modular form $f \in M_k(\Gamma)$ equals $k\,\mathrm{Vol}(\Gamma\backslash\mathfrak{H})/4\pi$, where just as in the case of Γ_1 we must count the zeros at elliptic fixed points or cusps of Γ with appropriate multiplicities. The same argument as for Corollary 1 of Proposition 2 then tells us $M_k(\Gamma)$ is finite dimensional and gives an explicit upper bound:

Proposition 3. *Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ for which $\Gamma\backslash\mathfrak{H}$ has finite volume V. Then* $\dim M_k(\Gamma) \le \dfrac{kV}{4\pi} + 1$ *for all $k \in \mathbb{Z}$.*

In particular, we have $M_k(\Gamma) = \{0\}$ for $k < 0$ and $M_0(\Gamma) = \mathbb{C}$, i.e., there are no holomorphic modular forms of negative weight on any group Γ, and the only modular forms of weight 0 are the constants. A further consequence is that any three modular forms on Γ are algebraically dependent. (If f, g, h were algebraically independent modular forms of positive weights, then for large k the dimension of $M_k(\Gamma)$ would be at least the number of monomials in f, g, h of total weight k, which is bigger than some positive multiple of k^2, contradicting the dimension estimate given in the proposition.) Equivalently, any two modular functions on Γ are algebraically dependent, since every modular function is a quotient of two modular forms. This is a special case of the general fact that there cannot be more than n algebraically independent algebraic functions on an algebraic variety of dimension n. But the most important consequence of Proposition 3 from our point of view is that it is the origin of the (unreasonable?) effectiveness of modular forms in number theory: if we have two interesting arithmetic sequences $\{a_n\}_{n\ge0}$ and $\{b_n\}_{n\ge0}$ and conjecture that they are identical (and clearly many results of number theory can be formulated in this way), then if we can show that both $\sum a_nq^n$ and $\sum b_nq^n$ are modular forms of the same weight and group, we need only verify the equality $a_n = b_n$ for a finite number of n in order to know that it is true in general. There will be many applications of this principle in these notes.

2 First Examples: Eisenstein Series and the Discriminant Function

In this section we construct our first examples of modular forms: the Eisenstein series $E_k(z)$ of weight $k > 2$ and the discriminant function $\Delta(z)$ of weight 12, whose definition is closely connected to the non-modular Eisenstein series $E_2(z)$.

2.1 Eisenstein Series and the Ring Structure of $M_*(\Gamma_1)$

There are two natural ways to introduce the Eisenstein series. For the first, we observe that the characteristic transformation equation (2) of a modular

Proposition 5. *The Fourier expansion of the Eisenstein series $\mathbb{G}_k(z)$ (k even, $k > 2$) is*

$$\mathbb{G}_k(z) \;=\; -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n\,, \tag{13}$$

where B_k is the kth Bernoulli number and where $\sigma_{k-1}(n)$ for $n \in \mathbb{N}$ denotes the sum of the $(k-1)$st powers of the positive divisors of n.

We recall that the Bernoulli numbers are defined by the generating function $\sum_{k=0}^{\infty} B_k x^k/k! = x/(e^x - 1)$ and that the first values of B_k ($k > 0$ even) are given by $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$, and $B_{14} = \frac{7}{6}$.

Proof. A well known and easily proved identity of Euler states that

$$\sum_{n \in \mathbb{Z}} \frac{1}{z+n} \;=\; \frac{\pi}{\tan \pi z} \qquad \bigl(\, z \in \mathbb{C} \setminus \mathbb{Z} \,\bigr)\,, \tag{14}$$

where the sum on the left, which is not absolutely convergent, is to be interpreted as a Cauchy principal value ($= \lim \sum_{-M}^{N}$ where M, N tend to infinity with $M - N$ bounded). The function on the right is periodic of period 1 and its Fourier expansion for $z \in \mathfrak{H}$ is given by

$$\frac{\pi}{\tan \pi z} \;=\; \pi\,\frac{\cos \pi z}{\sin \pi z} \;=\; \pi i\,\frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \;=\; -\pi i\,\frac{1+q}{1-q} \;=\; -2\pi i \left(\frac{1}{2} + \sum_{r=1}^{\infty} q^r\right),$$

where $q = e^{2\pi i z}$. Substitute this into (14), differentiate $k-1$ times and divide by $(-1)^{k-1}(k-1)!$ to get

