Cones, Crystals, and Patterns

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Introduction

Let G be a semisimple algebraic group and let be V a a simple G-module of highest weight λ. We consider the crystal graph C of V (or rather the crystal graph of the corresponding simple module for the quantum group $U_q(\mathrm{Lie}\, G)$). There are two obvious options for an algorithm to determine the graph: One is to develop a combinatorial theory that mimics the crystal base and the Kashiwara operators, the other is to encode more directly the information provided by the crystal graph. For example, one could fix for every element of the crystal a path in the graph that joins the given element with the highest weight element in the graph.

A combinatorial model can be found for example in Lusztig's articles (see [18], [19]). A different approach is given by the path model (see [13] for the path model and [9], [6] for the connection with the crystal graph). The latter can be viewed as a generalization of the classical theory of Young tableaux for $GL_n(\mathbb{C})$, it includes as special cases also the tableaux of Lakshmibai and Seshadri (see [12] and [16]) and of Kashiwara and Nakashima (see [10]). So it is natural to ask whether the other classical combinatorial tool, the Gelfand-Tsetlin patterns and its generalizations (see for example [2]), find a natural interpretation (and generalization) in the framework of crystal graphs.

The aim of this article is to give such an interpretation of the Gelfand-Tsetlin patterns and to generalize the notion to other groups. We will describe explicitly for each simple algebraic group (at least one type of) such a "generalized" pattern, and we will show how to "translate" for classical groups the new notion back into the patterns as they are described in [2].

The idea of the construction is rather obvious: Start with a crystal graph of a representation, let b be an element of the crystal basis. If b is not the highest weight element, then there is at least one incoming arrow with color a simple root α_1. Denote by n_1 the maximal integer such that $b_1 := e_{\alpha_1}^{n_1}(b) \neq 0$, where e_{α_1} denotes the Kashiwara operator on the crystal basis. By continuing with b_1 in the same way, we obtain a sequence of integers $(n_1, \ldots, n_r)$ such that $e_{\alpha_r}^{n_r} \ldots e_{\alpha_1}^{n_1}(b)$ is the highest weight element in the graph. To make the construction uniform for all elements of the crystal base, one fixes a reduced decomposition of the longest word in the Weyl group W and applies the root operators according to the appearance in this decomposition (for details see section 1). Let $\mathcal{S} \subset \mathbb{Z}^N$, N the length of the longest word in W, be the collection of such sequences, and denote by $C \subset \mathbb{R}^N$ the real cone spanned by $\mathcal{S}$. One can show that $\mathcal{S}$ is in fact the set of all integral points in this cone.

The defining inequalities for the cone C depend of course of the chosen decomposition. We show that there exists a special class of decompositions (the *nice* decompositions) such that the corresponding cone has a very simple description. For example, in the simply laced case, they are of the form: $\underline{a} = (a_1, \ldots, a_n) \in C$ if and only if $a_i \geq a_j$ for all $j > i$ such that $\beta_i > \beta_j$, where $\{\beta_1, \ldots, \beta_N\}$ are the positive roots, enumerated according to the fixed reduced decomposition, and "$>$" is the height ordering on the roots.

It is not clear how to describe the inequalities for an arbitrary decomposition. The fact that we can find a description in terms of the "root ordering" for all rank two Kac-Moody algebras suggests that it should be possible to find a similar description in general. The procedure described in section 1 to determine the set of inequalities has the disadvantage that the list will be far from being minimal. A conjecture for the set of inequalities defining C can be found in [21], they can prove the conjecture for all rank two cases, the group SL_n and the affine Kac-Moody algebra of type A^1_{n-1}.

The actual patterns, i.e., the integral sequences corresponding to elements of the crystal base of a given representation of highest weight λ, are the integral points of a polytope obtained as intersection of C with a set of half-spaces (depending on λ). We also describe a simple way to associate a basis to the patterns: Let $v_\lambda \in V_\lambda$ be a highest weight vector. Then the set of vectors

$$\{v_{\underline{a}} := F_{\alpha_1}^{(a_1)} \ldots F_{\alpha_p}^{(a_p)} v_\lambda \mid \underline{a} \text{ an element of the associated polytope } \}$$

forms a basis of (a Kostant lattice in) V_λ. Here the $F_\alpha^{(n)}$ are the usual divided powers of elements of a Chevalley basis of the Lie algebra of G.

Note that this description (or interpretation) of the Gelfand-Tsetlin patterns provides a canonical bijection between the various tableaux for classical groups and the corresponding Gelfand-Tsetlin patterns. Recall that it is rather simple to write down an explicit bijection between Young tableaux and the patterns for $G = GL_n(\mathbb{C})$, but already for the symplectic group this is quite a difficult task. Only recently J. Sheats [23] announced a symplectic Jeu de Tacquin that transforms the Zhelobenko patterns (or rather the corresponding King tableaux) into De Concini tableaux.

For the group $G = GL_n(\mathbb{C})$, the connection between the cone C and the Gelfand-Tsetlin patterns has been described already in [4], and the cone C has been studied before in the simply laced case in [3]. The basis for V_λ indicated above is a variation of the basis given in [11]. Another way to attach cones to crystals was studied in [22]. The author would like to thank A. Berenstein and A. Zelevinsky for very helpful and inspiring discussions, and R. Dehy for plentiful discussions and for her tremendous help to reduce the number of misprints to a minimum.

1. Graphs and adapted strings

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and denote by X its weight lattice. We fix an enumeration of the simple roots $\alpha_1, \ldots, \alpha_n$, and for a simple root α let $\alpha^\vee$ be its coroot. For the crystal graph of an irreducible highest weight representation and for the path model, we use the notation as in [8] and [13]. Once we have fixed a reduced decomposition $w = s_{i_1} \cdots s_{i_r}$ of an element w in the Weyl group W, we write just f_i, e_i for the (root-) operators $f_{\alpha_i}, e_{\alpha_i}$.

We identify (see [8] or [6]) in the following the crystal graph with its "realization" as path model. We recall quickly the notation introduced in [13]. Let $X_{\mathbb{R}}$ be the real vector space $X \otimes_{\mathbb{Z}} \mathbb{R}$, and denote by Π the set of all piecewise linear paths $\pi : [0,1] \to X_{\mathbb{R}}$ (modulo reparametrization) such that $\pi(0) = 0$ and $\pi(1) \in X$. Let Π^+ be the subset of paths contained in the dominant Weyl chamber.

Let $\lambda \in X^+$ be a dominant weight. We write $\lambda \gg 0$ for a generic regular dominant weight, i.e., $\lambda(\alpha^\vee) \gg 0$ for all simple roots α. We use the same notation for paths: We write $\pi \gg 0$ if $\pi \in \Pi^+$ and the endpoint of π is a generic regular weight: $\pi(1) \gg 0$.

We introduced in [13] some operators on the set $\Pi \cup \{0\}$. For every simple root α we defined two operators e_α and f_α. Roughly speaking, these operators contruct new paths out of the given one by replacing parts of the path by their image with respect to the simple reflection s_α. If it is not possible to apply the construction to a given path, then the image of the path is defined as the "extra" element 0. (This element has been introduced to underline the similarity with the crystal basis). Further, if $e_\alpha \pi \neq 0$, then on recovers the old path by applying the other operator: $f_\alpha e_\alpha \pi = \pi$, and if $f_\alpha \pi \neq 0$, then $e_\alpha f_\alpha \pi = \pi$.

