$\varphi^{2k}(q)$ lies is equal to 1. For $k > 4$, the dimension exceeds one, and so other modular forms play a role in formulas for $r_{2k}(n)$, making these formulas more complicated. In the latter part of the nineteenth century and the early part of the twentieth century, J. W. L. Glaisher wrote several papers on sums of squares. In particular, in [**99**], he derived formulas for $r_{2k}(n)$, $5 \le k \le 9$. In his paper [**186**], Ramanujan stated without proof a general formula for $r_{2k}(n)$, which was later proved by Mordell [**162**]. However, Ramanujan worked out the details for $1 \le k \le 12$ and recorded them in Table VI [**186**], [**192**, p. 159]. Using a modular transformation that connects $\varphi^{2k}(q)$ with the generating function $\eta^{24}(q)$ of $t_{2k}(n)$, Ramanujan [**187**], [**192**, pp. 179–199] derived corresponding formulas for $t_{2k}(n)$, although he did not give them explicitly. In her doctoral thesis [**183**], V. Ramamani gave new proofs of Ramanujan's formulas for $r_{2k}(n)$, $k \le 12$, and corrected two numerical errors in Ramanujan's formula for $r_{22}(n)$ given in his Table VI. A new class of formulas involving products of certain Eisenstein series has been conjectured by H. H. Chan and K. S. Chua [**69**].

The theorems of Jacobi have been enormously generalized by S. C. Milne [**156**]–[**158**], who also provides the most extensive literature survey known on formulas for sums of squares. K. Ono [**176**] and L. Long and Y. Yang [**150**] have given further proofs of some of Milne's results.

Elementary texts in number theory containing material related to that given in this chapter include those of Hardy and Wright [**112**, Chapter 20] and L. K. Hua [**127**, pp. 115–120, 204–210, 307–309]. The most extensive text on sums of squares is by Grosswald [**101**].

Chapter 4

Eisenstein Series

4.1. Bernoulli Numbers and Eisenstein Series

Appearing in the definition of the classical Eisenstein series are Bernoulli numbers, which we define first.

Definition 4.1.1. *The Bernoulli numbers B_n, $n \ge 0$, are defined by the generating function*

$$\frac{z}{e^z - 1} =: \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi. \tag{4.1.1}$$

(Note that the function $z/(e^z - 1)$ is analytic at $z = 0$, and its nearest singularities to 0 are $\pm 2\pi i$; thus, the series on the right side of (4.1.1) converges for $|z| < 2\pi$.) In particular,

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30},$$

$$B_5 = 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = -\frac{1}{30}, \quad B_9 = 0, \quad B_{10} = \frac{5}{66}.$$

It appears that $B_n = 0$ when n is odd and greater than 1. This is easy to prove.

Theorem 4.1.2. *For $n \ge 1$,*

$$B_{2n+1} = 0.$$

Proof. Define

(4.1.2) $$f(z) := \frac{z(e^z+1)}{2(e^z-1)} = \frac{z}{e^z-1} + \frac{1}{2}z = 1 - \sum_{n=2}^{\infty} B_n \frac{z^n}{n!}.$$

It is easily checked that

$$f(-z) = f(z),$$

i.e., $f(z)$ is an even function. Hence, from (4.1.2), we conclude that $B_n = 0$, when n is an odd integer at least equal to 3. □

Theorem 4.1.3. *For* $|x| < 2\pi$,

$$\frac{x}{2}\cot\frac{x}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}}{(2n)!} x^{2n}.$$

Proof. Using (4.1.1) with $z = -ix$, we find that

$$\begin{aligned}\frac{x}{2}\cot\frac{x}{2} &= -\frac{ix}{2} - \frac{ix}{e^{-ix}-1}\\ &= -\frac{ix}{2} + \sum_{m=0}^{\infty} B_m \frac{(-ix)^m}{m!}\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}}{(2n)!} x^{2n},\end{aligned}$$

because $B_1 = -\frac{1}{2}$ and $B_{2n+1} = 0, n \geq 1$, by Theorem 4.1.2. □

Eisenstein series are special cases of Ramanujan's more general function

(4.1.3) $$\Phi_{r,s}(q) := \sum_{k,n=1}^{\infty} k^r n^s q^{kn},$$

where r and s are nonnegative integers. Trivially, $\Phi_{r,s}(q) = \Phi_{s,r}(q)$. For our purposes, the most important special cases are when $r = 0$ and s is an odd positive integer, and when $r = 1$ and s is an even positive integer.