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} \;=\; \frac{(-1)^{k-1}}{(k-1)!}\,\frac{d^{k-1}}{dz^{k-1}}\left(\frac{\pi}{\tan \pi z}\right) \;=\; \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1}\, q^r \qquad (k \geq 2,\; z \in \mathfrak{H})\,,$$

an identity known as Lipschitz's formula. Now the Fourier expansion of G_k ($k > 2$ even) is obtained immediately by splitting up the sum in (10) into the terms with $m = 0$ and those with $m \neq 0$:

$$\begin{aligned} G_k(z) \;&=\; \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n^k} + \frac{1}{2} \sum_{\substack{m,\, n \in \mathbb{Z} \\ m \neq 0}} \frac{1}{(mz+n)^k} \;=\; \sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &=\; \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1}\, q^{mr} \\ &=\; \frac{(2\pi i)^k}{(k-1)!} \left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n \right), \end{aligned}$$

where in the last line we have used Euler's evaluation of $\zeta(k)$ ($k > 0$ even) in terms of Bernoulli numbers. The result follows.

The first three examples of Proposition 5 are the expansions

$$\begin{aligned}
\mathbb{G}_4(z) &= \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + 252q^6 + \cdots , \\
\mathbb{G}_6(z) &= -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + \cdots , \\
\mathbb{G}_8(z) &= \frac{1}{480} + q + 129q^2 + 2188q^3 + \cdots .
\end{aligned}$$

The other two normalizations of these functions are given by

$$\begin{aligned}
G_4(z) &= \frac{16\,\pi^4}{3!}\,\mathbb{G}_4(z) = \frac{\pi^4}{90}\,E_4(z)\,, \qquad E_4(z) = 1 + 240q + 2160q^2 + \cdots , \\
G_6(z) &= -\frac{64\,\pi^6}{5!}\,\mathbb{G}_6(z) = \frac{\pi^6}{945}\,E_6(z)\,, \quad E_6(z) = 1 - 504q - 16632q^2 - \cdots , \\
G_8(z) &= \frac{256\,\pi^8}{7!}\,\mathbb{G}_8(z) = \frac{\pi^8}{9450}\,E_8(z)\,, \quad E_8(z) = 1 + 480q + 61920q^2 + \cdots .
\end{aligned}$$

Remark. We have discussed only Eisenstein series on the full modular group in detail, but there are also various kinds of Eisenstein series for subgroups $\Gamma \subset \Gamma_1$. We give one example. Recall that a *Dirichlet character* modulo $N \in \mathbb{N}$ is a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$, extended to a map $\chi : \mathbb{Z} \to \mathbb{C}$ (traditionally denoted by the same letter) by setting $\chi(n)$ equal to $\chi(n \bmod N)$ if $(n, N) = 1$ and to 0 otherwise. If χ is a non-trivial Dirichlet character and k a positive integer with $\chi(-1) = (-1)^k$, then there is an Eisenstein series having the Fourier expansion

$$\mathbb{G}_{k,\chi}(z) \;=\; c_k(\chi) + \sum_{n=1}^{\infty}\biggl(\sum_{d|n}\chi(d)\,d^{k-1}\biggr)\,q^n$$

which is a "modular form of weight k and character χ on $\Gamma_0(N)$." (This means that $\mathbb{G}_{k,\chi}(\frac{az+b}{cz+d}) = \chi(a)(cz+d)^k\mathbb{G}_{k,\chi}(z)$ for any $z \in \mathfrak{H}$ and any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ with $c \equiv 0 \pmod N$.) Here $c_k(\chi) \in \overline{\mathbb{Q}}$ is a suitable constant, given explicitly by $c_k(\chi) = \frac{1}{2}L(1-k,\chi)$, where $L(s,\chi)$ is the analytic continuation of the Dirichlet series $\sum_{n=1}^{\infty}\chi(n)n^{-s}$.