For a path $\pi \in \Pi$ and a simple root α let m_α be the minimum of the function $[0,1] \to \mathbb{R}$, $t \mapsto \pi(t)(\alpha^\vee)$. Recall that if m and n are maximal such that $e_\alpha^m \pi \neq 0$ and $f_\alpha^n \pi \neq 0$, then $m \in \mathbb{N}$ is maximal such that $m \leq |m_\alpha|$, and $n \in \mathbb{N}$ is maximal such that $n \leq \pi(1)(\alpha^\vee) - m_\alpha$. As an immediate consequence we see that if $\pi \in \Pi^+$, then $e_\alpha \pi = 0$ for all simple roots α. Further, if $\pi \in \Pi$ is such that $e_\alpha \pi = 0$ and $n = \pi(1)(\alpha^\vee)$, then the set $\{\pi, f_\alpha \pi, \ldots, f_\alpha^n \pi\} \cup \{0\}$ is stable under the operators e_α, f_α, and the weights which occur as endpoints are exactly the weights on the α-string between $\lambda := \pi(1)$ and $s_\alpha(\lambda)$.

For $\pi \in \Pi^+$ we denote by $B(\pi)$ the set of paths obtained from π by applying the root operators, i.e., $B(\pi) \cup \{0\}$ is the smallest subset of Π that contains π and is stable under the root operators. Let $\lambda := \pi(1)$ be the endpoint of π. Recall that the multiplicity of the weight space $V_\lambda(\mu)$ in V_λ is equal to the number of paths in $B(\pi)$ that end in μ. Further, let $\mathcal{G}(\pi)$ be the colored directed graph having $B(\pi)$ as set of vertices, and we put an arrow $\eta \xrightarrow{\alpha} \eta'$ with color α between $\eta, \eta' \in B(\pi)$ if $f_\alpha(\eta) = \eta'$. Recall that $\mathcal{G}(\pi)$ depends only on the endpoint of π, in fact, $\mathcal{G}(\pi)$ is the crystal graph of the irreducible representation V_λ of highest weight $\lambda = \pi(1)$.

Let $\lambda \in X^+$ be a dominant weight. As the "path model" of the representation V_λ we will most of the time choose the set $B(\pi_\lambda)$, where π_λ is the straight line $\pi_\lambda : [0,1] \mapsto X_{\mathbb{R}}$, $t \mapsto t\lambda$, joining the origin with λ.

Suppose $\lambda, \mu \in X^+$ are dominant weights, and let $\pi \in B(\pi_\lambda)$ and $\eta \in B(\pi_\mu)$ be two paths of the corresponding path models. Recall ([13], Lemma 2.7) that the root operators act on the concatenation $\pi * \eta$ in the following way: $e_\alpha(\pi * \eta) = \pi * (e_\alpha \eta)$ if $\exists\ n \geq 1$ such that $e_\alpha^n \eta \neq 0$ but $f_\alpha^n \pi = 0$, and

$e_\alpha(\pi * \eta) = (e_\alpha\pi) * \eta$ else, and

$$f_\alpha(\pi * \eta) = \begin{cases} (f_\alpha\pi) * \eta & \text{if } \exists n \geq 1 \text{ such that } f_\alpha^n\pi \neq 0 \text{ but } e_\alpha^n\eta = 0; \\ \pi * (f_\alpha\eta), & \text{otherwise.} \end{cases}$$

Let $x := f_{\alpha_1} \dots f_{a_r}$ and $y := f_{\gamma_1} \dots f_{\gamma_s}$ be two monomials.

PROPOSITION 1.1. *If $x\pi_1 \neq 0$ and $x\pi_1 = y\pi_1$ for some $\pi_1 \in \Pi^+$, then $x\pi = y\pi$ for all $\pi \in \Pi^+$.*

Proof. For $\pi_2 \in \Pi^+$ choose $\nu \in X^+$ such that $\mu_1 := \nu - \pi_1(1)$ and $\mu_2 := \nu - \pi_2(1)$ are dominant weights. If $\rho_1, \rho_2 \in \Pi^+$ are such that $\rho_1(1) = \mu_1$ and $\rho_2(1) = \mu_2$, then we get:

$$x(\pi_1 * \rho_1) = (x\pi_1) * \rho_1, \quad y(\pi_1 * \rho_1) = (y\pi_1) * \rho_1,$$

and hence $x(\pi_1 * \rho_1) = y(\pi_1 * \rho_1)$. Since $\pi_1 * \rho_1(1) = \pi_2 * \rho_2(1)$, the corresponding graphs $\mathcal{G}(\pi_1 * \rho_1)$ and $\mathcal{G}(\pi_2 * \rho_2)$ are isomorphic. Hence:

$$x(\pi_2 * \rho_2) = y(\pi_2 * \rho_2). \tag{1.1}$$

If $x\pi_2 = y\pi_2 = 0$, then nothing is to prove. If (without loss of generality) $x\pi_2 \neq 0$, then $x(\pi_2 * \rho_2) = (x\pi_2) * \rho_2$. Set $\eta * \tau := y(\pi_2 * \rho_2)$, then (1.1) implies that $\eta = x\pi_2$, $\tau = \rho_2$, and hence: $y(\pi_2 * \rho_2) = (y\pi_2) * \rho_2$, so $y\pi_2 = x\pi_2$. □

Fix a reduced decomposition $w = s_{i_1} \dots s_{i_p}$ of $w \in W$. For a tuple $(\underline{a}) = (a_1, \dots, a_p) \in \mathbb{N}^p$ we write just $f^{\underline{a}}$ for

$$f^{\underline{a}} := f_{i_1}^{a_1} \cdots f_{i_r}^{a_r}.$$

Definition. A tuple $(\underline{a})$ is called an *adapted string* of $\eta := f^{\underline{a}}\pi$ if

$$\eta \neq 0 \text{ and } e_{i_1}(f_{i_2}^{a_2} f_{i_3}^{a_3} \dots f_{i_p}^{a_p}\pi) = 0, e_{i_2}(f_{i_3}^{a_3} \dots f_{i_p}^{a_p}\pi) = 0, \dots, e_{i_{p-1}}(f_{i_p}^{a_p}\pi) = 0.$$

Remark 1.2. Note that $\underline{a}$ is uniquely determined by η: a_1 is maximal such that $e_{i_1}^{a_1}(\eta) \neq 0$, a_2 is maximal such that $e_{i_2}^{a_2} e_{i_1}^{a_1}(\eta) \neq 0$, etc.

LEMMA 1.3. *Suppose $\eta = f^{\underline{a}}\pi_1$, $\pi_1 \in \Pi^+$, is an adapted string. If $\pi_2 \in \Pi^+$, then either $f^{\underline{a}}\pi_2 = 0$ or $\eta' := f^{\underline{a}}\pi_2$ is the adapted string.*

Proof. If $\eta' \neq 0$, then let $\eta' = f^{\underline{m}}\pi_2$ be the adapted string. Proposition 1.1 implies $\eta = f^{\underline{m}}\pi_1$. The equality $f^{\underline{m}}\pi_1 = f^{\underline{a}}\pi_1$ implies $a_1 \geq m_1$ by the definition of the adapted string of η, and $f^{\underline{m}}\pi_2 = f^{\underline{a}}\pi_2$ implies that $m_1 \geq a_1$. It follows that $a_1 = m_1$, and hence, by induction, $\underline{a} = \underline{m}$. □

Definition. Let $w = s_{i_1} \cdots s_{i_p}$ be a reduced decomposition, and for a dominant weight λ let $\pi_\lambda \in \Pi^+$ be such that $\pi_\lambda(1) = \lambda$. Denote by $\mathcal{S}_w$ the set of all $\underline{a} \in \mathbb{N}^p$ such that $\underline{a}$ is the adapted string of $f^{\underline{a}}\pi$ for some $\pi \gg 0$, and let $\mathcal{S}_w^\lambda$ be the subset $\{\underline{a} \in \mathcal{S}_w \mid f^{\underline{a}}\pi_\lambda \neq 0\}$.

