Introduction

> In der Theorie der Thetafunctionen ist es leicht, eine beliebig grosse Menge von Relationen aufzustellen, aber die Schwierigkeit beginnt da, wo es sich darum handelt, aus diesem Labyrinth von Formeln einen Ausweg zu finden.
>
> G. Frobenius

The content of this volume is more unified than those of the first two volumes of our attempts to provide proofs of the many beautiful theorems bequeathed to us by Ramanujan in his notebooks. Theta-functions provide the binding glue that blends Chapters 16–21 together. Although we provide proofs here for all of Ramanujan's formulas, in many cases, we have been unable to find the roads that led Ramanujan to his discoveries. It is hoped that others will attempt to discover the pathways that Ramanujan took on his journey through his luxuriant labyrinthine forest of enchanting and alluring formulas.

We first briefly review the content of Chapters 16–21. Although theta-functions play the leading role, several other topics make appearances as well.

Some of Ramanujan's most famous theorems are found in Chapter 16. The chapter begins with basic hypergeometric series and some q-continued fractions. In particular, a generalization of the Rogers–Ramanujan continued fraction and a finite version of the Rogers–Ramanujan continued fraction are found. Entry 7 offers an identity from which the Rogers–Ramanujan identities (found in Section 38) can be deduced as limiting cases, a fact that evidently Ramanujan failed to notice. The material on q-series ends with Ramanujan's celebrated ${}_1\psi_1$ summation. After stating the Jacobi triple product identity, which is a corollary of Ramanujan's ${}_1\psi_1$ summation, Ramanujan commences his work on theta-functions. Several of his results are classical and well known, but Ramanujan offers many interesting new results, especially in Sections 33–35. For an enlightening discussion of Ramanujan's contributions to basic hypergeometric series, as well as to hypergeometric series, see R. Askey's survey paper [8].

Chapter 17 begins with Ramanujan's development of some of the basic theory of elliptic functions highlighted by Entry 6, which provides the basic inversion formula relating theta-functions with elliptic integrals and hypergeometric functions. Section 7 offers many beautiful theorems on elliptic integrals. The following sections are devoted to a catalogue of formulas for the most well-known theta-functions and for Ramanujan's Eisenstein series, L, M, and N, evaluated at different powers of the argument. These formulas are of central importance in proving modular equations in Chapters 19–21.

Several topics are examined in Chapter 18, although most attention is given to the Jacobian elliptic functions. Approximations to π and the perimeter of an ellipse are found. More problems in geometry are discussed in this chapter than in any other chapter. The chapter ends with Ramanujan's initial findings about modular equations.

Chapters 19 and 20 are devoted to modular equations and associated theta-function identities. Most of the results in these two chapters are new and show Ramanujan at his very best. It is here that our proofs undoubtedly often stray from the paths followed by Ramanujan.

Chapter 21 occupies only 4 pages and is the shortest chapter in the second notebook. The content is not unlike that of the previous two chapters, but here the emphasis is on formulas for the series L, M, and N.

Since Ramanujan's death in 1920, there has been much speculation on the sources from which Ramanujan first learned about elliptic functions. In commenting on Ramanujan's paper [2] in Ramanujan's Collected Papers [10], L. J. Mordell writes "It would be extremely interesting to know if and how much Ramanujan is indebted to other writers." Mordell then conjectures that Ramanujan might have studied either Greenhill's [1] or Cayley's [1] books on elliptic functions. Greenhill's book can be found in the library at the Government College of Kumbakonam, but we have been unable to ascertain for certain if this book was in the library when Ramanujan lived in Kumbakonam. Hardy [3, p. 212] remarks that these two books were in the library at the University of Madras, where Ramanujan held a scholarship for nine months before departing for England. Hardy then quotes Littlewood's thoughts: "a sufficient, and I think necessary, explanation would be that Greenhill's very odd and individual Elliptic Functions was his text-book." Mordell, Hardy, and Littlewood surmised that Greenhill's book served as Ramanujan's source of knowledge partly because Greenhill's development avoids the theory of functions of a complex variable, a subject thought to have been never learned by Ramanujan. In particular, the double periodicity of elliptic functions is not mentioned by Greenhill until page 254. In the unorganized portions of the second notebook and in the third notebook, there is some evidence that Ramanujan knew a few facts about complex function theory. (See Berndt's book [11].) However, Ramanujan's development of the theory of elliptic functions did not need or depend on complex function theory.

Ramanujan also never mentions double periodicity. Because Cayley's book contains several sections on modular equations, it is reasonable to conjecture that this book might have been one of Ramanujan's sources of learning.

The origins of Ramanujan's knowledge of elliptic functions are probably not very important, since Ramanujan's development of the subject is uniquely and characteristically his own without a trace of influence by any other author. Ramanujan does not even use the standard notations for elliptic integrals and any of the classical elliptic functions. The content of Ramanujan's initial efforts overlaps with some of Jacobi's findings in his famous *Fundamenta Nova* [1], [2]. However, it is unlikely that Ramanujan had access to this work. Moreover, while the Jacobian elliptic functions were central in Jacobi's development, they play a far more minor role in Ramanujan's theory. (Our proofs in the pages that follow undoubtedly employ the Jacobian elliptic functions more than Ramanujan did.) Both Jacobi and Ramanujan extensively utilized theta-functions, but the evolution of Ramanujan's theory is quite different from that of Jacobi. The classical, general theta-function $\vartheta_3(z, q)$ may be defined by

$$\vartheta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz}, \tag{I1}$$

where $|q| < 1$ and z is any complex number. Ramanujan's general theta-function $f(a, b)$ is given by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{I2}$$

where $|ab| < 1$. The generalities of (I1) and (I2) are the same. To see this, set $a = q\exp(2iz)$ and $b = q\exp(2iz)$. For many purposes, the definition (I1) is superior. However, for Ramanujan's interests and theory, (I2) is definitely the preferred definition and was strongly instrumental in helping Ramanujan discover many new theorems in the subject.

Upon studying Ramanujan's development of the theory of modular equations in Chapters 18–21, we now are able to understand more clearly the rationale for Ramanujan's introduction of "modular equations" in Sections 15 and 16 of Chapter 15 of his second notebook [9], which we have previously described in Part II [9]. Before returning to this material, we need to define the generalized hypergeometric function ${}_{p+1}F_p$ by

$${}_{p+1}F_p(\alpha_1, \alpha_2, \ldots, \alpha_{p+1}; \beta_1, \beta_2, \ldots, \beta_p; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_{p+1})_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_p)_n} \frac{z^n}{n!},$$

where p is a nonnegative integer, $\alpha_1, \alpha_2, \ldots, \alpha_{p+1}$, $\beta_1, \beta_2, \ldots, \beta_p$ are complex numbers, $|z| < 1$, and

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1),$$

for each nonnegative integer n.