Definition 4.1.4. *For each positive integer* r, *define*

(4.1.4) $$S_r := -\frac{B_{r+1}}{2(r+1)} + \Phi_{0,r}(q) = -\frac{B_{r+1}}{2(r+1)} + \sum_{k=1}^{\infty} \frac{k^r q^k}{1-q^k}.$$

Definition 4.1.5. *The classical Eisenstein series* $E_{2n}(q)$, *where* n *is a positive integer, are defined by*

(4.1.5) $$E_{2n}(q) := 1 - \frac{4n}{B_{2n}} \sum_{k=1}^{\infty} \frac{k^{2n-1}q^k}{1-q^k}.$$

Thus

(4.1.6) $$E_{2n}(q) = -\frac{4n}{B_{2n}} S_{2n-1}, \qquad n \geq 1.$$

(Our notation here is slightly different from the classical notation; normally, $E_{2n}(\tau)$ is defined by the right side of (4.1.5) when $q = e^{2\pi i\tau}$ and $\operatorname{Im}\tau > 0$.) In Ramanujan's notation, the three most important special cases of (4.1.5) are

(4.1.7) $$P = P(q) := E_2(q) = 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} = -24S_1,$$

(4.1.8) $$Q = Q(q) := E_4(q) = 1 + 240\sum_{k=1}^{\infty} \frac{k^3q^k}{1-q^k} = 240S_3,$$

(4.1.9) $$R = R(q) := E_6(q) = 1 - 504\sum_{k=1}^{\infty} \frac{k^5q^k}{1-q^k} = -504S_5.$$

Our first goal is to derive two recurrence relations for S_{2n+1}, the first of which also involves $\Phi_{1,2n}$.

4.2. Trigonometric Series

Our goal in this section is to present Ramanujan's elementary method [**186**], [**192**, pp. 136–162] for deriving recurrence relations for Eisenstein series.

We begin with an exercise.

Exercise 4.2.1. *Verify (for example, by summing finite geometric series) the elementary identity*

(4.2.1) $$1 + 2\sum_{n=1}^{m-1} \cos(n\theta) + \cos(m\theta) = \cot\left(\tfrac{1}{2}\theta\right)\sin(m\theta),$$

valid for each positive integer m.

We use (4.2.1) when we square the left side of (4.2.2) below and collect terms. Note that the left side of (4.2.2) minus $\left(\frac{1}{4}\cot\frac{1}{2}\theta\right)^2$ is an even, continuously differentiable, periodic function of θ and so can be expressed as a Fourier cosine series. Accordingly, we write

$$\left(\frac{1}{4}\cot\frac{1}{2}\theta+\sum_{k=1}^{\infty}\frac{q^k\sin(k\theta)}{1-q^k}\right)^2=\left(\frac{1}{4}\cot\frac{1}{2}\theta\right)^2+\sum_{n=0}^{\infty}C_n\cos(n\theta), \tag{4.2.2}$$

where the Fourier coefficients C_n depend upon q but not upon θ. Squaring the left side of (4.2.2), using (4.2.1) for the terms involving $\cot(\frac{1}{2}\theta)\sin(k\theta)$, and using the identity $\sin^2(k\theta)=\frac{1}{2}(1-\cos(2k\theta))$, we find that

$$\begin{aligned}C_0&=\frac{1}{2}\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}+\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{q^k}{1-q^k}\right)^2\\&=\frac{1}{2}\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)^2}\\&=\frac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}mq^{mk}\\&=\frac{1}{2}\sum_{m=1}^{\infty}\frac{mq^m}{1-q^m}.\end{aligned} \tag{4.2.3}$$

To calculate $C_n, n\geq 1$, we need to be careful. First observe that we obtain contributions from

$$\begin{aligned}&\frac{q^j}{1-q^j}\frac{q^k}{1-q^k}\sin(j\theta)\sin(k\theta)\\&=\frac{q^j}{1-q^j}\frac{q^k}{1-q^k}\frac{1}{2}(\cos(j-k)\theta-\cos(j+k)\theta),\end{aligned}$$

for $j,k\geq 1$. We also gain contributions from the terms $\cot(\frac{1}{2}\theta)\sin(m\theta)$, for $m\geq n$. Thus, we find that, for $n\geq 1$,

$$\begin{aligned}C_n=&\frac{1}{2}\frac{q^n}{1-q^n}+\sum_{k=1}^{\infty}\frac{q^{n+k}}{1-q^{n+k}}+\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\frac{q^{n+k}}{1-q^{n+k}}\\&-\frac{1}{2}\sum_{k=1}^{n-1}\frac{q^k}{1-q^k}\frac{q^{n-k}}{1-q^{n-k}}.\end{aligned} \tag{4.2.4}$$