Remark 1.4. Lemma 1.3 implies that the definition above makes sense: If $\underline{a}$ is the adapted string of $f^{\underline{a}}\pi$ for some $\pi \in \Pi^+$, then $\underline{a}$ is the adapted string for all $f^{\underline{a}}\pi'$ such that $f^{\underline{a}}\pi' \neq 0$. Since $f^{\underline{a}}\pi' \neq 0$ for $\pi' \gg 0$, one could replace in the definition above "some" by "all $\pi \gg 0$".

Suppose $\mathfrak{g}$ is finite dimensional and let w_0 be the longest element in the Weyl group. For a dominant weight λ let $\mathcal{S}_{w_0}^\lambda \to B(\pi_\lambda)$ be the canonical map

$\underline{a} \mapsto f^{\underline{a}}\pi_\lambda$. Then [15] implies that the map $\mathcal{S}^\lambda_{w_0} \to B(\pi_\lambda)$ is a bijection, so we can naturally identify $\mathcal{S}^\lambda_{w_0}$ with the vertices of the crystal graph. For arbitrary w (and arbitrary $\mathfrak{g}$) we can identify in the same way $\mathcal{S}^\lambda_w$ with the vertices of the crystal graph of the Demazure module $V_\lambda(w)$ of V_λ [7].

Definition. Let $C_w \subset \mathbb{R}^p$ be the real cone spanned by $\mathcal{S}_w$. For a dominant weight λ let $C^w_\lambda \subset C_w$ be the polytope defined by: $a_p \le \langle \lambda, \alpha^\vee_{i_p}\rangle$ and

$$a_{p-1} \le \langle \lambda - a_p\alpha_{i_p}, \alpha^\vee_{i_{p-1}}\rangle, \ldots, a_1 \le \langle \lambda - a_p\alpha_{i_p} - \ldots - a_2\alpha_{i_2}, \alpha^\vee_{i_1}\rangle$$

PROPOSITION 1.5.

a) C_w *is a rational cone and* $\mathcal{S}_w$ *is the set of integral points in* C_w*.*

b) $\mathcal{S}^\lambda_w$ *is the set of integral points in the rational polytope* C^λ_w*.*

Proof. For $\eta \in \Pi$ and $k \in \mathbb{N}$ set $(k\eta)(t) := k\eta(t)$. Recall that $k(f_\alpha\eta) = f^k_\alpha(k\eta)$ and $k(e_\alpha\eta) = e^k_\alpha(k\eta)$ [13] . Further, if $\eta \in \mathbb{B}(\pi)$ for some $\pi \in \Pi^+$, then the minimum of the function $h^\eta_\alpha : t \mapsto \eta(t)(\alpha^\vee)$ is an integer for all simple roots, and $e_\alpha(\eta) = 0$ if and only if the minimum is equal to 0. In particular:

$$e_\alpha\eta = 0 \Leftrightarrow \min h^\eta_\alpha = 0 \Leftrightarrow \min h^{k\eta}_\alpha = 0 \Leftrightarrow e_\alpha(k\eta) = 0$$

for all $k \ge 1$. A vector $\underline{a}$ with integral co-ordinates is hence in $\mathcal{S}_w$ if and only if $k\underline{a} \in \mathcal{S}_w$ for some $k \ge 1$, so $\mathcal{S}_w$ is the set of integral points in C_w.

The description of $\mathcal{S}^\lambda_w$ as integral points of C^λ_w is an immediate consequence of the string property of the operators: Remember that if $\eta \in \Pi$ is a path such that $e_\alpha\eta = 0$, then $f^m_\alpha\eta \ne 0$ if and only if $0 \le m \le \langle \mu, \alpha^\vee\rangle$, where $\mu := \eta(1)$ is the endpoint of η. It remains to prove that C_w is rational, the rationality of C^λ_w follows then easily. We give a set of inequalities defining C_w, the construction is by induction on $l(w)$. If $w = s_\alpha$, then $C_w = \mathbb{R}^+_0$, the cone of non-negative real numbers.

Suppose now $w = s_{i_1}\ldots s_{i_p}$ is a reduced decomposition and $p > 1$. Let α be the simple root such that $\alpha = \alpha_{i_p}$, let ω be the corresponding fundamental weight, and fix a dominant weight λ such that $\lambda(\alpha^\vee) = 0$ but $\lambda(\gamma^\vee) \gg 0$ for all simple roots $\gamma \ne \alpha$. Set $w' := s_{i_1}\ldots s_{i_{p-1}}$.

The rules for the action of the root operators on the concatenation of two paths implies for $\underline{a} \in \mathbb{N}^p$ and $q \gg 0$:

$$f^{\underline{a}}(\pi_\lambda * \pi_{q\omega}) = (f^{\underline{a}'}\pi_\lambda) * (f^k_\alpha\pi_{q\omega}),\ \underline{a}' \in \mathbb{N}^{p-1},\ k \in \mathbb{N}.$$

where $a'_j = a_j$ for $\alpha_{i_j} \ne \alpha$, and $k \ge a_p$. The same rules imply: If $\underline{a} \in \mathcal{S}_w$, then $\underline{a}' \in \mathcal{S}_{w'}$. Guided by this fact, we define a sequence $\underline{m}^p, \ldots, \underline{m}^1$ with $\underline{m}^i \in \mathbb{Z}^i$: We set $\underline{m}_p := \underline{a}$, and we define $\underline{m}^{j-1} = (m^{j-1}_1, \ldots, m^{j-1}_{j-1})$ for $j \le p$ by

$$m^{j-1}_k := \min\{m^j_k, \Delta^j(k)\}$$

where $\Delta^j : \{1, \ldots, j-1\} \to \mathbb{Z}$ is the function defined as follows: For $k < \ell \le j$ let $s(k,\ell,j)$ be the sum $m^j_\ell - \sum_{k<s\le\ell} m^j_s\alpha_{i_s}(\alpha^\vee_{i_j})$, then

$$\Delta^j(k) := \begin{cases} \max\{s(k,\ell,j) \mid k < \ell \le j, \alpha_{i_\ell} = \alpha_{i_j}\}, & \text{if } \alpha_{i_k} = \alpha_{i_j}; \\ m^j_k, & \text{otherwise.} \end{cases}$$

LEMMA 1.6. $\underline{a} \in \mathcal{S}_w$ *if and only if* $\Delta^j(k) \ge 0\ \forall\ 2 \le j \le p, 1 \le k \le j-1$.

