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A New Symmetry of Partitions

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ABSTRACT

A simple combinatorial argument, based upon the graphic representation of partitions, leads to an explicit formula for the number of partitions of an integer with a given difference between the largest part and the number of parts. Euler's formula for the reciprocal of the partition generating function is an immediate corollary.

The "rank" of a partition of an integer into integer parts is defined as the greatest part minus the number of parts [1, 2]. Let $p(m, n)$ be the number of partitions of n into positive integer parts with rank m. Thus $p(m, n) = 0$ if either $m \geqslant n$ or $m \leqslant -n$. There is a well-known symmetry of partitions, called conjugation [3], which interchanges the greatest part with the number of parts, and thus implies

$$p(m, n) = p(-m, n). \tag{1}$$

In this paper we exhibit a second symmetry of partitions, distinct from (1), and deduce various consequences from the simultaneous existence of the two symmetries.

We define $q(m, n)$ to be the number of partitions of n into non-negative integer parts with rank m. As the addition of zero parts to a partition decreases the rank without changing the sum, we have

$$q(m, n) - q(m + 1, n) = p(m, n), \qquad n \geqslant 1. \tag{2}$$

However, it is important that (2) does not hold for $n = 0$. Since there are no partitions of zero into positive parts,

$$p(m, 0) = 0, \qquad \text{all} \quad m. \tag{3}$$

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On the other hand, there exists an infinite sequence of partitions of zero into non-negative (zero) parts, so that

$$q(m, 0) = 1, \qquad m \leqslant -1,$$
$$q(m, 0) = 0, \qquad m \geqslant 0. \tag{4}$$

Therefore (2) holds everywhere except for the single pair of values $m = -1$, $n = 0$. Tables 1 and 2 show values of $p(m, n)$ and $q(m, n)$, from which it is clear that the mismatch at the point $(-1, 0)$ is not accidental. The values of $p(m, n)$ are centered around the line $m = 0$, whereas those of $q(m, n)$ are centered around $m = -1$.

TABLE 1

VALUES OF $p(m, n)$

n \ m	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0	0	0	0
2	0	0	0	0	0	1	0	1	0	0	0	0	0
3	0	0	0	0	1	0	1	0	1	0	0	0	0
4	0	0	0	1	0	1	1	1	0	1	0	0	0
5	0	0	1	0	1	1	1	1	1	0	1	0	0
6	0	1	0	1	1	2	1	2	1	1	0	1	0
7	1	0	1	1	2	1	3	1	2	1	1	0	1

TABLE 2

VALUES OF $q(m, n)$

n \ m	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
0	1	1	1	1	1	1	0	0	0	0	0	0	0
1	1*	1	1	1	1	1	1	0	0	0	0	0	0
2	2	2	2*	2	2	2	1	1	0	0	0	0	0
3	3	3	3	3	3*	2	2	1	1	0	0	0	0
4	5	5	5	5	4	4	3*	2	1	1	0	0	0
5	7	7	7	6	6	5	4	3	2*	1	1	0	0
6	11	11	10	10	9	8	6	5	3	2	1*	1	0
7	15	14	14	13	12	10	9	6	5	3	2	1	1*

A closer inspection of Table 2 reveals that the values lie symmetrically about the column $m = -1$ on each line: $2n - m =$ constant. For example, the values marked with an asterisk have $2n - m = 8$. The second symmetry of partitions is thus

$$q(m, n) = q(-m - 2, n - m - 1). \tag{5}$$

We can give a simple combinatorial proof of (5). Let P be any partition of n into non-negative parts, with rank m. We can define an "adjoint" partition P' as follows. From P we derive a partition P_1 with strictly positive parts by adding 1 to each part in P. Let P_1' be the conjugate of P_1. Then P' is derived from P_1' by subtracting 1 from each part. This procedure establishes a one-to-one symmetric relationship between partitions P and P' into non-negative parts. The rank of P' is $(-m - 2)$ and its sum is $(n - m - 1)$. Thus (5) is merely a statement of the "adjoint" symmetry.

Let $p(n)$ be the total number of partitions of n into positive parts. As is customary, we make the convention $p(0) = 1$, in spite of (3). The definition of $q(m, n)$ then implies

$$q(m, n) = p(n), \qquad m \leqslant 1 - n, \tag{6}$$

$$q(m, n) = 0, \qquad m \geqslant n, \tag{7}$$

except for the mismatch which occurs when $n = 0$, $m = 0$ or 1. The symmetry (1) together with (2) gives

$$q(m, n) + q(1 - m, n) = q(1 + m, n) + q(-m, n), \tag{8}$$

except for $n = 0$, $m = \pm 1$. Consequently, either side of (8) is a number independent of m. The value of this number is $p(n)$ by (6) and (7). Thus

$$q(m, n) + q(1 - m, n) = p(n), \tag{9}$$

except for $n = 0$, $m = 0$ or 1. This together with (5) implies

$$q(m, n) = p(n) - q(m - 3, n + m - 2), \tag{10}$$

except for $n = 0$, $m = 0$ or 1, or equivalently

$$q(m, n) = p(n - m - 1) - q(m + 3, n - m - 1), \tag{11}$$

except for $m = -2$, $n = -1$ or $m = -3$, $n = -2$.