The simplest example, for $N = 4$ and $\chi = \chi_{-4}$ the Dirichlet character modulo 4 given by

$$\chi_{-4}(n) \;=\; \begin{cases} +1 & \text{if } n \equiv 1 \pmod 4\,, \\ -1 & \text{if } n \equiv 3 \pmod 4\,, \\ \;\;0 & \text{if } n \text{ is even} \end{cases} \tag{15}$$

and $k = 1$, is the series

$$\mathbb{G}_{1,\chi_{-4}}(z) = c_1(\chi_{-4}) + \sum_{n=1}^{\infty}\Biggl(\sum_{d|n}\chi_{-4}(d)\Biggr)\,q^n = \frac{1}{4} + q + q^2 + q^4 + 2q^5 + q^8 + \cdots . \tag{16}$$

(The fact that $L(0,\chi_{-4}) = 2c_1(\chi_{-4}) = \frac{1}{2}$ is equivalent via the functional equation of $L(s,\chi_{-4})$ to Leibnitz's famous formula $L(1,\chi_{-4}) = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$.) We will see this function again in §3.1.

♠ Identities Involving Sums of Powers of Divisors

We now have our first explicit examples of modular forms and their Fourier expansions and can immediately deduce non-trivial number-theoretic identities. For instance, each of the spaces $M_4(\Gamma_1)$, $M_6(\Gamma_1)$, $M_8(\Gamma_1)$, $M_{10}(\Gamma_1)$ and $M_{14}(\Gamma_1)$ has dimension exactly 1 by the corollary to Proposition 2, and is therefore spanned by the Eisenstein series $E_k(z)$ with leading coefficient 1, so we immediately get the identities

$$E_4(z)^2 = E_8(z)\,, \quad E_4(z)E_6(z) = E_{10}(z)\,,$$
$$E_6(z)E_8(z) = E_4(z)E_{10}(z) = E_{14}(z)\,.$$

Each of these can be combined with the Fourier expansion given in Proposition 5 to give an identity involving the sums-of-powers-of-divisors functions $\sigma_{k-1}(n)$, the first and the last of these being

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{\sigma_7(n)-\sigma_3(n)}{120}\,,$$
$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_9(n-m) = \frac{\sigma_{13}(n)-11\sigma_9(n)+10\sigma_3(n)}{2640}\,.$$

Of course similar identities can be obtained from modular forms in higher weights, even though the dimension of $M_k(\Gamma_1)$ is no longer equal to 1. For instance, the fact that $M_{12}(\Gamma_1)$ is 2-dimensional and contains the three modular forms E_4E_8, E_6^2 and E_{12} implies that the three functions are linearly dependent, and by looking at the first two terms of the Fourier expansions we find that the relation between them is given by $441E_4E_8 + 250E_6^2 = 691E_{12}$, a formula which the reader can write out explicitly as an identity among sums-of-powers-of-divisors functions if he or she is so inclined. It is not easy to obtain any of these identities by direct number-theoretical reasoning (although in fact it can be done). ♡

2.3 The Eisenstein Series of Weight 2

In §2.1 and §2.2 we restricted ourselves to the case when $k > 2$, since then the series (9) and (10) are absolutely convergent and therefore define modular forms of weight k. But the final formula (13) for the Fourier expansion of $\mathbb{G}_k(z)$ converges rapidly and defines a holomorphic function of z also for $k = 2$, so in this weight we can simply *define* the Eisenstein series $\mathbb{G}_2$, G_2 and E_2 by equations (13), (12), and (11), respectively, i.e.,

$$\begin{aligned}\mathbb{G}_2(z) \;&=\; -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)\, q^n \;=\; -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \cdots\,,\\ G_2(z) \;&=\; -4\pi^2\,\mathbb{G}_2(z)\,, \quad E_2(z) \;=\; \frac{6}{\pi^2}\,G_2(z) \;=\; 1 - 24q - 72q^2 - \cdots\,.\end{aligned} \tag{17}$$

Moreover, the same proof as for Proposition 5 still shows that $G_2(z)$ is given by the expression (10), if we agree to carry out the summation over n first and then over m:

$$G_2(z) \;=\; \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} \;+\; \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2}\,. \tag{18}$$

The only difference is that, because of the non-absolute convergence of the double series, we can no longer interchange the order of summation to get the modular transformation equation $G_2(-1/z) = z^2 G_2(z)$. (The equation $G_2(z+1) = G_2(z)$, of course, still holds just as for higher weights.) Nevertheless, the function $G_2(z)$ and its multiples $E_2(z)$ and $\mathbb{G}_2(z)$ do have some modular properties and, as we will see later, these are important for many applications.