Note that the m_k^j depend on the a_ℓ in a piecewise linear way, and the functions Δ^j are piecewise linear. It follows that C_w is defined by a finite number of rational, piecewise linear inequalities, and hence C_w is a finite union of rational convex cones, which finishes the proof of the proposition. □

Proof of the lemma. By induction it suffices to prove:

$$\underline{a} \in \mathcal{S}_w \quad \Longleftrightarrow \quad \underline{m}^{p-1} \in \mathcal{S}_{w'} \text{ and } \Delta^p(k) \geq 0 \text{ for all } 1 \leq k \leq p-1$$

To simplify the notation we write in the following just $\underline{m}$ and Δ for $\underline{m}^{p-1}$ and Δ^p. Suppose first that $\underline{a} \in \mathcal{S}_w$. We will show:

$$f^{\underline{a}}(\pi_\lambda * \pi_{q\omega}) = (f^{\underline{m}}\pi_\lambda) * (f_\alpha^k \pi_{q\omega}),$$

where $k = a_p + (a_{p-1} - m_{p-1}^{p-1}) + \ldots + (a_1 - m_1^{p-1})$. This implies of course $\underline{m} \in \mathcal{S}_{w'}$ and $\Delta(k) \geq 0$.

Remember that $a'_j = a_j = m_j$ and $\Delta(j) = a_j \geq 0$ for $\alpha_{i_j} \neq \alpha$. Assume now that $\alpha_{i_\ell} = \alpha$ and $a'_s = m_s$ for all $s > \ell$. Since $\underline{a}' \in \mathcal{S}_{w'}$, this implies for $\underline{m}$: $e_\alpha(f_{i_{\ell+1}}^{m_{\ell+1}} \ldots f_{i_{p-1}}^{m_{p-1}} \pi_\lambda) = 0$ and

$$f_{i_{\ell+1}}^{a_{\ell+1}} \ldots f_{i_p}^{a_p}(\pi_\lambda * \pi_{q\omega}) = (f_{i_{\ell+1}}^{m_{\ell+1}} \ldots f_{i_{p-1}}^{m_{p-1}} \pi_\lambda) * (f_\alpha^t \pi_{q\omega}),\ t = a_p + \sum_{s>\ell}(a_s - m_s).$$

Now $\underline{a} \in \mathcal{S}_w$ implies $e_\alpha(f_{i_{\ell+1}}^{a_{\ell+1}} \ldots f_{i_p}^{a_p}(\pi_\lambda * \pi_{q\omega})) = 0$. The equation above and the rules for the action on the concatenation of two paths imply that this is only possible if $t \leq \sum_{\ell<s<p}(-m_s\alpha_{i_s})(\alpha^\vee)$. It follows that $\Delta(\ell) \geq 0$ because:

$$\sum_{\ell<s<p}(-m_s\alpha_{i_s})(\alpha^\vee) - t = \sum_{\substack{\ell<s<p \\ \alpha_{i_s}\neq\alpha}} a_s|\alpha_{i_s}(\alpha^\vee)| - \sum_{\substack{\ell<s<p \\ \alpha_{i_s}=\alpha}} m_s - \sum_{\substack{\ell<s\leq p \\ \alpha_{i_s}=\alpha}} a_s = \Delta(\ell)$$

Hence we get:

$$f_\alpha^{a_\ell}(f_{\alpha_{i_{\ell+1}}}^{m_{\ell+1}} \ldots f_{\alpha_{p-1}}^{m_{p-1}} \pi_\lambda * f_\alpha^t \pi_{q\omega}) = (f_\alpha^{m_\ell} \ldots f_{\alpha_{p-1}}^{m_{p-1}} \pi_\lambda) * (f_\alpha^{t'} \pi_{q\omega}),$$

where $m_\ell = \min\{a_\ell, \Delta(\ell)\}$ and $t' = t + a_\ell - m_\ell$, so $m_\ell = a'_\ell$. It follows now by induction that $\underline{a} \in \mathcal{S}_w$ implies $\underline{m} \in \mathcal{S}_{w'}$ and $\underline{a}' = \underline{m}$.

Suppose now $\underline{m} \in \mathcal{S}_{w'}$. We have to show that $\underline{a} \in \mathcal{S}_w$ and $\underline{a}' = \underline{m}$. Assume $j \leq p-1$ and $a'_s = m_s$ for all $s > j$. If $\alpha_{i_j} \neq \alpha$, then

$$e_{i_j}(f_{i_{j+1}}^{a_{j+1}} \ldots f_{i_p}^{a_p}(\pi_\lambda * \pi_{q\omega})) = (e_{i_j}(f_{i_{j+1}}^{m_{j+1}} \ldots f_{i_{p-1}}^{m_{p-1}} \pi_\lambda)) * (f_\alpha^t \pi_{q\omega}) = 0,$$

because $\underline{m} \in \mathcal{S}_{w'}$. Note further that we get obviously: $a'_j = a_j = m_j$. If $\alpha_{i_j} = \alpha$, then $\underline{m} \in \mathcal{S}_{w'}$ implies $\Delta(j) \geq 0$ and

$$e_\alpha(f_{i_{j+1}}^{m_{j+1}} \ldots f_{i_p}^{m_p} \pi_\lambda) = 0,\ e_\alpha(f_{i_{j+1}}^{m_{j+1}} \ldots f_{i_p}^{m_p} \pi_\lambda * f_\alpha^t \pi_{q\omega}) = 0,$$

where $t = a_p + \sum_{s>j}(a_s - m_s)$, and hence: $e_\alpha(f_{i_{j+1}}^{a_{j+1}} \ldots f_{i_p}^{a_p}(\pi_\lambda * q\pi_\omega)) = 0$. Further:

$$f_\alpha^{a_j}(f_{i_{j+1}}^{m_{j+1}} \ldots f_{i_p}^{m_p} \pi_\lambda * f_\alpha^t \pi_{q\omega}) = (f_\alpha^{m_j} \ldots f_{i_{p-1}}^{m_{p-1}} \pi_\lambda) * (f_\alpha^{t'} \pi_{q\omega}),$$

where $m_j = \min\{a_j, \Delta(j)\} = a'_j$ and $t' = t + a_j - m_j$. It follows that $\underline{a} \in \mathcal{S}_w$ and $\underline{a}' = \underline{m}$, which finishes the proof of the lemma. □

The proof above shows that C_w is a finite union of rational convex cones. In fact, it can be shown that C_w is convex. Though we will not need the convexity of C_w, we will roughly outline the proof as it was pointed out to the author by A. Berenstein and A. Zelevinsky. The main point in the demonstration is the theorem below, which proves that $\mathcal{S}_w$ is a semi-group.

In the case where the root system is simply laced this has been proved in [3] (Theorem 2.1, Proposition 4.1), the proof in the general case will appear in a forthcoming paper by A. Berenstein. The author would like to thank A. Berenstein and A. Zelevinsky for helpful discussions and for communicating the proof of the semi-group property.

Theorem 1.7. (A. Berenstein, A. Zelevinsky) *$\mathcal{S}_w$ is a semi-group.*

Corollary 1. *C_w and C_w^λ are convex.*

Proof of the corollary. To prove the convexity of C_w, it is sufficient to consider rational points in C_w. By Proposition 1.5, we know that $\mathcal{S}_w$ coincides with the set of integral points in C_w, and hence it is sufficient to prove that for any two integral points $\underline{a}, \underline{b} \in C_w$ and any rational number r, $0 \leq r \leq 1$, $r\underline{a}+(1-r)\underline{b} \in C_w$. Since C_w is a cone, this is the same as to say that for any two non negative integers p, q and any two points $\underline{a}, \underline{b} \in \mathcal{S}_w$, the point $p\underline{a}+q\underline{b} \in C_w$. Since this is again an integral point in C_w, the claim is equivalent to the claim that $\mathcal{S}_w$ is a semi-group. The fact that C_w^λ is a convex rational polytope follows then immediately from the convexity of C_w. □

The algorithm to determine the defining equations of the cone described above is rather involved, and the set of equations one gets seems to be far from being a minimal set. In the following we will study a special class of reduced decompositions (as mentioned in the introduction) where the cone has a particularly simple structure. These "nice decompositions" are associated to a sequence of Levi subgroups $G \supset L_1 \supset L_2 \ldots \supset L_n = T$, where the Levi subgroups correspond to so-called braidless fundamental weights.