Using the easily verified elementary identities

$$\frac{q^{n+k}}{1-q^{n+k}}\left(1+\frac{q^k}{1-q^k}\right)=\frac{q^n}{1-q^n}\left(\frac{q^k}{1-q^k}-\frac{q^{n+k}}{1-q^{n+k}}\right)$$

and

$$\frac{q^n}{(1-q^k)(1-q^{n-k})}=\frac{q^n}{1-q^n}\left(1+\frac{q^k}{1-q^k}+\frac{q^{n-k}}{1-q^{n-k}}\right)$$

in (4.2.4), we find that

$$\begin{aligned}C_n=&\frac{q^n}{2(1-q^n)}+\frac{q^n}{1-q^n}\sum_{k=1}^{\infty}\left(\frac{q^k}{1-q^k}-\frac{q^{n+k}}{1-q^{n+k}}\right)\\&-\frac{q^n}{2(1-q^n)}\sum_{k=1}^{n-1}\left(1+\frac{q^k}{1-q^k}+\frac{q^{n-k}}{1-q^{n-k}}\right)\\=&\frac{q^n}{2(1-q^n)}+\left(\frac{q^n}{1-q^n}\right)^2-\frac{(n-1)q^n}{2(1-q^n)}\\=&\frac{q^n}{1-q^n}\left(\frac{1}{1-q^n}-\frac{n}{2}\right).\end{aligned} \tag{4.2.5}$$

Substituting (4.2.3) and (4.2.5) in (4.2.2), we find that

$$\begin{aligned}\left(\frac{1}{4}\cot\frac{1}{2}\theta+\sum_{k=1}^{\infty}\frac{q^k\sin(k\theta)}{1-q^k}\right)^2=&\left(\frac{1}{4}\cot\frac{1}{2}\theta\right)^2+\sum_{k=1}^{\infty}\frac{q^k\cos(k\theta)}{(1-q^k)^2}\\&+\frac{1}{2}\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}(1-\cos(k\theta)).\end{aligned} \tag{4.2.6}$$

This is the first of the two primary trigonometric series identities that we need. Using (4.2.6), we establish a recurrence formula for the functions S_r. Appearing in the recurrence relation are the functions from (4.1.3)

$$\Phi_{1,2n}(q)=\sum_{m,k=1}^{\infty}mk^{2n}q^{mk}=\sum_{k=1}^{\infty}\frac{k^{2n}q^k}{(1-q^k)^2}. \tag{4.2.7}$$

Theorem 4.2.2. *For* $n\geq 1$,

$$\frac{(2n+3)}{2(2n+1)}S_{2n+1}-\Phi_{1,2n}=\sum_{k=1}^{n}\binom{2n}{2k-1}S_{2k-1}S_{2n-2k+1}. \tag{4.2.8}$$

Proof. From the elementary identity

$$\cot^2\theta = -\left(1 + \frac{d}{d\theta}\cot\theta\right) \tag{4.2.9}$$

and Theorem 4.1.3, we find that

$$\frac{1}{16}\cot^2\frac{1}{2}\theta = \frac{1}{4\theta^2} - \frac{1}{24} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n B_{2n+2}(2n+1)}{(2n+2)!}\theta^{2n}. \tag{4.2.10}$$

Using Theorem 4.1.3, (4.2.10), and the Maclaurin series for $\sin(k\theta)$ and $\cos(k\theta)$ in (4.2.6), we deduce that

(4.2.11)

$$\left(\frac{1}{2\theta} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n B_{2n}}{(2n)!}\theta^{2n-1} + \sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\sum_{n=0}^{\infty}\frac{(-1)^n k^{2n+1}}{(2n+1)!}\theta^{2n+1}\right)^2$$
$$= \frac{1}{4\theta^2} - \frac{1}{24} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n B_{2n+2}}{(2n)!(2n+2)}\theta^{2n}$$
$$+ \sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)^2}\sum_{n=0}^{\infty}\frac{(-1)^n k^{2n}}{(2n)!}\theta^{2n}$$
$$+ \frac{1}{2}\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}k^{2n}}{(2n)!}\theta^{2n}.$$

Collecting coefficients of like powers of θ in (4.2.11) and recalling the representations of S_r and $\Phi_{r,s}(q)$ given in (4.1.4) and (4.2.7), respectively, we find that

$$\left(\frac{1}{2\theta} + \frac{S_1}{1!}\theta - \frac{S_3}{3!}\theta^3 + \frac{S_5}{5!}\theta^5 - \cdots\right)^2$$
$$= \frac{1}{4\theta^2} + S_1 - \frac{\Phi_{1,2}}{2!}\theta^2 + \frac{\Phi_{1,4}}{4!}\theta^4 - \cdots$$
$$+ \frac{1}{2}\left(\frac{S_3}{2!}\theta^2 - \frac{S_5}{4!}\theta^4 + \frac{S_7}{6!}\theta^6 - \cdots\right).$$