Proposition 6. *For* $z \in \mathfrak{H}$ *and* $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ *we have*

$$G_2\left(\frac{az+b}{cz+d}\right) \;=\; (cz+d)^2\,G_2(z) \;-\; \pi i c(cz+d)\,. \tag{19}$$

Proof. There are many ways to prove this. We sketch one, due to Hecke, since the method is useful in many other situations. The series (10) for $k = 2$ does not converge absolutely, but it is just at the edge of convergence, since $\sum_{m,n} |mz+n|^{-\lambda}$ converges for any real number $\lambda > 2$. We therefore modify the sum slightly by introducing

$$G_{2,\varepsilon}(z) \;=\; \frac{1}{2}{\sum_{m,\,n}}'\frac{1}{(mz+n)^2\,|mz+n|^{2\varepsilon}} \qquad (z\in\mathfrak{H},\ \varepsilon>0)\,. \tag{20}$$

(Here $\sum'$ means that the value $(m,n) = (0,0)$ is to be omitted from the summation.) The new series converges absolutely and transforms by $G_{2,\varepsilon}\left(\frac{az+b}{cz+d}\right) = (cz+d)^2|cz+d|^{2\varepsilon}G_{2,\varepsilon}(z)$. We claim that $\lim\limits_{\varepsilon\to 0} G_{2,\varepsilon}(z)$ exists and equals $G_2(z) - \pi/2y$, where $y = \Im(z)$. It follows that each of the three non-holomorphic functions

$$G_2^*(z) \;=\; G_2(z) - \frac{\pi}{2y}\,, \quad E_2^*(z) \;=\; E_2(z) - \frac{3}{\pi y}\,, \quad \mathbb{G}_2^*(z) \;=\; \mathbb{G}_2(z) + \frac{1}{8\pi y} \tag{21}$$

transforms like a modular form of weight 2, and from this one easily deduces the transformation equation (19) and its analogues for E_2 and $\mathbb{G}_2$. To prove the claim, we define a function I_ε by

$$I_\varepsilon(z) \;=\; \int_{-\infty}^{\infty} \frac{dt}{(z+t)^2\,|z+t|^{2\varepsilon}} \qquad \bigl(z\in\mathfrak{H},\ \varepsilon>-\tfrac12\bigr)\,.$$

Then for $\varepsilon>0$ we can write

$$\begin{aligned} G_{2,\varepsilon} - \sum_{m=1}^{\infty} I_\varepsilon(mz) \;&=\; \sum_{n=1}^{\infty} \frac{1}{n^{2+2\varepsilon}} \\ &+ \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\biggl[\frac{1}{(mz+n)^2\,|mz+n|^{2\varepsilon}} \;-\; \int_n^{n+1}\frac{dt}{(mz+t)^2|mz+t|^{2\varepsilon}}\biggr]\,. \end{aligned}$$

Both sums on the right converge absolutely and locally uniformly for $\varepsilon>-\frac12$ (the second one because the expression in square brackets is $\mathrm{O}\bigl(|mz+n|^{-3-2\varepsilon}\bigr)$ by the mean-value theorem, which tells us that $f(t)-f(n)$ for any differentiable function f is bounded in $n\le t\le n+1$ by $\max_{n\le u\le n+1}|f'(u)|$), so the limit of the expression on the right as $\varepsilon\to0$ exists and can be obtained simply by putting $\varepsilon=0$ in each term, where it reduces to $G_2(z)$ by (18). On the other hand, for $\varepsilon>-\frac12$ we have

$$\begin{aligned} I_\varepsilon(x+iy) \;&=\; \int_{-\infty}^{\infty}\frac{dt}{(x+t+iy)^2\,((x+t)^2+y^2)^\varepsilon} \\ &=\; \int_{-\infty}^{\infty}\frac{dt}{(t+iy)^2\,(t^2+y^2)^\varepsilon} \;=\; \frac{I(\varepsilon)}{y^{1+2\varepsilon}}\,, \end{aligned}$$

where $I(\varepsilon)=\int_{-\infty}^{\infty}(t+i)^{-2}(t^2+1)^{-\varepsilon}dt$, so $\sum_{m=1}^{\infty} I_\varepsilon(mz) = I(\varepsilon)\zeta(1+2\varepsilon)/y^{1+2\varepsilon}$ for $\varepsilon>0$. Finally, we have $I(0)=0$ (obvious),

$$I'(0) \;=\; -\int_{-\infty}^{\infty}\frac{\log(t^2+1)}{(t+i)^2}\,dt \;=\; \biggl(\frac{1+\log(t^2+1)}{t+i} \;-\; \tan^{-1}t\biggr)\bigg|_{-\infty}^{\infty} \;=\; -\,\pi\,,$$

and $\zeta(1+2\varepsilon)=\dfrac{1}{2\varepsilon}+\mathrm{O}(1)$, so the product $I(\varepsilon)\zeta(1+2\varepsilon)/y^{1+2\varepsilon}$ tends to $-\pi/2y$ as $\varepsilon\to0$. The claim follows.