Ramanujan begins his study of "modular equatons" in Chapter 15 by defining

$$F(x) := (1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} x^n = {}_1F_0(\tfrac{1}{2}; x), \qquad |x| < 1. \tag{I3}$$

He then states the trivial identity

$$F\left(\frac{2t}{1+t}\right) = (1+t)F(t^2). \tag{I4}$$

After setting $\alpha = 2t/(1+t)$ and $\beta = t^2$, Ramanujan offers the "modular equation of degree 2,"

$$\beta(2-\alpha)^2 = \alpha^2, \tag{I5}$$

which is readily verified. The factor $(1+t)$ in (I4) is called the multiplier. He then derives some modular equations of higher degree and offers some general remarks. We emphasize that this definition of modular equation has no connection with any of the standard definitions, but we shall draw some parallels shortly.

There are many definitions of a modular equation in the literature. See Ramanathan's paper [10] or our expository introduction to Ramanujan's modular equations [7] for discussions of some of these alternative definitions. We now give the definition of a modular equation that Ramanujan employed and the one that we shall use in the sequel. First, the complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2), \tag{I6}$$

where $0 < k < 1$ and where the series representation in (I6) is found by expanding the integrand in a binomial series and integrating termwise. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and ℓ', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{I7}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and ℓ which is implied by (I7). Ramanujan writes his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = \ell^2$. We shall often say that β has degree n. As we shall see in Section 6 of Chapter 17, modular equations can alternatively be expressed as identities involving theta-functions. In fact, often one first proves a theta-function identity and then transcribes it into an equivalent modular equation by using the formulas in Entries 10–12 in Chapter 17. Ramanujan undoubtedly used this procedure

in proving most of his modular equations, and we shall proceed in the same fashion. The multiplier m for a modular equation of degree n is defined by

$$m = \frac{K}{L}. \tag{I8}$$

Ramanujan also established many "mixed" modular equations in which four distinct moduli appear. See the introduction of Chapter 20 for the definition of "mixed" modular equation.

For those not familiar with modular equations, these definitions may appear to be arbitrary and unmotivated. The raison d'etre can be found in the first six sections of Chapter 17. In particular, we note that the base q in the classical theory of elliptic functions is defined by $q = \exp(-\pi K'/K)$. Often one seeks relations among theta-functions where the arguments appearing are q and q^n, for some integer n. Further motivation can be found in two survey articles (Berndt [7], [8]).

Before offering some historical remarks about modular equations, we point out the analogies between Ramanujan's definition of a "modular equation" in Chapter 15 and the standard definition arising from (I7) that we have given above. The function $F(x)$ in (I3) is an analogue of $K(k)$ in (I6). Note that if one of the parameters $\frac{1}{2}$ of ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$ in (I6) is replaced by 1, then this hypergeometric function reduces to ${}_1F_0(\frac{1}{2}; k^2)$, which appears in (I6) with $x = k^2$. Observe that (I5) is a relation between the "moduli" α and β. Furthermore, note that the multiplier $1 + \sqrt{\beta}$ in (I4) is analogous to the multiplier defined in (I8).

One could argue, as we did in [7], that the theory of modular equations began in 1771 and 1775 with the appearance of J. Landen's two papers [1], [2] in which Landen's transformation was introduced. Strictly speaking, the theory commenced when A. M. Legendre [2] derived a modular equation of degree 3 in 1825 and C. G. J. Jacobi established modular equations of degrees 3 and 5 in his *Fundamenta Nova* [1], [2] in 1829. Subsequently, in the century that followed, contributions were made by many mathematicians including C. Guetzlaff, L. A. Sohncke, H. Schröter, L. Schläfli, F. Klein, A. Hurwitz, E. Fiedler, A. Cayley, R. Fricke, R. Russell, and H. Weber. Classical texts containing much material on modular equations include those of Enneper [1], Weber [2], [3], Klein [2], [3], and Fricke [3]. Enneper's book [1] and Hanna's paper [1] contain many references to the literature. As we shall see in the remainder of this book, Ramanujan's contributions in the area of modular equations are immense. He discovered many of the classical modular equations found by the aforementioned authors, but he derived many more new ones as well. With little or no exaggeration, we suggest that perhaps Ramanujan found more modular equations than all of his predecessors discovered together. After approximately a half century of dormancy, modular equations have become prominent once again. They arise in the theory of elliptic curves, in the hard hexagon models of lattice gases (Joyce [1]), and in algorithms for the rapid calculation of π (J. M. Borwein [1]; J. M. and P. B. Borwein [1]–[6]; J. M. Borwein, P. B. Borwein, and D. H. Bailey [1]). H. Cohn [1]–[8] and Cohn and J. Deutsch [1] have returned to the classical viewpoints but with a more modern approach and with computer algebra. Further references and applications of modular equations are discussed in our expository survey paper [7]. A briefer and more elementary introduction to modular equations has been given by us in [8]. T. Kondo and T. Tasaka [1], [2], G. Köhler [1], [2], and I. J. Zucker [3] have recently discovered some new beautiful theta-function identities in the spirit of those arising in the theory of modular equations.

Many algebraic, analytic, and elementary methods have been devised to prove modular equations. Except for H. Schröter, we have not found the methods of others helpful in proving Ramanujan's modular equations. Watson (Hardy [3, p. 220]) has declared that "when dealing with Ramanujan's modular equations generally, it has always seemed to me that knowledge of other people's work is a positive disadvantage in that it tends to put one off the shortest track."