Equating coefficients of θ^{2n}, $n \geq 1$, on both sides above, we find that

$$\frac{(-1)^{n-1}}{2}\frac{S_{2n+1}}{(2n)!} + \frac{(-1)^n}{(2n)!}\Phi_{1,2n}$$
$$= (-1)^{n-1}\left(\frac{S_1}{1!}\frac{S_{2n-1}}{(2n-1)!} + \frac{S_3}{3!}\frac{S_{2n-3}}{(2n-3)!} + \cdots + \frac{S_{2n-1}}{(2n-1)!}\frac{S_1}{1!}\right)$$
$$+ \frac{(-1)^n S_{2n+1}}{(2n+1)!},$$

or, upon simplification,

$$\frac{(2n+3)}{2(2n+1)}S_{2n+1} - \Phi_{1,2n} = \sum_{k=1}^{n}\binom{2n}{2k-1}S_{2k-1}S_{2n-2k+1},$$

which is (4.2.8). □

Our next task is to use Theorem 4.2.2 to derive the three important first order differential equations satisfied by P, Q, and R. From (4.1.5),

$$q\frac{dE_{2n}(q)}{dq} = q\frac{d}{dq}\left(1 - \frac{4n}{B_{2n}}\sum_{k=1}^{\infty}\frac{k^{2n-1}q^k}{1-q^k}\right)$$
$$= -\frac{4n}{B_{2n}}q\sum_{k=1}^{\infty}\left(\frac{k^{2n}q^{k-1}}{1-q^k} + \frac{k^{2n}q^{2k-1}}{(1-q^k)^2}\right)$$
$$= -\frac{4n}{B_{2n}}\sum_{k=1}^{\infty}\frac{k^{2n}q^k}{(1-q^k)^2}$$
$$= -\frac{4n}{B_{2n}}\Phi_{1,2n}(q), \tag{4.2.12}$$

by (4.2.7). In particular, from our table of Bernoulli numbers at the beginning of this chapter and the definitions (4.1.7)–(4.1.9),

$$q\frac{dP}{dq} = -24\Phi_{1,2}(q), \tag{4.2.13}$$

$$q\frac{dQ}{dq} = 240\Phi_{1,4}(q), \tag{4.2.14}$$

$$q\frac{dR}{dq} = -504\Phi_{1,6}(q). \tag{4.2.15}$$

Putting $n=1$ in (4.2.8), we find that

$$\text{(4.2.16)} \qquad \frac{5}{6}S_5 - \Phi_{1,2} = 2S_1^2,$$

or

$$\text{(4.2.17)} \qquad 288\Phi_{1,2} = Q - P^2;$$

putting $n=2$ in (4.2.8), we find that

$$\frac{7}{10}S_5 - \Phi_{1,4} = 8S_1S_3,$$

or

$$\text{(4.2.18)} \qquad 720\Phi_{1,4} = PQ - R;$$

and putting $n=3$ in (4.2.8), we find that

$$\frac{9}{14}S_7 - \Phi_{1,6} = 12S_1S_5 + 20S_3^2,$$

or, with the use of the identity $Q^2 = 480S_7$, which we shall prove below in (4.2.38),

$$\text{(4.2.19)} \qquad 1008\Phi_{1,6} = Q^2 - PR.$$

Hence, putting (4.2.17)–(4.2.19) in (4.2.13)–(4.2.15), respectively, we deduce the following important theorem.

Theorem 4.2.3. *We have*

$$\text{(4.2.20)} \qquad q\frac{dP}{dq} = \frac{P^2-Q}{12},$$

$$\text{(4.2.21)} \qquad q\frac{dQ}{dq} = \frac{PQ-R}{3},$$

$$\text{(4.2.22)} \qquad q\frac{dR}{dq} = \frac{PR-Q^2}{2}.$$

The next theorem is also of enormous importance.

Theorem 4.2.4. *We have*

$$\text{(4.2.23)} \qquad Q^3(q) - R^2(q) = 1728q(q;q)_\infty^{24}.$$

Proof. By a straightforward use of logarithmic differentiation, we easily find that

$$q\frac{d}{dq}\log\left(q(q;q)_\infty^{24}\right) = P(q).$$

On the other hand, using straightforward differentiation along with Theorem 4.2.3, we find that

$$q\frac{d}{dq}\log\left(Q^3(q) - R^2(q)\right) = P(q).$$

The last two equalities imply that

$$\frac{d}{dq}\log\left(q(q;q)_\infty^{24}\right) = \frac{d}{dq}\log\left(Q^3(q) - R^2(q)\right),$$

or

$$\text{(4.2.24)} \qquad q(q;q)_\infty^{24} = c\left(Q^3(q) - R^2(q)\right),$$

for some constant c. Equating coefficients of q on each side of (4.2.24), we deduce that

$$1 = c(3\cdot 240 + 2\cdot 504) = 1728c;$$

and so with the value $c = 1/1728$, Theorem 4.2.4 follows from (4.2.24). ☐

To obtain a second recurrence relation for S_r, we begin with analogues of (4.2.1) and (4.2.6).