In the following we will first study the rank two case which will help us understand the structure of the cone for the nice decomposition. In the exmples later we will for each simple algebraic group describe explicitely at least one cone associated to a nice decomposition.

2. The rank two case

In this section we suppose that $\mathfrak{g}$ is a symmetrizable Kac–Moody algebra of rank two. We write only $f_i, s_i, \ldots$ for $f_{\alpha_i}, s_{\alpha_i} \ldots$. We fix a decomposition of an element $w = s_1 s_2 \ldots \in W$. Let $\geq$ be the usual order on the set of weights defined by $\lambda \geq \mu$ if $\lambda - \mu$ is a sum of positive roots. The following proposition expresses the inequalities defining the cone of adapted strings in terms of the usual ordering on the set of roots. The proposition is a reformulation of the results in [7] and the proof is a variation of the proof given in [13].

Proposition 2.1. *$\underline{a} \in C_w$ if and only if*

$$a_2 s_1(\alpha_2) \geq a_3\alpha_1,\ a_3 s_1 s_2(\alpha_1) \geq a_4 s_1(\alpha_2),\ a_4 s_1 s_2 s_1(\alpha_2) \geq a_5 s_1 s_2(\alpha_1),\ \ldots.$$

Proof. Set $x := \langle \alpha_1, \alpha_2^\vee\rangle\langle \alpha_2, \alpha_1^\vee\rangle$, $y_0 := 1$ and $y_j := 1 - (1/xy_{j-1})$. One checks easily by induction that

$$(s_is_j)^k(\alpha_i) = (x^{k-1}|\langle \alpha_i, \alpha_j^\vee\rangle| \prod_{l=1}^{2k-2} y_l)\alpha_j + (x^k \prod_{l=1}^{2k-1} y_l)\alpha_i,$$
$$s_j(s_is_j)^k(\alpha_i) = (x^k|\langle \alpha_i, \alpha_j^\vee\rangle| \prod_{l=1}^{2k} y_l)\alpha_j + (x^k \prod_{l=1}^{2k-1} y_l)\alpha_i.$$

So $s_j(s_is_j)^k(\alpha_i)$ is obtained from $(s_is_j)^k(\alpha_i)$ by multiplying the coefficient of α_j by $xy_{2k-1}y_{2k}$, and $(s_is_j)^{k+1}(\alpha_i)$ is obtained from $s_j(s_is_j)^k(\alpha_i)$ by multiplying the coefficient of α_i by $xy_{2k}y_{2k+1}$. Since $y_p \geq y_q$ for $q \geq p$ (see [7] or [13]), it follows that: If $a, b > 0$ are such that $a(s_is_j)^k(\alpha_i) \geq bs_i(s_js_i)^{k-1}(\alpha_j)$, then $a(s_is_j)^l(\alpha_i) \geq bs_i(s_js_i)^{l-1}(\alpha_j)$ and $as_j(s_is_j)^l(\alpha_i) \geq b(s_js_i)^l(\alpha_j)$ for all $l < k$.

The proof of the proposition is by induction on $l(w)$. Suppose $l(w) > 1$ and $\pi \gg 0$, we may assume that the proposition is true for $w' \in W$ such that $l(w') = l(w) - 1$. Let $f^{(\underline{a})}\pi$ be such that $e_2(f_1^{a_3}f_2^{a_4}\dots\pi) = 0$, $e_1(f_2^{a_4}\dots\pi) = 0$, ... The discussion above shows that to prove the proposition it is sufficient to prove that, under these conditions, $e_1(f_2^{a_2}f_1^{a_3}\dots\pi) = 0$ is equivalent to:

$$a_2s_1(\alpha_2) \geq a_3\alpha_1,\ a_3s_1s_2(\alpha_1) \geq a_4s_1(\alpha_2),\ a_4s_1s_2s_1(\alpha_2) \geq a_5s_1s_2(\alpha_1),\ \dots$$

Denote by $\phi^k : [0,1] \to \mathbb{R}$ the map such that

$$f_i^{a_k}f_j^{a_{k+1}}...\pi(t) = f_j^{a_{k+1}}...\pi(t) - \phi^k(t)\alpha_i.$$

The integrality property implies $e_1(f_2^{a_2}\dots\pi) = 0 \Leftrightarrow \langle f_2^{a_2}\dots\pi(t), \alpha_1^\vee\rangle \geq 0$ for all $t \in [0,1]$. But this is equivalent to

$$\min\{\langle f_2^{a_4}\dots\pi(t) - \phi^3(t)\alpha_1 - \phi^2(t)\alpha_2, \alpha_1^\vee\rangle \mid t \in [0,1]\} \geq 0.$$

Suppose the minimum is attained at $t = t_0$. Since $\langle f_2^{a_4}\dots\pi(t), \alpha_1^\vee\rangle \geq 0$ by assumption and $\langle \alpha_2, \alpha_1^\vee\rangle < 0$, we can choose t_0 such that ϕ^3 is not locally constant in t_0. But then $\langle f_2^{a_4}\dots\pi(t_0), \alpha_1^\vee\rangle = \phi^3(t_0)$ (by the definition of ϕ^3), so the minimum above is equal to

$$\min\{-\langle \alpha_2, \alpha_1^\vee\rangle\phi^2(t) - \phi^3(t) \mid t \in [0,1]\}.$$

A map $\varphi : [0,1] \to \mathbb{R}^2$ can be written as $\varphi(t) = \varphi_1(t)\alpha_1 + \varphi_2(t)\alpha_2$, where φ_1, φ_2 are $\mathbb{R}$-valued functions. We write $\varphi \geq 0$ if $\varphi_1(t), \varphi_2(t) \geq 0$ for all $t \in [0,1]$. So the condition above can be written as:

$$\phi^2(t)s_1(\alpha_2) - \phi^3(t)\alpha_1 \geq 0. \tag{2.1}$$

For $t = 1$ we get $\phi^2(1) = a_2$, $\phi^3(1) = a_3$ and thus the necessary condition:

$$a_2s_1(\alpha_2) \geq a_3\alpha_1.$$

Note that $\langle f_1^{a_3}f_2^{a_4}\dots\pi(t), \alpha_2^\vee\rangle \geq \phi^2(t)$, so

$$\langle f_1^{a_3}f_2^{a_4}\dots\pi(t), \alpha_2^\vee\rangle s_1(\alpha_2) - \phi^3(t)\alpha_1 \geq \phi^2(t)s_1(\alpha_2) - \phi^3(t)\alpha_1.$$

Suppose the right side attains its minimum in $t_0 \in [0,1]$. Since $\phi^2, \phi^3 \geq 0$, we may assume that $t_0 = 1$ or ϕ^2 is not locally constant in t_0. But then

$$\phi^2(t_0) = \langle f_1^{a_3}f_2^{a_4}\dots\pi(t_0), \alpha_2^\vee\rangle,$$

so the minima of the left and right side are the same. It follows that (2.1) is satisfied if and only if:

$$a_2 s_1(\alpha_2) \geq a_3\alpha_1 \text{ and } \langle f_1^{a_3} f_2^{a_4} \ldots \pi(t), \alpha_2^\vee\rangle s_1(\alpha_2) - \phi^3(t)\alpha_1 \geq 0.$$