In attempting to establish Ramanujan's modular equations, we have utilized three approaches. The first relies on the theory of theta-functions and frequently employs Schröter's formulas, first established in his dissertation [1] in 1854. Schröter's primary theorem is a formula representing a product of theta-functions as a linear combination of products of other theta-functions. Schröter's formulas can be found in the books of Hardy [3, p. 219], Tannery and Molk [1, pp. 163–167], Enneper [1, p. 142], and J. M. and P. B. Borwein [2, p. 111], as well as in a recent paper by Kondo and Tasaka [1]. In our applications, we need to slightly modify Schröter's formulas and obtain related representations for $f(a, b)f(c, d) \pm f(-a, -b)f(-c, -d)$. All of the requisite formulas are proved in detail in Section 36 of Chapter 16. Schröter [1]–[4] utilized his formulas to find several modular equations, although, except for his thesis [1], he never published complete proofs of his results. Ramanujan, to our knowledge, has not explicitly stated Schröter's formulas in any of his published papers, notebooks, or unpublished manuscripts. However, it seems clear, from the theory of theta-functions and modular equations that he did develop, that Ramanujan must have been aware of these formulas or at least of the principles that yield the many special cases that Ramanujan doubtless used. However, Schröter's formulas are applicable in only a small minority of instances. We conjecture that Ramanujan possessed other general formulas or procedures involving theta-functions that are unknown to us. In particular, we think that he had derived a formula involving quotients of theta-functions that he did not record in his notebooks and that we have been unable to find elsewhere in the literature as well. Watson [5, p. 150] asserted that "a prolonged study of his modular equations has convinced me that he was in possession of a general formula by means of which modular equations can be constructed in almost terrifying numbers." Watson then intimates that Ramanujan's "general formula" is, in fact, Schröter's most general formula. However, as pointed out above, Schröter's formulas cannot be used in most instances. Further efforts should be made in attempting to discover Ramanujan's analytical methods.

The second method exploits previously derived modular equations and may involve a heavy dosage of elementary algebra. The primary idea is to find parametric representations for α and β which are then employed along with elementary algebra to verify a given modular equation. Ramanujan probably used such methods, especially for small values of the degree n. The algebraic difficulties normally increase very rapidly with n. Some of our algebraic proofs are very tedious, and it is doubtful that Ramanujan would have employed such drudgery. Ramanujan, with his great skills in spotting algebraic relationships, could undoubtedly discover modular equations using algebraic manipulation, but, particularly in Chapters 19 and 20, the reader will see that some of the proofs presented here could not have been accomplished without knowing the modular equation in advance.

Our third method employs the theory of modular forms. In some ways, this represents the best approach. First, the theory of modular forms provides the theoretical basis which explains why certain identities among theta-functions exist. Second, this approach usually does not become too much more complicated with increasing n, and so proofs remain comparatively short, after the requisite theory has been developed. The primary disadvantage to this method is that the modular equation must be known in advance, and so, as in the second approach, the proofs are more properly called verifications. The principal idea is to show that the multiplier systems of certain modular forms agree and that the coefficients in the expansion of a certain modular form are equal to zero up to a certain prescribed point. We then can conclude that the modular form must identically be equal to zero. This approach has been used by A. J. Biagioli [1], S. Raghavan [1], [2], Raghavan and S. S. Rangachari [1], and R. J. Evans [1] in establishing several of Ramanujan's theta-function identities. It might be argued that Ramanujan used a variant of this method by comparing coefficients in the expansions of theta-functions. This is extremely doubtful, however, because Ramanujan would not have discovered the identities by this procedure.

An earlier version of Chapter 16, coauthored with C. Adiga, S. Bhargava, and G. N. Watson, was published in "Chapter 16 of Ramanujan's second notebook: Theta-functions and q-series," *Memoirs of the American Mathematical Society*, vol. 53, no. 315, 1985. The revised version appears here by permission of the American Mathematical Society. A substantial majority of the theorems and proofs appearing in Chapters 17–21 have not heretofore appeared in print. B. C. Berndt, A. J. Biagioli, and J. M. Purtilo [1]–[3] have proved some of Ramanujan's modular equations in journals commemorating the centenary of Ramanujan's birth. A brief description of Ramanujan's work on Eisenstein series in Chapter 21 was given by us in [10]. Some of Ramanujan's work on modular equations has also been examined by K. G. Ramanathan [9], [10], V. R. Thiruvenkatachar and K. Venkatachaliengar [1], and K. Venkatachaliengar [1].

To help readers find modular equations of certain degrees, we offer a table indicating the chapter and sections where the desired modular equations may be found.

Degree	Chapter	Sections
3	19	5, 7
5	19	11, 13
7	19	18, 19
	20	21
11	20	7
13	20	8
15	20	21
17	20	12
19	20	16
23	20	15
31	20	22
47	20	23
71	20	23
3, 9	20	3
5, 25	19	15
3, 5, 15	20	11
3, 7, 21	20	13
3, 9, 27	20	5
3, 11, 33	20	14
3, 13, 39	20	19, 21
3, 21, 63	20	20
3, 29, 87	20	24
5, 7, 35	20	18, 19
5, 11, 55	20	19, 21
5, 19, 95	20	20
5, 27, 135	20	24
7, 9, 63	20	19, 21
7, 17, 119	20	20
7, 25, 175	20	24
9, 15, 135	20	20
9, 23, 207	20	24
11, 13, 143	20	20
11, 21, 231	20	24
13, 19, 247	20	24
15, 17, 255	20	24

Each of Chapters 16–20 in the second notebook contains 12 pages, while Chapter 21 has only 4 pages. The number of theorems, corollaries, and examples found in each chapter is listed in the following table.

Chapter	Number of Results
16	134
17	162
18	135
19	185
20	173
21	45
Total	834

Many of the theorems that Ramanujan communicated in his letters of January 16, 1913 and February 27, 1913 to G. H. Hardy may be found in Chapters 16–21. We list these results in the following table.

Location in Collected Papers	Location in Notebooks
p. xxviii, (1)	Chapter 16, Entry 15 and corollary, Entry 39 (i)
p. xxviii, (6)	Chapter 20, Entry 20 (i)
p. xxix, (15)	Chapter 18, Corollary in Section 12
p. xxix, (20) (i), (v)	Chapter 20, Entries 11 (i), (ii), (xiv)
p. xxix, (21)	Chapter 20, Entry 19 (iii)
p. 350, (3)	Chapter 18, Entry 12 (ii)
p. 353, (20) (ii), (iii), (iv), (vi)	Chapter 20, Entries 11 (iii), (iv), (v), (xv)
p. 353, (21)	Chapter 20, Entry 19 (iii)
p. 353, (22)	Chapter 20, Entry 24 (i)

A few of Ramanujan's published papers and questions posed to readers of the *Journal of the Indian Mathematical Society* have their origins in Chapters 16–21 of the second notebook. In some cases, only a small portion of the paper actually arises from material in the notebooks. The following table lists those papers and the corresponding locations in the notebooks.