Exercise 4.2.5. *First, prove that*

$$\text{(4.2.25)} \qquad \cot^2\left(\tfrac{1}{2}\theta\right)(1-\cos(n\theta)) = (2n-1) - 4\sum_{k=1}^{n-1}(n-k)\cos(k\theta) + \cos(n\theta).$$

Exercise 4.2.6. (*This is a very difficult exercise.*) *Second, using* (4.2.25), *prove that*

$$\text{(4.2.26)} \qquad L(\theta) := \left(\frac{1}{8}\cot^2\frac{1}{2}\theta + \frac{1}{12} + \sum_{k=1}^\infty \frac{kq^k}{1-q^k}(1-\cos(k\theta))\right)^2$$
$$= \left(\frac{1}{8}\cot^2\frac{1}{2}\theta + \frac{1}{12}\right)^2 + \frac{1}{12}\sum_{k=1}^\infty \frac{k^3q^k}{1-q^k}(5+\cos(k\theta)) =: R(\theta).$$

Theorem 4.2.7. *For each positive integer n,*

$$\text{(4.2.27)} \qquad \frac{(n-1)(2n+5)}{12(n+1)(2n+1)}S_{2n+3} = \sum_{k=1}^{n-1}\binom{2n}{2k}S_{2k+1}S_{2n-2k+1}.$$

Proof. Using (4.2.10), we find that

$$\text{(4.2.28)}\qquad \frac{1}{8}\csc^2\frac{1}{2}\theta+\frac{1}{12}=\frac{1}{2\theta^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nB_{2n+2}(2n+1)}{(2n+2)!}\theta^{2n}.$$

Using (4.2.28) and the Maclaurin series for $\cos(k\theta)$, we find that the left side of (4.2.26) has the form

$$\text{(4.2.29)}\qquad L(\theta)=\left(\frac{1}{2\theta^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nB_{2n+2}}{(2n+2)(2n)!}\theta^{2n}\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}k^{2n}}{(2n)!}\theta^{2n}\right)^2.$$

For $n\geq 1$, the coefficient of θ^{2n} within parentheses above is equal to

$$\frac{(-1)^nB_{2n+2}}{2(2n+2)(2n)!}+\frac{(-1)^{n+1}}{(2n)!}\sum_{k=1}^{\infty}\frac{k^{2n+1}q^k}{1-q^k}=\frac{(-1)^{n+1}}{(2n)!}S_{2n+1},$$

by (4.1.4). Thus, the left side of (4.2.26), i.e., (4.2.29), is equal to

$$\text{(4.2.30)}\qquad L(\theta)=\left(\frac{1}{2\theta^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}S_{2n+1}\theta^{2n}\right)^2.$$

Returning to (4.2.9), we find that

$$\frac{d^2}{d\theta^2}\cot\theta=2\cot\theta\csc^2\theta,$$
$$\frac{d^3}{d\theta^3}\cot\theta=-2-8\cot^2\theta-6\cot^4\theta$$
$$=6+8\frac{d}{d\theta}\cot\theta-6\cot^4\theta,$$

or

$$\text{(4.2.31)}\qquad \cot^4\theta=1+\frac{4}{3}\frac{d}{d\theta}\cot\theta-\frac{1}{6}\frac{d^3}{d\theta^3}\cot\theta.$$

Thus, by (4.2.31), (4.2.9), and Theorem 4.1.3,

(4.2.32)

$$\frac{1}{64}\cot^4\theta+\frac{1}{48}\cot^2\theta+\frac{1}{144}$$
$$=\frac{1}{64}\left(1+\frac{4}{3}\frac{d}{d\theta}\cot\theta-\frac{1}{6}\frac{d^3}{d\theta^3}\cot\theta\right)+\frac{1}{48}\left(-1-\frac{d}{d\theta}\cot\theta\right)+\frac{1}{144}$$
$$=\frac{1}{2^6\cdot 3^2}-\frac{1}{2^7\cdot 3}\frac{d^3}{d\theta^3}\cot\theta$$
$$=\frac{1}{2^6\cdot 3^2}-\frac{1}{2^7\cdot 3}\sum_{n=0}^{\infty}\frac{(-1)^nB_{2n}2^{2n}(2n-1)(2n-2)(2n-3)}{(2n)!}\theta^{2n-4}$$
$$=\frac{1}{2^6\cdot 3^2}-\frac{1}{2^7\cdot 3}\left(-\frac{6}{\theta^4}-\frac{2}{15}+\sum_{n=1}^{\infty}\frac{(-1)^nB_{2n+4}2^{2n+4}}{(2n+4)(2n)!}\theta^{2n}\right)$$
$$=\frac{1}{2^6\theta^4}+\frac{1}{2^6\cdot 3\cdot 5}-\frac{1}{2^3\cdot 3}\sum_{n=1}^{\infty}\frac{(-1)^nB_{2n+4}2^{2n}}{(2n+4)(2n)!}\theta^{2n},$$

where in the penultimate line, we used the fact that $B_4=-\frac{1}{30}$. Using (4.2.32) with θ replaced by $\frac{1}{2}\theta$ and the Maclaurin series for $\cos(k\theta)$, we find that the right side of (4.2.26) is equal to

$$\text{(4.2.33)}\qquad R(\theta)=\frac{1}{4\theta^4}+\frac{1}{2^6\cdot 3\cdot 5}-\frac{1}{2^3\cdot 3}\sum_{n=1}^{\infty}\frac{(-1)^nB_{2n+4}}{(2n+4)(2n)!}\theta^{2n}$$
$$+\frac{1}{12}\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}\left(6+\sum_{n=1}^{\infty}\frac{(-1)^nk^{2n}}{(2n)!}\theta^{2n}\right).$$