The latter can be rewritten as:

$$\begin{aligned}&\big(\langle f_1^{a_5} \ldots \pi(t), \alpha_2^\vee\rangle - \phi^3(t)\langle\alpha_1, \alpha_2^\vee\rangle - \phi^4(t)\langle\alpha_2, \alpha_2^\vee\rangle\big)s_1(\alpha_2) - \phi^3(t)\alpha_1\\ &= \big(\langle f_1^{a_5} \ldots \pi(t), \alpha_2^\vee\rangle - \phi^4(t)\big)s_1(\alpha_2) + \phi^3(t)s_1s_2(\alpha_1) - \phi^4(t)s_1(\alpha_2) \geq 0.\end{aligned}$$

As above, by choosing an appropriate $t_0 \in [0,1]$, one proves that this map is non-negative if and only if

$$\phi^3(t)s_1s_2(\alpha_1) - \phi^4(t)s_1(\alpha_2) \geq 0. \tag{2.2}$$

For $t = 1$ we get: $a_3s_1s_2(\alpha_1) \geq a_4s_1(\alpha_2)$. Proceeding in an same way, we see that the condition $e_1(f_2^{a_2} \ldots \pi) = 0$ is equivalent to the conditions:

$$a_2s_1(\alpha_2) \geq a_3\alpha_1,\ a_3s_1s_2(\alpha_1) \geq a_4s_1(\alpha_2),\ \ldots.$$

□

Suppose now that $\mathfrak{g}$ finite dimensional and $w = w_0$ is the longest element W. We write only C_1 or C_2 for the cone of adapted strings, depending on the chosen decomposition $w_0 = s_1s_2\ldots$ or $w_0 = s_2s_1\ldots$ (α_2 is the long root).

Corollary 2.

i) If $\mathfrak{g}$ is of type $\mathtt{A}_2$ and $w_0 = s_1s_2s_1 = s_2s_1s_2$, then:

$$C_1 = C_2 = \{(\underline{a}) \in \mathbb{R}^3 \mid a_2 \geq a_3\},$$

ii) If $\mathfrak{g}$ is of type $\mathtt{C}_2$ and $w_0 = s_1s_2s_1s_2 = s_2s_1s_2s_1$, then:

$$C_1 = \{(\underline{a}) \in \mathbb{R}^4 \mid 2a_2 \geq a_3 \geq 2a_4\},\ C_2 = \{(\underline{a}) \in \mathbb{R}^4 \mid a_2 \geq a_3 \geq a_4\}.$$

iii) If $\mathfrak{g}$ of type $\mathtt{G}_2$ and $w_0 = s_1s_2s_1s_2s_1s_2 = s_2s_1s_2s_1s_2s_1$, then

$$\begin{aligned}C_1 &= \{(\underline{a}) \in \mathbb{R}^6 \mid 6a_2 \geq 2a_3 \geq 3a_4 \geq 2a_5 \geq 6a_6\}\\ C_2 &= \{(\underline{a}) \in \mathbb{R}^6 \mid 2a_2 \geq 2a_3 \geq a_4 \geq 2a_5 \geq 2a_6\}.\end{aligned}$$

We have a canonical bijection $\phi : C_1 \to C_2$, where $\phi(\underline{a})$ is such that $f^{\underline{a}}\pi = f^{\phi(\underline{a})}\pi$ for $\pi \gg 0$. We will now describe this bijection. The same arguments as in the proof of Proposition 2.1 (compare [13]) yield for $w \in W$:

Lemma 2.2. *Suppose $e_1(f_2^{a_2}f_1^{a_3} \ldots \pi) = 0, e_2(f_1^{a_3} \ldots \pi) = 0, \ldots$. If m is maximal such that $e_2^m(f_1^{a_1}f_2^{a_2}f_1^{a_3} \ldots \pi) \neq 0$, then $(-m)$ is the minimum of the coefficients of $a_1s_2(\alpha_1) - a_2\alpha_2,\ a_2s_2s_1(\alpha_2) - a_3s_2(\alpha_1),\ \ldots$.*

If $\mathfrak{g}$ is of type $\mathtt{A}_2$, then the lemma above implies together with a simple weight argument (see also [3]):

Proposition 2.3. *The bijection $C_1 \to C_2$ is the piecewise linear map:*

$$(a_1, a_2, a_3) \mapsto (\max\{a_3, a_2 - a_1\}, a_1 + a_3, \min\{a_2 - a_3, a_1\}).$$

Proposition 2.4. *If $\mathfrak{g}$ is of type $\mathtt{C}_2$, then the bijection $C_1 \to C_2$ is the piecewise linear map:*

$$n_1 = \max\{p_4, p_3 - p_2, p_2 - p_1\},\ n_2 = \max\{p_3, p_1 - 2p_2 + 2p_3, p_1 + 2p_4\}$$
$$n_3 = \min\{p_2, 2p_2 - p_3 + p_4, p_4 + p_1\},\ n_4 = \min\{p_1, 2p_2 - p_3, p_3 - 2p_4\}.$$

and its inverse is the piecewise linear map $C_2 \to C_1$:

$$p_1 = \max\{n_4, 2n_3 - n_2, n_2 - 2n_1\},\ p_3 = \min\{n_2, 2n_2 - 2n_3 + n_4, n_4 + 2n_1\}$$
$$p_2 = \max\{n_3, n_1 + n_4, n_1 + 2n_3 - n_2\},\ p_4 = \min\{n_1, n_3 - n_4, n_2 - n_3\}.$$

Proof. The formula for p_1 follows from Lemma 2.2, and since $p_1 + p_3 = n_2 + n_4$, we get also the formula for p_3. If $q \gg 0$ and $\underline{n} \in C_2$, then Lemma 2.2 and [13], Lemma 2.7, imply: $f_2^{n_1} f_1^{n_2} f_2^{n_3} f_1^{n_4}(\pi_{q\omega_1} * \pi_{q\omega_2}) =$

$$f_2^{\min\{n_1, n_2 - n_4 - (n_3 - n_4)\}} f_1^{n_2} f_2^{n_4} f_1^{n_4} \pi_{q\omega_1} * f_2^{n_3 - n_4 + n_1 - \min\{n_1, n_2 - n_4 - (n_3 - n_4)\}} \pi_{q\omega_2}$$

and for $\underline{p} \in C_1$ we get: $f_1^{p_1} f_2^{p_2} f_1^{p_3} f_2^{p_4}(\pi_{q\omega_1} * \pi_{q\omega_2}) =$

$$f_1^{p_1} f_2^{\min\{p_2, p_3 - p_4\}} f_1^{p_3} \pi_{q\omega_1} * f_2^{p_4 + p_2 - \min\{p_2, p_3 - p_4\}} \pi_{q\omega_2}.$$

This implies: $\max\{p_4, p_2 - p_3 + 2p_4\} = \max\{n_3 - n_4, n_1 - n_2 + 2n_3 - n_4\}$. Since $p_2 + p_4 = n_1 + n_3$, it follows:

$$p_4 = \max\{n_3 - n_4, n_1 - n_2 + 2n_3 - n_4\} - \max\{0, n_1 + n_3 - p_3\}.$$

But $p_3 = n_2 + n_4 - p_1 = \min\{n_2, 2(n_2 - n_3) + n_4, 2n_1 + n_4\}$, so we get:

$$\begin{aligned} p_4 = -n_4 + \max\{n_3, n_1 - n_2 + 2n_3\} - n_1 - n_3 + \\ \min\{n_1 + n_3, n_2, 2n_2 - 2n_3 + n_4, 2n_1 + n_4\} \\ = \min\{n_1 + n_3 - n_4, n_2 - n_4, 2n_2 - 2n_3, 2n_1\} - \min\{n_1, n_2 - n_3\}. \end{aligned}$$

By discussing the cases $n_1 \leq n_2 - n_3$ and $n_1 \geq n_2 - n_3$ separately, one sees easily that $p_4 = \min\{n_1, n_3 - n_4, n_2 - n_3\}$. Since $p_2 + p_4 = n_1 + n_3$, this yields also the formula for p_2, so this finishes the proof for the second map.