Paper	Location in Notebooks
Squaring the circle	Chaper 18, Entry 20 (i)
Modular equations and approximations to π	Chapter 18, Entry 3, Corollary in Section 3; Chapter 21
Question 584	Chapter 16, Entries 38 (i), (ii)
Some definite integrals	Chapter 16, Entry 14
Question 662	Chapter 19, Entry 7 (iv) (first part)
On certain arithmetical functions	Chapter 16, Section 35; Chapter 17, Entry 13
Question 755	Chapter 18, Corollary (ii) of Section 19
Proof of certain identities in combinatory analysis	Chapter 16, Entries 38 (i), (ii)

In the sequel, equation numbers refer to equations in the same chapter, unless another chapter is indicated. Unless otherwise stated, page numbers refer to pages in the pagination of the Tata Institute's publication of Ramanujan's second notebook [9]. Page numbers unattended by any reference number always refer to Ramanujan's second notebook. Parts I and II refer to the author's accounts [5] and [9], respectively, of Ramanujan's notebooks.

We mention some standard notations that will be used in the sequel. The rational integers, the rational numbers, the real numbers, and the complex numbers are denoted by $\mathbb{I}$, Q, $\mathbb{R}$, and $\mathscr{C}$, respectively. The residue of a meromorphic function f at a pole α is denoted by R_α, if the identity of the function f is understood.

I am very grateful to many mathematicians for the proofs and suggestions that they have supplied. I am most indebted to G. N. Watson for the notes that he compiled on Chapters 16–21. In particular, many of the proofs in Chapters 19–21 are due to Watson. F. J. Dyson [1, p. 7] has affirmed that "Watson was chief gardener in the 1930's and worked hard to develop and elucidate Ramanujan's ideas." Evidently, Watson was very careful about whom he would permit to stroll through this garden. However, through the extensive notes that he left behind, he has allowed me to view many of the flowers in the garden, and I am very appreciative.

I owe special thanks to the following mathematicians. C. Adiga and S. Bhargava made many contributions in their coauthoring an earlier version of Chapter 16 with me. The quality of Chapter 16 has greatly been enhanced by the many suggestions offered by R. A. Askey. A. J. Biagioli and J. M. Purtilo provided invaluable and necessary help in the theory of modular forms and MACSYMA, respectively. R. J. Evans [1] furnished beautiful proofs of some of Ramanujan's most intractable theta-function identities, and we have reproduced in the sequel much of his paper. L. Jacobsen has contributed several helpful remarks and suggestions on continued fractions.

For their comments and suggestions, I am also obliged to G. Almkvist, G. E. Andrews, J. M. and P. B. Borwein, J. Brillhart, R. L. Lamphere, R. Müller, C. Rama Murthy, K. G. Ramanathan, K. Stolarsky, M. Villarino, H. Waadeland, J. Wetzel, and I. J. Zucker.

The author bears the responsibility for all errors and wishes to be notified of such, whether they be minor or serious.

Most of the manuscript for this book was typed by Dee Wrather, and I thank her for her very accurate and rapid typing.

The figures in Chapters 18 and 19 were drawn by Jonathan Manton using the graphics of Mathematica.

A perusal of the references at the conclusion of this book indicates that several are obscure. Nancy Anderson, the mathematics librarian at the University of Illinois, helped to unearth many of these, and I owe her special thanks.

Lastly, I express my deep gratitude to James Vaughn and the Vaughn Foundation, and to the National Science Foundation for their financial support during several summers.

CHAPTER 16

q-Series and Theta-Functions

In Chapter 16, Ramanujan develops two closely related topics, q-series and theta-functions. The first 17 sections are devoted primarily to q-series, while the latter 22 sections constitute a very thorough development of the theory of theta-functions.

Ramanujan begins by stating some mostly familiar theorems in the theory of q-series. In particular, Ramanujan rediscovered some of Heine's famous theorems including his q-analogue of Gauss' theorem. However, several results appear to be new. Perhaps most noteworthy in this respect are the continued fractions in Sections 10–13. (Entry 10 is not a q-continued fraction and is more properly placed in Chapter 12 among other theorems of this type.) Entry 13 was later generalized by Ramanujan in his "lost notebook" [11]. Entry 16 is a "finite" form of what is now generally known as the "Rogers–Ramanujan continued fraction" and was first established in print by Hirschhorn [1] in 1972 while being unaware that the result is found in Ramanujan's notebooks.

As is to be expected, Ramanujan's findings in the theory of theta-functions contain many of their classical properties. In particular, he rediscovered several theorems found in Jacobi's epic *Fundamenta Nova* [1], [2]. In Entry 27, Ramanujan records transformation formulas for the modular transformation: $T(\tau) = -1/\tau$. He did not discover more general transformation formulas. In Entry 19, Ramanujan gives the famous Jacobi triple product identity of which he made numerous applications. Because several of our proofs employ Watson's quintuple product identity, it would seem that Ramanujan had discovered it. Indeed, the quintuple product identity can be found in Ramanujan's "lost notebook" [11]. Results in the last part of Chapter 16 indicate that Ramanujan had found Schröter's formulas [1]. Although Ramanujan does not give these formulas in their most general form, he does offer several special cases and deductions from them.

But more importantly, Ramanujan discovered several new and deep theorems in the theory of theta-functions. For example, the beautiful theorems in Sections 33–35 appear to be new, as well as Entry 38(iv) and the corollaries in Section 37.

In closing our brief survey of the content of Chapter 16, we would like to mention that this chapter contains four results that are due originally to Ramanujan and for which he is justly famous. Entry 14 offers Ramanujan's q-analogue of the beta-function. The evaluation of this integral was first recorded by Ramanujan in [4], [10, p. 57]. There are now at least four distinct verifications. In Entry 17 we find "Ramanujan's ${}_1\psi_1$ summation." Several proofs, including a new one offered here, now exist. Ramanujan found many applications for his ${}_1\psi_1$ summation, including a proof of Jacobi's triple product identity. The remarkable Rogers–Ramanujan identities are found in Entries 38(i), (ii), and the "Rogers–Ramanujan continued fraction" in Entry 38(iii). It might be remarked that this continued fraction is the only continued fraction proved in Ramanujan's published papers. However, he did submit several formulas containing continued fractions to the problems section of the *Journal of the Indian Mathematical Society*. Also, Ramanujan's letters to Hardy contain many beautiful theorems on continued fractions.