The constant term in (4.2.33) equals

$$\text{(4.2.34)}\qquad \frac{1}{2^6\cdot 3\cdot 5}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}=\frac{1}{2}S_3,$$

by (4.1.4). For $n\geq 1$, the coefficient of θ^{2n} in (4.2.33) equals

$$\text{(4.2.35)}\qquad \frac{(-1)^{n-1}B_{2n+4}}{24(2n+4)(2n)!}+\frac{(-1)^n}{12(2n)!}\sum_{k=1}^{\infty}\frac{k^{2n+3}q^k}{1-q^k}=\frac{(-1)^n}{12(2n)!}S_{2n+3},$$

by (4.1.4). Hence, (4.2.33) can be rewritten in the shape

$$\text{(4.2.36)}\qquad R(\theta)=\frac{1}{4\theta^4}+\frac{1}{2}S_3+\frac{1}{12}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}S_{2n+3}\theta^{2n}.$$

Hence, by (4.2.30) and (4.2.36), we deduce that

$$\left(\frac{1}{2\theta^2}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}S_{2n+1}\theta^{2n}\right)^2$$
$$=\frac{1}{4\theta^4}+\frac{1}{2}S_3-\frac{1}{12}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}S_{2n+3}\theta^{2n}.$$

Equating coefficients of θ^{2n}, $n\geq 1$, on both sides above, we find that

$$\frac{(-1)^n}{12(2n)!}S_{2n+3}=\frac{(-1)^n}{(2n)!}\left\{\binom{2n}{2}S_3S_{2n-1}+\binom{2n}{4}S_5S_{2n-3}+\cdots\right.$$
$$\left.+\binom{2n}{2n-2}S_{2n-1}S_3\right\}+\frac{(-1)^n}{(2n-2)!}S_{2n+3}, \tag{4.2.37}$$

or

$$\frac{(n-1)(2n+5)}{12(n+1)(2n+1)}S_{2n+3}=\sum_{k=1}^{n-2}\binom{2n}{2k}S_{2k+1}S_{2n-2k+1},$$

which completes the proof of Theorem 4.2.7. □

For example, putting $n=2$ in (4.2.27) and using (4.1.6) and (4.1.8) yields

$$\frac{1}{20}S_7=6S_3^2 \quad \text{or} \quad E_8=Q^2. \tag{4.2.38}$$

Putting $n=3$ in (4.2.27) yields

$$E_{10}=E_4E_6=QR. \tag{4.2.39}$$

Using (4.2.27), the definitions (4.1.8) and (4.1.9), and induction on r, we show that, for each nonnegative integer r,

$$S_{2r+1}=\sum_{\substack{m,n\geq 0\\ 4m+6n=2r+2}}c_{m,n}Q^mR^n, \tag{4.2.40}$$

where the numbers $c_{m,n}$ are constants. It is clear that (4.2.40) is valid for $r=1,2$ by the definition (4.1.6) of S_{2r+1}, and for $r=3,4$ by (4.2.38) and (4.2.39), respectively. Assume that (4.2.40) holds. We prove (4.2.40) with r replaced by $r+1$. By Theorem 4.2.7 and the induction hypothesis,

$$S_{2r+3}=\sum_{k=1}^{r-1}\sum_{\substack{m',n'\geq 0\\ 4m'+6n'=2k+2}}c_{m',n'}Q^{m'}R^{n'}$$
$$\times\sum_{\substack{m'',n''\geq 0\\ 4m''+6n''=2r-2k+2}}c_{m'',n''}Q^{m''}R^{n''}$$
$$=\sum_{\substack{m=m'+m'',\,n=n'+n''\geq 0\\ 4m+6n=2r+4}}c_{m,n}Q^mR^n.$$

Exercise 4.2.8. *Similarly, using (4.2.8), along with the definitions (4.1.7)–(4.1.9), use induction on r to prove that, for each positive integer r,*

$$\Phi_{1,2r}=\sum_{\substack{\ell,m,n\geq 0,\,\ell\leq 2\\ 2\ell+4m+6n=2r+2}}c_{\ell,m,n}P^{\ell}Q^mR^n, \tag{4.2.41}$$

where the numbers $c_{\ell,m,n}$ are constants. For the case $r=1$, see (4.2.16).