By comparing $f_1^{p_1} f_2^{p_2} f_1^{p_3} f_2^{p_4}(\pi_{q\omega_2} * \pi_{q\omega_1})$ and $f_2^{n_1} f_1^{n_2} f_2^{n_3} f_1^{n_4}(\pi_{q\omega_2} * \pi_{q\omega_1})$, the same arguments yield the formula for the inverse map. □

Using the Young tableaux in [14], a similar kind of formula can be deduced for $\mathfrak{g}$ of type $\mathtt{G}_2$. We state here just the formula for the first term, the formula is a consequence of Lemma 2.2.

Lemma 2.5. *Suppose $\mathfrak{g}$ is of type $\mathtt{G}_2$, $\underline{p} \in C_1$ and $\underline{n} \in C_2$ (α_2 is the long root) are such that $f^{\underline{n}}\pi = f^{\underline{p}}\pi$ for $\pi \gg 0$. Then*

$$p_1 = \max\{n_6, 3n_5 - n_4, 2n_4 - 3n_3, 3n_3 - 2n_2, n_2 - 3n_1\}$$
$$n_1 = \max\{p_6, p_5 - p_4, 2p_4 - p_3, p_3 - 2p_2, p_2 - p_1\}.$$

3. The braidless weights

Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Denote by Φ the root system, by X the weight lattice, and fix a set of simple roots Δ. We call a fundamental weight ω *braidless* if for any $\tau \in W$ the following holds:

($*$) *If* $\alpha, \gamma \in \Delta$ *are such that* $\langle \tau(\omega), \alpha^\vee \rangle > 0, \langle \tau(\omega), \gamma^\vee \rangle > 0$*, then* $\langle \gamma, \alpha^\vee \rangle = 0$.

If $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ is a decomposition of $\mathfrak{g}$ into simple ideals, then ω is braidless for $\mathfrak{g}$ if and only if ω is braidless for one of the $\mathfrak{g}_i$. Denote by α_ω the unique simple root such that $\langle \omega, \alpha^\vee \rangle \neq 0$, let P_ω be the associated maximal parabolic subgroup, and denote by N_ω the nilpotent radical of the Lie algebra $\operatorname{Lie} P_\omega$. We use the enumeration of the fundamental weights as in [1].

Lemma 3.1. *A fundamental weight* ω *of a simple Lie algebra* $\mathfrak{g}$ *is braidless if and only if* ω *is a minuscule weight of* $\mathfrak{g}$*, or* $\omega = \omega_1$ *for* $\mathfrak{g}$ *of type* $\mathtt{B}_n$*, or* $\omega = \omega_n$ *for* $\mathfrak{g}$ *of type* $\mathtt{C}_n$*, or* ω *is arbitrary and* $\mathfrak{g}$ *is of rank at most two.*

Proof. Recall that ω is called minuscule if $|\langle \tau(\omega), \beta^\vee \rangle| \leq 1$ for all positive roots and all $\tau \in W$. Now $\langle \tau(\omega), \alpha^\vee \rangle > 0$ and $\langle \tau(\omega), \gamma^\vee \rangle > 0$ for two simple roots such that $\langle \alpha, \gamma^\vee \rangle \neq 0$ would imply $|\langle \tau(\omega), s_\alpha(\gamma)^\vee \rangle| > 1$. So $\langle \alpha, \gamma^\vee \rangle = 0$, and minuscule weights are braidless.

It is obvious that ω is braidless if $\operatorname{rk} \mathfrak{g} = 2$. Suppose now $\operatorname{rk} \mathfrak{g} > 2$ and ω is braidless but not minuscule. Let β be a positive root such that $\langle \omega, \beta^\vee \rangle > 0$, i.e., the root space $\mathfrak{g}_\beta$ corresponding to β is contained in N_ω. By Weyl group invariance we may assume: $\langle \beta, \alpha^\vee \rangle \geq 0$ for all $\alpha \neq \alpha_\omega$.

Suppose $\langle \beta, \alpha_\omega^\vee \rangle < 0$, so the root system Φ' spanned by β and α_ω is of type $\mathtt{A}_2$ or $\mathtt{B}_2$. Note if we take the intersection of the real span of Φ' with Φ, then we get again a root system of rank two, which can only be the same root system. So Φ' is a root subsystem of Levi type, i.e., there exists a $\tau \in W$ such that $\tau(\alpha_\omega)$ and $\tau(\beta)$ are simple roots. It follows that $\langle \tau(\omega), \tau(\alpha_\omega^\vee) \rangle > 0$, $\langle \tau(\omega), \tau(\beta^\vee) \rangle > 0$, in contradiction to the assumption that ω is braidless.

So β is hence a dominant root. If $\mathfrak{g}$ is of simply laced type, then β is the highest root. Since β is unique, this implies that β is conjugate to α_ω by W_ω, i.e., β is of the form $\alpha_\omega + \sum_{\alpha \neq \alpha_\omega} a_\alpha \alpha$, and ω is hence minuscule.

If $\mathfrak{g}$ is not simply laced and if the representation of $L_\omega \subset P_\omega$ on N_ω is irreducible, then the highest root is of the form $\alpha_\omega + \sum_{\alpha \neq \alpha_\omega} a_\alpha \alpha$. So (using for example the tables in [1]) one finds that $\omega = \omega_1$ for $\mathfrak{g}$ of type $\mathtt{B}_n$, or $\omega = \omega_n$ for $\mathfrak{g}$ of type $\mathtt{C}_n$. One checks easily that these two are braidless weights.