We conclude our introduction with several remarks on notation. For those reading this book in conjunction with the notebooks, it seems best to retain Ramanujan's notation $f(a, b)$ for the theta-functions (see (18.1)). We remark that $f(a, b) = \vartheta_3(z, \tau)$, where $ab = e^{2\pi i\tau}$, $a/b = e^{4iz}$, and $\vartheta_3(z, \tau)$ denotes the classical theta-function in the notation of Whittaker and Watson [1]. Most of the results in the sequel are, in fact, more easily stated in the notation $f(a, b)$ rather than in the notation $\vartheta_3(z, \tau)$. Ramanujan uses x to denote his primary variable. Since q is almost universally used today instead of x, we have adopted the more standard designation. It is assumed throughout the sequel that $|q| < 1$. As usual, for any complex number a, we write

$$(a)_k := (a; q)_k := (1 - a)(1 - aq)(1 - aq^2)\cdots(1 - aq^{k-1})$$

and

$$(a)_\infty := (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k).$$

Ramanujan writes $\prod(-a, x)$ for $(a)_\infty$, where $x = q$. The basic hypergeometric series ${}_{s+1}\varphi_s$ is defined by

$$ {}_{s+1}\varphi_s\left[\begin{matrix} a_1, a_2, \ldots, a_{s+1} \\ b_1, b_2, \ldots, b_s \end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_{s+1})_k}{(b_1)_k(b_2)_k\cdots(b_s)_k} \frac{x^k}{(q)_k}, \tag{0.1}$$

where $|x| < 1$ and $a_1, a_2, \ldots, a_{s+1}$, $b_1, b_2, \ldots, b_s$ are arbitrary, except that, of course, $(b_j)_k \neq 0$, $1 \le j \le s$, $0 \le k < \infty$. If s is "small," we shall write ${}_{s+1}\varphi_s(a_1, \ldots, a_{s+1}; b_1, \ldots, b_s; x)$ in place of the notation at the left side of (0.1). Finally, to denote the dependence on the base q, we may write ${}_{s+1}\varphi_s(a_1, \ldots, a_{s+1}; b_1, \ldots, b_s; q; x)$.

Entry 1. *Let q be real with $|q| < 1$, and suppose that a and x are any complex numbers. Let the principal branches of $(1 - a)^x$ and $(1 - q)^x$ be chosen. Then*

(i)
$$\lim_{q\to 1} \frac{(a)_\infty}{(aq^x)_\infty} = (1 - a)^x,$$

(ii)
$$\lim_{q\to 1} \frac{(q)_\infty}{(1 - q)^x(q^{x+1})_\infty} = \Gamma(x + 1),$$

(iii)
$$(a)_\infty = \prod_{k=0}^{n-1} (aq^k; q^n)_\infty,$$

and

(iv)
$$(a)_\infty = \frac{(a; \sqrt{q})_\infty}{(a\sqrt{q}; q)_\infty}.$$

Proof. First assume that $|a| < 1$. Apply (2.1) below with a and t replaced by aq^x and q^{-x}, respectively. Hence,

$$\lim_{q\to 1} \frac{(a)_\infty}{(aq^x)_\infty} = 1 + \sum_{k=1}^{\infty} \frac{(-x)(-x + 1)\cdots(-x + k - 1)}{k!} a^k = (1 - a)^x,$$

by the binomial theorem. The general result follows by analytic continuation.

The following proof of (ii) is due to R. W. Gosper, Write

$$\lim_{q\to 1} \frac{(q)_\infty}{(1 - q)^x(q^{x+1})_\infty} = \lim_{q\to 1} \prod_{k=1}^{\infty} \frac{1 - q^k}{1 - q^{k+x}} \left(\frac{1 - q^{k+1}}{1 - q^k}\right)^x$$
$$= \prod_{k=1}^{\infty} \frac{k}{k + x} \left(\frac{k + 1}{k}\right)^x = \Gamma(x + 1).$$

Identity (iii) follows easily by regrouping the factors on the left side. To prove (iv), let $n = 2$ in (iii) and replace q by $\sqrt{q}$.

The q-gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1 - q)^{1-x}. \tag{1.1}$$

Thus, Entry 1(ii) may be rewritten in the form

$$\lim_{q\to 1-} \Gamma_q(x + 1) = \Gamma(x + 1).$$

Gosper's proof of Entry 1(ii) may also be found in Andrews' monograph [18, p. 109]. Our proofs of Entries 1(i), (ii) are not completely rigorous, because limits were taken without justification under the summation and product signs, respectively. T. H. Koornwinder [1] has indeed justified these formal processes and provided rigorous proofs.

Ramanujan's proof of Entry 2 below can be found in his paper [4] [10, pp. 57–58].

Entry 15. *If* $|q| < 1$, *then*

$$\frac{\sum_{k=0}^{\infty} \frac{b^k q^{k^2}}{(aq)_k (q)_k}}{\sum_{k=0}^{\infty} \frac{b^k q^{k(k+1)}}{(aq)_k (q)_k}} = 1 + \frac{bq}{1 - aq +} \frac{bq^2}{1 - aq^2 +} \frac{bq^3}{1 - aq^3 +} \cdots.$$

Proof. Let $f(b, a)$ be defined by (13.2). Replacing a by aq in (13.4) and adding the result to (13.3), we find that

$$f(b, a) = (1 - aq)f(bq, aq) + bqf(bq^2, aq^2).$$

Replacing a by aq^{n-1} and b by bq^n, we may rewrite the previous equality in the form

$$\frac{f(bq^n, aq^{n-1})}{f(bq^{n+1}, aq^n)} = 1 - aq^n + \cfrac{bq^{n+1}}{\cfrac{f(bq^{n+1}, aq^n)}{f(bq^{n+2}, aq^{n+1})}}, \qquad n \geq 1. \tag{15.1}$$

Using (13.4), (15.1), and iteration, we deduce that

$$\frac{f(b, a)}{f(bq, a)} = 1 + \cfrac{bq}{\cfrac{f(bq, a)}{f(bq^2, aq)}} = 1 + \frac{bq}{1 - aq +} \cfrac{bq^2}{\cfrac{f(bq^2, aq)}{f(bq^3, aq^2)}}$$

$$= 1 + \frac{bq}{1 - aq +} \frac{bq^2}{1 - aq^2 +} \frac{bq^3}{1 - aq^3 +} \cdots.$$

That this continued fraction converges and that it converges, indeed, to $f(b, a)/f(bq, a)$ follow as in the proof of Entry 13. If $b \neq 0$ and $aq^n = 1$ for some positive integer n, then equality holds with the convention that we take the limit of both sides as a tends to $1/q^n$. If $b = 0$ and $aq^n = 1$, we interpret both sides as equaling 1. This completes the proof.