Remark. The transformation equation (18) says that G_2 is an example of what is called a *quasimodular* form, while the functions G_2^*, E_2^* and $\mathbb{G}_2^*$ defined in (21) are so-called *almost holomorphic modular forms* of weight 2. We will return to this topic in Section 5.

2.4 The Discriminant Function and Cusp Forms

For $z\in\mathfrak{H}$ we define the *discriminant function* $\Delta(z)$ by the formula

$$\Delta(z) \;=\; e^{2\pi iz}\prod_{n=1}^{\infty}\bigl(1-e^{2\pi inz}\bigr)^{24}\,. \tag{22}$$

The point is now that the two transformations $z \mapsto z+1$ and $z \mapsto -1/4z$ generate a subgroup of $\mathrm{SL}(2,\mathbb{R})$ which is commensurable with $\mathrm{SL}(2,\mathbb{Z})$, so (30) implies that the function $\theta(z)$ is a modular form of weight 1/2. (We have not defined modular forms of half-integral weight and will not discuss their theory in these notes, but the reader can simply interpret this statement as saying that $\theta(z)^2$ is a modular form of weight 1.) More specifically, for every $N \in \mathbb{N}$ we have the "congruence subgroup" $\Gamma_0(N) \subseteq \Gamma_1 = \mathrm{SL}(2,\mathbb{Z})$, consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ with c divisible by N, and the larger group $\Gamma_0^+(N) = \langle \Gamma_0(N), W_N \rangle = \Gamma_0(N) \cup \Gamma_0(N)W_N$, where $W_N = \frac{1}{\sqrt{N}}\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ ("Fricke involution") is an element of $\mathrm{SL}(2,\mathbb{R})$ of order 2 which normalizes $\Gamma_0(N)$. The group $\Gamma_0^+(N)$ contains the elements $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and W_N for any N. In general they generate a subgroup of infinite index, so that to check the modularity of a given function it does not suffice to verify its behavior just for $z \mapsto z+1$ and $z \mapsto -1/Nz$, but for $N = 4$ (like for $N = 1$!) they generate the full group and this *is* sufficient. The proof is simple. Since $W_N^2 = -1$, it is sufficient to show that the two matrices T and $\widetilde{T} = W_4 T W_4^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ generate the image of $\Gamma_0(4)$ in $\mathrm{PSL}(2,\mathbb{R})$, i.e., that any element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ is, up to sign, a word in T and $\widetilde{T}$. Now a is odd, so $|a| \neq 2|b|$. If $|a| < 2|b|$, then either $b+a$ or $b-a$ is smaller than b in absolute value, so replacing γ by $\gamma \cdot T^{\pm 1}$ decreases a^2+b^2. If $|a| > 2|b| \neq 0$, then either $a+4b$ or $a-4b$ is smaller than a in absolute value, so replacing γ by $\gamma \cdot \widetilde{T}^{\pm 1}$ decreases a^2+b^2. Thus we can keep multiplying γ on the right by powers of T and $\widetilde{T}$ until $b = 0$, at which point $\pm\gamma$ is a power of $\widetilde{T}$.

Now, by the principle "a finite number of q-coefficients suffice" formulated at the end of Section 1, the mere fact that $\theta(z)$ is a modular form is already enough to let one prove non-trivial identities. (We enunciated the principle only in the case of forms of integral weight, but even without knowing the details of the theory it is clear that it then also applies to half-integral weight, since a space of modular forms of half-integral weight can be mapped injectively into a space of modular forms of the next higher integral weight by multiplying by $\theta(z)$.) And indeed, with almost no effort we obtain proofs of two of the most famous results of number theory of the 17th and 18th centuries, the theorems of Fermat and Lagrange about sums of squares.