4.3. A Class of Series from Ramanujan's Lost Notebook Expressible in Terms of P, Q, and R

On pages 188 and 369 of his lost notebook, in the pagination of [**104**], Ramanujan examines the series defined for $|q|<1$ and each nonnegative integer k by

$$T_{2k}:=T_{2k}(q)$$
$$:=1+\sum_{n=1}^{\infty}(-1)^n\left\{(6n-1)^{2k}q^{n(3n-1)/2}+(6n+1)^{2k}q^{n(3n+1)/2}\right\}. \tag{4.3.1}$$

Note that the exponents $n(3n\pm 1)/2$ are the generalized pentagonal numbers and that when $k=0$, (4.3.1) reduces to the series in the pentagonal number theorem, Corollary 1.3.5. Ramanujan records formulas for T_{2k}, $k=1,2,\ldots,6$, in terms of the Eisenstein series P, Q, and R, defined by (4.1.7)–(4.1.9). Ramanujan's formulations of

An Elementary Proof that $p(11n+6) \equiv 0(mod\ 11)$

By: Syrous Marivani, Ph.D.

LSU Alexandria

Mathematics Department

8100 HWY 71S

Alexandria, LA 71302

Email: smarivani@lsua.edu

Abstract

In this paper I give an elementary proof of congruence identity $p(11n+6) \equiv 0(mod 11)$.

Let n be a positive integer, let p(n), denotes the number of unrestricted representations of n as a sum of positive integers, where representations with different orders of the same summands are not regarded as distinct. We call p(n) the partition function.

In 1919, Ramanujan [1], [2, pp.210-213] announced that he had found three simple congruences satisfied by p(n), namely,

$$p(5n+4) \equiv 0(mod\ 5)$$
$$p(7n+5) \equiv 0(mod\ 7)$$
$$p(11n+6) \equiv 0(mod\ 11)$$

He gave the proofs of the first two of the above congruences in [1] and later in a short one page note [3], [2, p.230] announced that he had also found a proof of the third identity above. Of the proofs given for the third identity above, the most elementary proof is due to L. Winquist [4] and uses Winquist's Identity. Another elementary approach of proving the third identity has been devised by Berndt, S. H. Chan, Z.-G. Liu, and H. Yesilyurt [5], who established a new identity for $(q;q)_\infty^{10}$. Hirschhorn [6] has devised a common approach to proving all three congruences.

In this paper, I prove the third congruence identity above on the same line of the elementary proofs of the first two congruence identities which are given in [7,pp.103-104]. To do so, I give a review of q-series, P, Q, and R series and several Lemmas and a Theorem whose proof can be found in [7].

Definition 1: Define,

$(a)_0 := (a;q)_0 := 1, \quad (a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),\ n \geq 1,$

$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \qquad\qquad |q| < 1$

We call q the base.

The generating function for p(n), due to Euler, is given by

(1.1) $\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{1-q}\frac{1}{1-q^2}\cdots\frac{1}{1-q^k}\cdots,$

where we define $p(0) = 1$. To see this, observe that the factor

$$\frac{1}{1-q^k} = \sum_{j=0}^{\infty} q^{jk}$$

generates the number of k's that appear in a particular partition of n. Each partition of n appears only once if the right side of (1.1) is interpreted in this manner.

Here also is an introduction to P, Q, and R series.

Definition 2: For each positine integer r, define

(1.2) $S_r = -\frac{B_{r+1}}{2(r+1)} + \sum_{k=1}^{\infty}\frac{k^r q^r}{1-q^k}$

where B_r is the r-th Bernoulli number.

Definition 3: The classical Eisenstein series $E_{2n}(q)$, where n is a positive integer, are defined by

(1.3) $$E_{2n}(q) = -\frac{4n}{B_{2n}} S_{2n-1}$$

Also,

(1.4) $P = P(q) = E_2(q) = 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} = -24S_1$,

(1.5) $Q = Q(q) = E_4(q) = 1 + 240\sum_{k=1}^{\infty} \frac{k^3q^k}{1-q^k} = 240S_3$,

(1.6) $R = R(q) = E_6(q) = 1 - 504\sum_{k=1}^{\infty} \frac{k^5q^k}{1-q^k} = -504S_5$.

Lemma 1: We have

(1.7) $q\frac{dP}{dq} = \frac{P^2-Q}{12}$,

(1.8) $q\frac{dQ}{dq} = \frac{PQ-R}{3}$,

(1.9) $q\frac{dR}{dq} = \frac{PR-Q^2}{2}$.

Proof: [7, pp.89-92].

Lemma 2: We have

(1.10) $E_{10} = QR$.

Proof: [7, pp.93-96].

Theorem 1: We have

(1.11) $Q^3(q) - R^2(q) = 1728q(q;q)_{\infty}^{24}$.

Proof: [7, pp.92-93].

We shall see congrences of the form,

(1.12) $\sum a_n q^n \equiv \sum b_n q^n (mod\ m)$

The latter is equivalent to the condition $a_n \equiv b_n (mod\ m)$, for every integer n appearing as an index in either power series.

We also denote a power series in q with integer coeficients $J = J(q)$.

So, here is our Theorem and its proof.

Theorem 2: We have

(1.13) $p(11n + 6) \equiv 0 (mod\ 11)$.