Corollary. *If* $|q| < 1$, *then*

$$\frac{\sum_{k=0}^{\infty} \frac{a^k q^{k(k+1)}}{(q)_k}}{\sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(q)_k}} = \frac{1}{1 +} \frac{aq}{1 +} \frac{aq^2}{1 +} \frac{aq^3}{1 +} \cdots.$$

Proof. Set $a = 0$ in Entry 15 and then replace b by a. The corollary now readily follows.

The continued fractions of both Entry 15 and its corollary were mentioned by Ramanujan [10, p. xxviii] in his second letter to Hardy. The corollary was established earlier by Rogers [1, p. 328, Eq. (4)] and then later by Watson [3]. The special case $a = 1$ is Entry 38(iii) and is discussed in detail in Section 38.

Ramamani [1] has given a similar proof of Entry 15 by obtaining functional relations for the function

$$\sum_{k=0}^{\infty} \frac{(-1)^k (b/a)_k a^k q^{k(k+1)/2}}{(q)_k}$$

and using Entry 9. Entry 9 is not used in our proof. Entry 16 below provides a finite version of Entry 15. Thus, an alternative proof of Entry 15 is obtained by letting n tend to ∞ in Entry 16. Hirschhorn [1], [4] has given a proof of Entry 16; perhaps our proof is somewhat simpler.

Entry 16. *For each positive integer* n, *let*

$$\mu = \mu_n(a, q) = \sum_{k=0}^{[(n+1)/2]} \frac{a^k q^{k^2} (q)_{n-k+1}}{(q)_k (q)_{n-2k+1}}$$

and

$$\nu = \nu_n(a, q) = \sum_{k=0}^{[n/2]} \frac{a^k q^{k(k+1)} (q)_{n-k}}{(q)_k (q)_{n-2k}}.$$

Then

$$\frac{\mu}{\nu} = 1 + \frac{aq}{1 +} \frac{aq^2}{1 +} \cdots + \frac{aq^n}{1}.$$

Proof. For each nonnegative integer r, define

$$F_r = \sum_{k=0}^{[(n-r+1)/2]} \frac{a^k q^{k(r+k)} (q)_{n-r-k+1}}{(q)_k (q)_{n-r-2k+1}}.$$

Observe that $F_0 = \mu$, $F_1 = \nu$, $F_n = 1$, and $F_{n-1} = 1 + aq^n$. A straightforward calculation shows that

$$F_r - F_{r+1} = aq^{r+1} F_{r+2}, \qquad r \geq 0. \tag{16.1}$$

Using (16.1), iteration, and the special cases pointed out above, we find that

$$\frac{\mu}{\nu} = \frac{F_0}{F_1} = 1 + \frac{aq}{F_1/F_2} = 1 + \frac{aq}{1 +} \frac{aq^2}{F_2/F_3}$$

$$= 1 + \frac{aq}{1 +} \frac{aq^2}{1 +} \cdots + \frac{aq^{n-1}}{F_{n-1}/F_n}$$

$$= 1 + \frac{aq}{1 +} \frac{aq^2}{1 +} \cdots + \frac{aq^{n-1}}{1 +} \frac{aq^n}{1},$$

which is the required result.

Entry 17 offers another famous discovery of Ramanujan known as "Ramanujan's summation of the ${}_1\psi_1$." It was first brought before the mathematical world by Hardy [3, pp. 222, 223] who described it as "a remarkable formula with many parameters." Hardy did not supply a proof but indicated that a proof could be constructed from the q-binomial theorem. The first published proofs appear to be by W. Hahn [1] and M. Jackson [1] in 1949 and 1950, respectively. Other proofs have been given by Andrews [2], [3], Andrews and Askey [2], Askey [2], Ismail [1], Fine [1, pp. 19–20], and Mimachi [1]. The short proof of Entry 17 that we offer below has been motivated by Askey's paper [2] and has been discovered independently by K. Venkatachaliengar [1]. See also his monograph with V. R. Thiruvenkatachar [1]. Askey [4] has discussed our proof along with a "proof" of a "false theorm" to illustrate certain pitfalls in formally manipulating Laurent series.

We emphasize that Entry 17 is an extremely useful result, and several applications of it will be made in the sequel. Fine [1] and Bhargava and Adiga [5] have employed Entry 17 in their work on sums of squares. For a connection between Entries 14 and 17, see Askey's paper [2]. Further applications of Entry 17 have been made by Andrews [10], [18, Chap. 5], Askey [3], [5], and Moak [1]. A generalization of Entry 17 has been found by Andrews [12, Theorem 6].

As we shall see in Section 19, the Jacobi triple product identity is a special case of Ramanujan's ${}_1\psi_1$ summation. In 1972, I. Macdonald [1] found multi-dimensional analogues of the Jacobi triple product identity, which can also be considered as analogues of Entry 17, and which are now called the Macdonald identities. One of the Macdonald identities is, in fact, the quintuple product identity, discussed in detail in Section 38. More elementary proofs of some of Macdonald's identities have been found by S. Milne [1]. These considerations partly motivated Milne [2]–[4] to develop multiple sum generalizations of Ramanujan's ${}_1\psi_1$ sum. R. Gustafson [1] has found further analogues of the ${}_1\psi_1$ summation. Lastly, we mention that D. Stanton [1] has developed an elementary approach to the Macdonald identities.

Entry 17. *Suppose that* $|\beta q| < |z| < 1/|\alpha q|$. *Then*

$$1 + \sum_{k=1}^{\infty} \frac{(1/\alpha; q^2)_k(-\alpha q)^k}{(\beta q^2; q^2)_k} z^k + \sum_{k=1}^{\infty} \frac{(1/\beta; q^2)_k(-\beta q)^k}{(\alpha q^2; q^2)_k} z^{-k}$$
$$= \left\{\frac{(-qz; q^2)_\infty(-q/z; q^2)_\infty}{(-\alpha qz; q^2)_\infty(-\beta q/z; q^2)_\infty}\right\}\left\{\frac{(q^2; q^2)_\infty(\alpha\beta q^2; q^2)_\infty}{(\alpha q^2; q^2)_\infty(\beta q^2; q^2)_\infty}\right\}. \quad (17.1)$$

PROOF. Let $f(z)$ denote the former expression in curly brackets on the right side of (17.1). Since $f(z)$ is analytic in the annulus, $|\beta q| < |z| < 1/|\alpha q|$, we may set

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k, \qquad |\beta q| < |z| < 1/|\alpha q|.$$

From the definition of f, it is easy to see that

$$(\beta + qz)f(q^2z) = (1 + \alpha qz)f(z),$$

provided that also $|\beta q| < |q^2 z|$. Thus, in the sequel we assume that $|\beta/q| < |z| < 1/|\alpha q|$.