Either (10) or (11) gives at once an explicit formula for $q(m, n)$ in terms of $p(n)$. Using (11) repeatedly, we find

$$\begin{aligned} q(m, n) &= p(n - m - 1) - p(n - 2m - 5) + q(m + 6, n - 2m - 5) \\ &= p(n - m - 1) - p(n - 2m - 5) + p(n - 3m - 12) \\ &\quad - p(n - 4m - 22) + \cdots \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} p(n - km - \tfrac{1}{2}k(3k - 1)). \end{aligned} \tag{12}$$

The series comes to an end after a finite number of terms, as soon as $\frac{1}{2}k(3k - 1) + km > n$. In the same way (10) gives

$$\begin{aligned} q(m, n) &= p(n) - p(n + m - 2) + p(n + 2m - 7) \\ &\quad - p(n + 3m - 15) + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k p(n + km - \tfrac{1}{2}k(3k + 1)). \end{aligned} \tag{13}$$

Both (12) and (13) hold subject to the proviso that the exceptional case should not arise in (11) or (10), respectively. The exceptional case in (11) will arise only if for some $k \geqslant 1$

$$n = -\tfrac{1}{2}k(3k - 1), \qquad m = 1 - 3k,$$

or

$$n = -\tfrac{1}{2}k(3k + 1), \qquad m = -3k. \tag{14}$$

In case (14) holds, the term $(-1)^{k-1}\, p(0)$ is to be omitted from the right side of (12). Similarly, the exceptional case in (10) will arise only if for some $k \geqslant 0$

$$n = -\tfrac{1}{2}k(3k - 1), \qquad m = 3k,$$

or

$$n = -\tfrac{1}{2}k(3k + 1), \qquad m = 3k + 1. \tag{15}$$

In case (15) holds, the term $(-1)^k\, p(0)$ on the right side of (13) is to be omitted.

In particular, both (12) and (13) hold as written for all m if n is positive, (12) holds for all n if m is non-negative, and (13) holds for all n if m is negative.

It is now easy to deduce from (12) and (13) the explicit form of the generating functions

$$Q_m(x) = \sum_{n=0}^{\infty} q(m, n) x^n. \tag{16}$$

We write also

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n. \tag{17}$$

Then (12), with the correction applied whenever (14) holds, gives the result

$$Q_m(x) = P(x) \sum_{k=1}^{\infty} (-1)^{k-1} x^{km+\frac{1}{2}k(3k-1)}, \tag{18}$$

if either $m \geqslant 0$ or $m \equiv 2 \pmod 3$, and

$$Q_m(x) = P(x) \sum_{k=1}^{\infty} (-1)^{k-1} x^{km+\frac{1}{2}k(3k-1)} + (-1)^p x^{-(1/6)m(m-1)}, \tag{19}$$

if $m < 0$ and $m = 1 + 3p$ or $m = 3p$. Similarly, (13) with (15) gives

$$Q_m(x) = P(x) \sum_{k=0}^{\infty} (-1)^k x^{-km+\frac{1}{2}k(3k+1)}, \tag{20}$$

if either $m < 0$ or $m \equiv 2 \pmod 3$, and

$$Q_m(x) = P(x) \sum_{k=0}^{\infty} (-1)^k x^{-km+\frac{1}{2}k(3k+1)} - (-1)^p x^{-(1/6)m(m-1)}, \tag{21}$$

if $m \geqslant 0$ and $m = 1 + 3p$ or $m = 3p$.

The simple generating functions for $q(m, n)$ are (18) for $m \geqslant 0$ and (20) for $m < 0$. These are consistent with the generating functions for $p(m, n)$ derived earlier [1, 2]. Also, (18) and (20) are related through the identity

$$Q_m(x) = x^{1+m} Q_{-m-2}(x), \tag{22}$$

which is a restatement of the basic symmetry (5).

Finally we can equate (19) with (20) or (18) with (21). The result is

$$P(x) \sum_{k=-\infty}^{\infty} (-1)^k x^{-km+\frac{1}{2}k(3k+1)} = (-1)^p x^{-(1/6)m(m-1)}, \tag{23}$$

when $m = 1 + 3p$ or $m = 3p$. In particular, taking $m = p = 0$, we have Euler's formula

$$[P(x)]^{-1} = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{1}{2}k(3k+1)}. \tag{24}$$

This combinatorial derivation of Euler's formula is less direct, but perhaps more illuminating, than the well-known combinatorial proof by Franklin [4].

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