♠ Sums of Two and Four Squares

Let $r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 \mid a^2+b^2 = n\}$ be the number of representations of an integer $n \geq 0$ as a sum of two squares. Since $\theta(z)^2 = \big(\sum_{a\in\mathbb{Z}} q^{a^2}\big)\big(\sum_{b\in\mathbb{Z}} q^{b^2}\big)$, we see that $r_2(n)$ is simply the coefficient of q^n in $\theta(z)^2$. From Proposition 9 and the just-proved fact that $\Gamma_0(4)$ is generated by $-\mathrm{Id}_2$, T and $\widetilde{T}$, we find that the function $\theta(z)^2$ is a "modular form of weight 1 and character χ_{-4} on $\Gamma_0(4)$" in the sense explained in the paragraph preceding equation (15), where χ_{-4} is the Dirichlet character modulo 4 defined by (15).

Since the Eisenstein series $\mathbb{G}_{1,\chi_{-4}}$ in (16) is also such a modular form, and since the space of all such forms has dimension at most 1 by Proposition 3 (because $\Gamma_0(4)$ has index 6 in $\mathrm{SL}(2,\mathbb{Z})$ and hence volume 2π), these two functions must be proportional. The proportionality factor is obviously 4, and we obtain:

Proposition 10. *Let n be a positive integer. Then the number of representations of n as a sum of two squares is 4 times the sum of $(-1)^{(d-1)/2}$, where d runs over the positive odd divisors of n.*

Corollary (Theorem of Fermat). *Every prime number $p \equiv 1 \pmod 4$ is a sum of two squares.*

Proof of Corollary. We have $r_2(p) = 4\left(1 + (-1)^{(p-1)/4}\right) = 8 \neq 0$.

The same reasoning applies to other powers of θ. In particular, the number $r_4(n)$ of representations of an integer n as a sum of four squares is the coefficient of q^n in the modular form $\theta(z)^4$ of weight 2 on $\Gamma_0(4)$, and the space of all such modular forms is at most two-dimensional by Proposition 3. To find a basis for it, we use the functions $\mathbb{G}_2(z)$ and $\mathbb{G}_2^*(z)$ defined in equations (17) and (21). We showed in §2.3 that the latter function transforms with respect to $\mathrm{SL}(2,\mathbb{Z})$ like a modular form of weight 2, and it follows easily that the three functions $\mathbb{G}_2^*(z)$, $\mathbb{G}_2^*(2z)$ and $\mathbb{G}_2^*(4z)$ transform like modular forms of weight 2 on $\Gamma_0(4)$ (exercise!). Of course these three functions are not holomorphic, but since $\mathbb{G}_2^*(z)$ differs from the holomorphic function $\mathbb{G}_2(z)$ by $1/8\pi y$, we see that the linear combinations $\mathbb{G}_2^*(z) - 2\mathbb{G}_2^*(2z) = \mathbb{G}_2(z) - 2\mathbb{G}_2(2z)$ and $\mathbb{G}_2^*(2z) - 2\mathbb{G}_2^*(4z) = \mathbb{G}_2(2z) - 2\mathbb{G}_2(4z)$ are holomorphic, and since they are also linearly independent, they provide the desired basis for $M_2(\Gamma_0(4))$. Looking at the first two Fourier coefficients of $\theta(z)^4 = 1 + 8q + \cdots$, we find that $\theta(z)^4$ equals $8\,(\mathbb{G}_2(z) - 2\mathbb{G}_2(2z)) + 16\,(\mathbb{G}_2(2z) - 2\mathbb{G}_2(4z))$. Now comparing coefficients of q^n gives:

Proposition 11. *Let n be a positive integer. Then the number of representations of n as a sum of four squares is* 8 *times the sum of the positive divisors of n which are not multiples of* 4.

Corollary (Theorem of Lagrange). *Every positive integer is a sum of four squares.* ♡

For another simple application of the q-expansion principle, we introduce two variants $\theta_M(z)$ and $\theta_F(z)$ ("M" and "F" for "male" and "female" or "minus sign" and "fermionic") of the function $\theta(z)$ by inserting signs or by shifting the indices by 1/2 in its definition:

$$\theta_M(z) \;=\; \sum_{n\in\mathbb{Z}} (-1)^n\, q^{n^2} \;=\; 1 - 2q + 2q^4 - 2q^9 + \cdots\,,$$
$$\theta_F(z) \;=\; \sum_{n\in\mathbb{Z}+1/2} q^{n^2} \;=\; 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots\,.$$