Equating coefficients of z^k, $-\infty < k < \infty$, on both sides, we find that

$$\beta q^{2k}c_k + q^{2k-1}c_{k-1} = c_k + \alpha q c_{k-1}. \quad (17.2)$$

Hence,

$$c_k = -\frac{\alpha q(1 - q^{2k-2}/\alpha)c_{k-1}}{1 - \beta q^{2k}}, \qquad 1 \le k < \infty,$$

and

$$c_{-k} = -\frac{\beta q(1 - q^{2k-2}/\beta)c_{-k+1}}{1 - \alpha q^{2k}}, \qquad 1 \le k < \infty,$$

where, to get the latter equality, we replaced k by $1 - k$ in (17.2). Iterating the last two equalities, we deduce that, respectively,

$$c_k = \frac{(-\alpha q)^k(1/\alpha; q^2)_k c_0}{(\beta q^2; q^2)_k}, \qquad 1 \le k < \infty, \quad (17.3)$$

and

$$c_{-k} = \frac{(-\beta q)^k(1/\beta; q^2)_k c_0}{(\alpha q^2; q^2)_k}, \qquad 1 \le k < \infty.$$

Examining (17.1), we see that, to complete the proof, it suffices to show that

$$c_0 = \frac{(\alpha q^2; q^2)_\infty(\beta q^2; q^2)_\infty}{(q^2; q^2)_\infty(\alpha\beta q^2; q^2)_\infty}. \quad (17.4)$$

Now let $\varphi(z)$ and $\psi(z)$ denote, respectively, the two infinite series on the left side of (17.1). Now $f(z)$ has a simple pole at $z = -1/\alpha q$, and since $\psi(z)$ is analytic for $|z| > |\beta q|$, we find that

$$\lim_{z\to -1/\alpha q} (1 + \alpha qz)f(z) = \lim_{z\to -1/\alpha q} (1 + \alpha qz)\varphi(z) = \lim_{n\to\infty} \frac{(-1)^n c_n}{(\alpha q)^n}, \quad (17.5)$$

by Abel's theorem. Using the definition of $f(z)$ and (17.3), we may rewrite (17.5) in the form

$$\frac{(1/\alpha; q^2)_\infty(\alpha q^2; q^2)_\infty}{(q^2; q^2)_\infty(\alpha\beta q^2; q^2)_\infty} = \frac{(1/\alpha; q^2)_\infty c_0}{(\beta q^2; q^2)_\infty}.$$

Equality (17.4) obviously follows, and so the proof of Entry 17 is complete for $|\beta/q| < |z| < 1/|\alpha q|$. By analytic continuation, (17.1) is valid for $|\beta q| < |z| < 1/|\alpha q|$.

Entry 17 can be reformulated in a more compact setting. We first extend the definition of $(c; q)_k$ by defining

$$(c)_k = (c; q)_k = \frac{(c; q)_\infty}{(cq^k; q)_\infty},$$

for *every* integer k. In Entry 17, now replace α, β, and z by $1/a, b/q^2$, and $-az/q$, respectively. Lastly, replace q^2 by q. Then (17.1) can be written in the form

$$\sum_{k=-\infty}^{\infty} \frac{(a)_k}{(b)_k} z^k = \frac{(az)_\infty (q/az)_\infty (q)_\infty (b/a)_\infty}{(z)_\infty (b/az)_\infty (b)_\infty (q/a)_\infty}, \tag{17.6}$$

where $|b/a| < |z| < 1$.

For another proof of (17.4), see the monograph of Thiruvenkatachar and Venkatachaliengar [1].

Corollary. *If* $|nq| < |z| < 1/|nq|$, *then*

$$1 + \sum_{k=1}^{\infty} \frac{(1/n; q^2)_k (-nq)^k (z^k + z^{-k})}{(nq^2; q^2)_k} = \frac{(-qz; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty (n^2q^2; q^2)_\infty}{(-nqz; q^2)_\infty (-nq/z; q^2)_\infty (nq^2; q^2)_\infty^2}.$$

PROOF. Set $\alpha = \beta = n$ in Entry 17.

The remainder of Chapter 16 is devoted to the theta-function

$$f(a, b) = 1 + \sum_{k=1}^{\infty} (ab)^{k(k-1)/2} (a^k + b^k) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \tag{18.1}$$

where $|ab| < 1$. If we set $a = qe^{2iz}$, $b = qe^{-2iz}$, and $q = e^{\pi i\tau}$, where z is complex and $\text{Im}(\tau) > 0$, then $f(a, b) = \vartheta_3(z, \tau)$, where $\vartheta_3(z, \tau)$ denotes one of the classical theta-functions in its standard notation (Whittaker and Watson [1, p. 464]). Thus, all of Ramanujan's theorems on $f(a, b)$ may be reformulated in terms of $\vartheta_3(z, \tau)$. It seems preferable, however, to retain Ramanujan's notation. Not only will the reader find it easier to follow our presentation in conjunction with Ramanujan's, but Ramanujan's theorems are more simply and elegantly stated in his notation.

Entry 18. *We have*

(i) $f(a, b) = f(b, a)$,

(ii) $f(1, a) = 2f(a, a^3)$,

(iii) $f(-1, a) = 0$,

and, if n is an integer,

(iv) $f(a, b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n})$.

Ramanujan remarks that (iv) is approximately true when n is not an integer. We have not been able to give a mathematically precise formulation of this statement. Repeated use of (iv) will be made in the sequel.

PROOF. First, (i) is obvious.

Second,

$$\begin{aligned} f(1, a) &= 2 + \sum_{k=1}^{\infty} a^{k(k+1)/2} + \sum_{k=2}^{\infty} a^{k(k-1)/2} \\ &= 2\left(1 + \sum_{k=1}^{\infty} a^{k(k+1)/2}\right) \\ &= 2\left(1 + \left\{\sum_{\substack{k=1\\k \text{ even}}}^{\infty} + \sum_{\substack{k=1\\k \text{ odd}}}^{\infty}\right\} a^{k(k+1)/2}\right) \\ &= 2\left(1 + \sum_{k=1}^{\infty} a^{k(2k+1)} + \sum_{k=1}^{\infty} a^{k(2k-1)}\right) \\ &= 2\left(1 + \sum_{k=1}^{\infty} a^{k(k-1)/2}(a^3)^{k(k+1)/2} + \sum_{k=1}^{\infty} a^{k(k+1)/2}(a^3)^{k(k-1)/2}\right) \\ &= 2f(a, a^3). \end{aligned}$$

Third,

$$f(-1, a) = \sum_{k=2}^{\infty} (-1)^{k(k+1)/2} a^{k(k-1)/2} + \sum_{k=1}^{\infty} (-1)^{k(k-1)/2} a^{k(k+1)/2} = 0,$$

upon the replacement of k by $k + 1$ in the first sum on the right side.

