-i-1}\sum_{n\geq 0}\frac{x^{(k-1)n}q^{(k-1)n^2+(3k-i-2)n}}{(q)_n}J_{k-1,i-1}(0;xq^{2n+2};q)$$
$$+ (xq)^{i-1}\sum_{n\geq 0}\frac{x^{(k-1)n}q^{(k-1)n^2+(k+i-2)n}}{(q)_n}(J_{k-1,k-i+1}(0;xq^{2n+1};q)$$
$$- (xq^{2n+2})^{k-i}J_{k-1,i-1}(0;xq^{2n+2};q))$$
$$= x^{k-1}q^{2k-i-1}\sum_{n\geq 0}\frac{x^{(k-1)n}q^{(k-1)n^2+(3k-i-2)n}}{(q)_n}J_{k-1,i-1}(0;xq^{2n+2};q)$$
$$+ (xq)^{i-1}\sum_{n\geq 0}(xq)^{(k-1)n}q^{(k-1)n^2+(k-(k-i+1))n}J_{k-1,k-i+1}(0;xq^{2n+1};q)$$
$$- x^{k-1}q^{2k-i-1}\sum_{n\geq 0}\frac{x^{(k-1)n}q^{(k-1)n^2+(3k-i-2)n}}{(q)_n}J_{k-1,i-1}(0;xq^{2n+2};q)$$
$$= (xq)^{i-1}R_{k,k-i+1}(xq,q). \tag{7.3.12}$$

Recalling that the coefficients in the expansion of $J_{k,i}(0; x; q)$ were uniquely determined by (7.3.4), (7.3.5), and (7.3.6), we conclude that since $R_{k,i}(x; q)$ satisfies (7.3.10), (7.3.11), and (7.3.12), and thus its coefficients must satisfy (7.3.4), (7.3.5), and (7.3.6), therefore $R_{k,i}(x; q) = J_{k,i}(0; x; q)$ for $0 \leqslant i \leqslant k$. Thus we have (7.3.9) and with it Theorem 7.8. ■

COROLLARY 7.9 (Eq. (7.1.6)).

$$1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots$$
$$= \prod_{n=0}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1}.$$

Proof. Set $k = i = 2$ in Theorem 7.8. ■

COROLLARY 7.10 (Eq. (7.1.7)).

$$1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} + \cdots$$
$$= \prod_{n=0}^{\infty}(1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1}.$$

Proof. Set $k = 2$, $i = 1$ in Theorem 7.8. ■

7.4 The Göllnitz-Gordon Identities and Their Generalization

To appreciate the general area of partition identities such as the Rogers–Ramanujan identities, we turn to results discovered independently in the 1960s