Proof: Using (1.11), (1.1), definition 1, and Fermat's Little theorem, we have

(1.14) $(Q^3 - R^2)^5 \equiv (q^{121}; q^{121})_\infty \sum_{n=0}^{\infty} p(n)q^{n+5} (mod\ 11)$.

Using Definition 1.2 and (1.10), we have

(1.15) $QR \equiv 1 (mod\ 11)$.

Also, differentiating both sides of (1.15), we have

(1.16) $Q\frac{dR}{dq} + R\frac{dQ}{dq} \equiv 0 (mod\ 11)$.

From now congruence means mod 11.

Using (1.15) and Binomial theorem, we have,

(1.17) $(Q^3 - R^2)^5 \equiv Q^{15} - 5Q^{10} + 10Q^5 + 1 + 5R^5 - R^{10}$.

Using (1.9), (1.15) and (1.16), we have

$-2qQ^{11}\frac{dQ}{dq} \equiv PQ^{12} - Q^{15}$.

Similarly,

$-2qQ^6\frac{dQ}{dq} \equiv PQ^7 - Q^{10}$,

$-2qQ\frac{dQ}{dq} \equiv PQ^2 - Q^5$.

Using the latter 3 identities, we have

(1.18) $Q^{15} - 5Q^{10} + 10Q^5 \equiv PQ^{12} - 5PQ^7 + 10PQ^2 + 2q(Q^{11} - 5Q^6 + 10Q)\frac{dQ}{dq}$.

Using (1.8) and like the derivation of the latter identity, we have

(1.19) $PQ^{12} - 5PQ^7 + 10PQ^2 \equiv Q^{10} - 5Q^5 + 10 + 3q(Q^{11} - 5Q^6 + 10Q)\frac{dQ}{dq}$

Using (1.9), (1.15), (1.16), and (1.8) again we get

(1.20) $Q^{10} - 5Q^5 \equiv PQ^7 - 5PQ^2 + 2q(Q^6 - 5Q)\frac{dQ}{dq}$

(1.21) $PQ^7 - 5PQ^2 \equiv Q^5 - 5 + 3q(Q^6 - 5Q)\frac{dq}{dq}$

(1.22) $Q^5 \equiv PQ^2 + 2qQ\frac{dQ}{dq}$

(1.23) $PQ^2 \equiv 1 + 3qQ\frac{dQ}{dq}$.

Combining (1.18)-(1.23), we have

(1.24) $Q^{15} - 5Q^{10} + 10Q^5 + 1 \equiv -4 + q(5Q^{11} + 2Q^6 - 3Q)\frac{dQ}{dq}$.

Similarly,

(1.25) $-R^{10}+5R^5 \equiv -PR^8+5PR^3+3q(-R^7+5R^2)\frac{dR}{dq}$

(1.26) $-PR^8+5PR^3 \equiv -R^5+5+2q(-R^7+5R^2)\frac{dR}{dq}$

(1.27) $-R^5 \equiv -PR^3-3qR^2\frac{dR}{dq}$

(1.28) $-PR^3 \equiv -1-2qR^2\frac{dR}{dq}$.

Combining (1.25)-(1.28), we have

(1.29) $-R^{10}+5R^5 \equiv 4+q(-5R^7-2R^2)\frac{dR}{dq}$.

Using (1.24), and (1.29) in (1.17) we have,

$(Q^3-R^2)^5 \equiv q(5Q^{11}+2Q^6-3Q)\frac{dQ}{dq}+q(-5R^7-2R^2)\frac{dR}{dq}$.

Multipling both sides of the latter equation by 8(3)(7)=168 and using the Power Rule, we have

(1.30) $3(Q^3-R^2)^5 \equiv q\frac{dJ}{dq}$.

Using (1.14) and (1.30), we have,

$3(q^{121};q^{121})_\infty \sum_{k=0}^{\infty} p(11k+6)q^{11k+11} \equiv 0(mod\ 11)$.

The conclusion follows from the latter.•

References

1. S. Ramanujan, Some properties of p(n), the number of patitions of n, Proc. Cambridge Philos. Soc. 19 (1919), 210-213.

2. S. Ramanujan, Collected Papers, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.

3. S. Ramanujan, Congruence properties of partitions, Proc. London Math. Soc. 18 (1920), records for 13 March 1919, xix.

4. L. Winquist, An elementary proof of $p(11m+6) \equiv 0 (mod 11)$, J. Comb. Thy. 6 (1969), 56-59.

5. B. C. Berndt, S. H. Chan, Z.-G. Liu, and H. Yesilyurt, A new identity for $(q;q)_\infty^{10}$ with an application to Ramanujan's partition congruence modulo 11, Quart. J. Math. (Oxford) 55 (2004), 13-30.

6. M. D. Hirschhorn, Ramanujan's partition congruences, Discrete Math. 131 (1994), 351-355.

7. B. C. Berndt, Number Theory in the Spirit of Ramanujan, American Mathematical Society, Providence, RI 2006.