Fourth, replacing k by $k + n$ on the far right side of (18.1), we find that

$$\begin{aligned} f(a, b) &= \sum_{k=-\infty}^{\infty} a^{(k+n)(k+n+1)/2} b^{(k+n)(k+n-1)/2} \\ &= a^{n(n+1)/2} b^{n(n-1)/2} \sum_{k=-\infty}^{\infty} a^{k(k+2n+1)/2} b^{k(k+2n-1)/2} \\ &= a^{n(n+1)/2} b^{n(n-1)/2} \sum_{k=-\infty}^{\infty} \{a(ab)^n\}^{k(k+1)/2} \{b(ab)^{-n}\}^{k(k-1)/2}, \end{aligned}$$

which completes the proof of (iv).

Entry 19. *We have*

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

PROOF. In Entry 17, let $qz = a$, $q/z = b$, and $\alpha = \beta = 0$.

In the notebooks [9, Vol. 2, p. 197], Ramanujan informs us how he proved Entry 19 by remarking: "This result can be got like XVI. 17 Cor. or as follows. We see from iv. that if $a(ab)^n$ or $b(ab)^n$ be equal to -1 then $f(a, b) = 0$ and also if $(ab)^n = 1$, $f(a, b)\{1 - (a/b)^{n/2}\} = 0$ and hence $f(a, b) = 0$. Therefore $(-a; ab)_\infty$, $(-b; ab)_\infty$, and $(ab; ab)_\infty$ are the factors of $f(a, b)$." (We have slightly altered Ramanujan's notation.) The product and series in Entry 19 converge only when $|ab| < 1$, but there is even a more serious objection to Ramanujan's argument. It is not clear that the *only* factors of $f(a, b)$ are $(-a; ab)_\infty$, $(-b; ab)_\infty$, and $(ab; ab)_\infty$.

Entry 19 is Jacobi's famous triple product identity, established in his *Fundamenta Nova* [1], [2] but, in fact, first proved by Gauss [3, p. 464]. See the texts of Andrews [9, pp. 21, 22] and Hardy and Wright [1, pp. 282, 283] for other proofs.

Entry 20. *If* $\alpha\beta = \pi$, $\operatorname{Re}(\alpha^2) > 0$, *and* n *is any complex number, then*

$$\sqrt{\alpha} f(e^{-\alpha^2+n\alpha}, e^{-\alpha^2-n\alpha}) = e^{n^2/4}\sqrt{\beta} f(e^{-\beta^2+in\beta}, e^{-\beta^2-in\beta}).$$

Entry 20 is a formulation of the classical transformation formula for the theta-function $\vartheta_3(z, \tau)$ (Whittaker and Watson [1, p. 475]). This entry is also recorded in Chapter 14 [9, Vol. 2, p. 169, Entry 7]. A proof via the Poisson summation formula is sketched in our book [9, p. 253].

Entry 21. *If* $|q|, |a|, |b| < 1$, *then*

$$\operatorname{Log}(-a; q)_\infty = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}a^k}{k(1-q^k)} \tag{21.1}$$

and

$$\operatorname{Log} f(a, b) = \operatorname{Log}(ab; ab)_\infty + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(a^k + b^k)}{k(1-a^kb^k)}. \tag{21.2}$$

Proof. For $|q|, |a| < 1$,

$$\operatorname{Log}(-a; q) = \sum_{n=0}^{\infty} \operatorname{Log}(1 + aq^n) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(aq^n)^k}{k}$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}a^k}{k} \sum_{n=0}^{\infty} (q^k)^n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}a^k}{k(1-q^k)}.$$

Equality (21.2) follows immediately from Entry 19 and (21.1).

Entry 22. *If* $|q| < 1$, *then*

(i) $$\varphi(q) := f(q, q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (-q^2; q^2)_\infty},$$

(ii) $$\psi(q) := f(q, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty},$$

(iii) $$f(-q) := f(-q, -q^2) = \sum_{k=0}^{\infty} (-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty} (-1)^k q^{k(3k+1)/2}$$
$$= (q; q)_\infty,$$

and

(iv) $$\chi(q) := (-q; q^2)_\infty.$$

Observe that $\varphi(q) = \vartheta_3(0, \tau)$, where $q = e^{\pi i\tau}$. If $q = e^{2\pi i\tau}$, then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind eta-function. Equality (iii) is a statement of Euler's famous pentagonal number theorem [1], [5]. See Andrews' book [9, pp. 9–12, 14] for an elementary proof and further references. Note that (iv) is only a *definition* of $\chi(q)$.

Proof of (i). The first equality follows immediately from the definition (18.1) of $f(a, b)$.

From Entry 19,

$$f(q, q) = (-q; q^2)_\infty^2 (q^2; q^2)_\infty. \tag{22.1}$$

Now,

$$(-q; q^2)_\infty = \prod_{n=1}^{\infty} (1 + q^{2n-1}) = \prod_{n=1}^{\infty} \frac{1 + q^n}{1 + q^{2n}}$$
$$= \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{(1 - q^n)(1 + q^{2n})} = \frac{1}{(q; q^2)_\infty (-q^2; q^2)_\infty}, \tag{22.2}$$

which is a famous identity of Euler. Substituting (22.2) into (22.1), we complete the proof of (i).

Observe that (22.2) may be rewritten in the form

$$(-q; q)_\infty = \frac{1}{(q; q^2)_\infty}. \tag{22.3}$$

The equality (22.3) is the analytic equivalent of Euler's famous theorem: the number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts.

Using (22.3) in Entry 22(i), we derive the useful representations

$$\varphi(-q) = (q; q)_\infty (q; q^2)_\infty = \frac{(q; q)_\infty}{(-q; q)_\infty}. \tag{22.4}$$

Proof of (ii). For $|q| < 1$,

$$f(q, q^3) = 1 + \sum_{k=1}^{\infty} q^{2k(2k-1)/2} + \sum_{k=1}^{\infty} q^{2k(2k+1)/2}$$
$$= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} q^{k(k+1)/2} + \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} q^{k(k+1)/2}$$
$$= \sum_{k=0}^{\infty} q^{k(k+1)/2},$$

which proves the first equality.