Suppose now the representation is not irreducible. The arguments above show that the positive roots in N_ω are conjugate by W_ω to the dominant roots, so $N_\omega = N_1 \bigoplus N_2$, where $N_1 = \oplus \mathfrak{g}_\alpha$ is the sum of all root spaces corresponding to short roots in N_ω and $N_2 = \oplus \mathfrak{g}_\alpha$ is the sum of all root spaces in N_ω of the long roots. Now if $\mathfrak{g}$ is of type $\mathtt{C}_n$, one sees easily that the submodule generated by the highest root contains always the root spaces of short roots and long roots unless $\omega = \omega_1$, which is a minuscule weight. If $\mathfrak{g}$ is of type $\mathtt{B}_n$, then the submodule generated by the root α contains always the root spaces of short roots and long roots unless $\omega = \omega_n$, which is a minuscule weight. It remains to consider the case $\mathtt{F}_4$. The decomposition of $N_\omega = N_1 \bigoplus N_2$ implies that the highest root is of the form $\beta = 2\alpha_\omega + \sum_{\alpha \neq \alpha_\omega} a_\alpha \alpha$, so $\omega = \omega_1, \omega_4$. But the L_ω-submodule generated by α_ω contains in both cases long and short roots. $\square$

Lemma 3.2. *Suppose* ω *is braidless. Denote by* ${}^\omega W$ *the minimal representatives of the left* W_ω*-classes in* W*. If* $\tau \in {}^\omega W$*, then a reduced decomposition of* τ *is unique up to the exchange of orthogonal simple reflections.*

Proof. It is well-known that one can pass from one reduced decomposition to another by exchanging orthogonal simple reflections or by using the braid relations, i.e., in a reduced decomposition one replaces

$$s_{\alpha_1}\dots s_{\alpha_r}(s_a s_\gamma s_a\dots)s_{\alpha_{r+1}}\dots s_{\alpha_t} \quad \text{by} \quad s_{\alpha_1}\dots s_{\alpha_r}(s_\gamma s_\alpha s_\gamma\dots)s_{\alpha_{r+1}}\dots s_{\alpha_t}.$$

But such a braid relation can occur in a reduced decomposition if and only if

$$\langle s_{\alpha_r}\dots s_{\alpha_1}(\omega),\alpha^\vee\rangle>0\ ,\langle s_{\alpha_r}\dots s_{\alpha_1}(\omega),\gamma^\vee\rangle>0,$$

which is not possible because ω is a braidless weight. □

Let w be the longest word in W_ω and let τ be the longest word in ${}^\omega W$. Fix a reduced decomposition $\tau=s_{\alpha_1}\dots s_{\alpha_r}$. The set of positive roots $N(\Phi)$ below is exactly the set of positive roots in N_ω:

$$N(\Phi):=\{\beta_1:=w(\alpha_1),\beta_2:=ws_{\alpha_1}(\alpha_2),\dots,\beta_r:=ws_{\alpha_1}\dots s_{\alpha_{r-1}}(\alpha_r)\}$$

Lemma 3.3.

i) *If* $\langle\alpha_i,\alpha_{i+1}^\vee\rangle=0$ *and both have the same length, then* β_i *and* β_{i+1} *are not comparable in the usual partial order.*

ii) $\langle\alpha_i,\alpha_{i+1}^\vee\rangle=\langle\alpha_{i+1},\alpha_i^\vee\rangle=-1\Longrightarrow\beta_i>\beta_{i+1}$.

iii) $\langle\alpha_i,\alpha_{i+1}^\vee\rangle=-1$, $\langle\alpha_{i+1},\alpha_i^\vee\rangle=-2$, $\alpha_{i+2}=\alpha_i^\vee\Longrightarrow 2\beta_i>\beta_{i+1}>2\beta_{i+2}$.

iv) $\langle\alpha_i,\alpha_{i+1}^\vee\rangle=-2$, $\langle\alpha_{i+1},\alpha_i^\vee\rangle=-1$ *and* $\alpha_{i+2}=\alpha_i^\vee\Longrightarrow\beta_i>\beta_{i+1}>\beta_{i+2}$.

Proof. We prove first *ii*). Note that

$$(*)\qquad \langle s_{\alpha_{i-1}}\dots s_{\alpha_1}(\omega),\alpha_i^\vee\rangle>0 \text{ implies } \langle s_{\alpha_{i-1}}\dots s_{\alpha_1}(\omega),\alpha_{i+1}^\vee\rangle=0.$$

The latter can not be positive because ω is braidless. Similarly, if the latter is negative, then there exists a reduced decomposition:

$$s_{\alpha_1}\dots s_{\alpha_{i-1}}=s_{\alpha'_1}\dots s_{\alpha'_{i-2}}s_{\alpha_{i+1}},$$

and hence $s_{\alpha'_1}\dots s_{\alpha_{i+1}}s_{\alpha_i}s_{\alpha_{i+1}}$ would be a reduced decomposition for an element in ${}^\omega W$, which leads again to a contradiction. Now $(*)$ implies that

$$s_{\alpha_1}\dots s_{\alpha_{i-1}}s_{\alpha_{i+1}}=s_\gamma s_{\alpha_1}\dots s_{\alpha_{i-1}}$$

for some simple root $\gamma\neq\alpha_\omega$. Note that $s_\gamma=s_{\alpha_1}\dots s_{\alpha_{i-1}}s_{\alpha_{i+1}}s_{\alpha_{i-1}}\dots s_{\alpha_1}$ implies $\gamma=s_{\alpha_1}\dots s_{\alpha_{i-1}}(\alpha_{i+1})$. It follows that

$$\beta_{i+1}=ws_{\alpha_1}\dots s_{\alpha_i}(\alpha_{i+1})=ws_{\alpha_1}\dots s_{\alpha_{i-1}}(\alpha_i+\alpha_{i+1})=\beta_i+w(\gamma)<\beta_i,$$

because $w(\gamma)$ is the negative of a simple root. As above one proves for *iii), iv)*:

$$s_{\alpha_1}\dots s_{\alpha_{i-1}}s_{\alpha_{i+1}}=s_\gamma s_{\alpha_1}\dots s_{\alpha_{i-1}}$$

for some simple root $\gamma\neq\alpha_\omega$. So for *iii)* we get:

$$\beta_{i+1}=ws_{\alpha_1}\dots s_{\alpha_i}(\alpha_{i+1})=ws_{\alpha_1}\dots s_{\alpha_{i-1}}(2\alpha_i+\alpha_{i+1})=2\beta_i+w(\gamma),$$
$$\beta_{i+2}=ws_{\alpha_1}\dots s_{\alpha_i}s_{\alpha_{i+1}}(\alpha_{i+2})=ws_{\alpha_1}\dots s_{\alpha_{i-1}}(\alpha_i+\alpha_{i+1})=\beta_i+w(\gamma).$$

This implies: $2\beta_i>\beta_{i+1}$. Since $w(\gamma)$ is the negative of a simple root we get:

$$\beta_{i+1}=2\beta_i+w(\gamma)>2\beta_i+2w(\gamma)=2\beta_{i+2}$$

and hence $2\beta_i > \beta_{i+1} > 2\beta_{i+2}$. Similarly, we get for *iv)*:

$$\beta_{i+1} = ws_{\alpha_1}\dots s_{\alpha_i}(\alpha_{i+1}) = ws_{\alpha_1}\dots s_{\alpha_{i-1}}(\alpha_i+\alpha_{i+1}) = \beta_i + w(\gamma),$$
$$\beta_{i+2} = ws_{\alpha_1}\dots s_{\alpha_i}s_{\alpha_{i+1}}(\alpha_{i+2}) = ws_{\alpha_1}\dots s_{\alpha_{i-1}}(\alpha_i+2\alpha_{i+1}) = \beta_i + 2w(\gamma).$$

and hence $\beta_i > \beta_{i+1} > \beta_{i+2}$. It remains to prove *i)* for $\operatorname{rk}\mathfrak{g} > 2$. By exchanging orthogonal reflections if necessary, we may assume that the decomposition

$$s_{\alpha_1}\dots s_{\alpha_j}(s_{\alpha_{j+1}}\dots s_{\alpha_k})(s_{\alpha_{k+1}}\dots s_{\alpha_i})s_{\alpha_{i+1}}(s_{\alpha_{i+2}}\dots)$$

is such that the following sets of roots are pairwise orthogonal to each other:

$$\Delta_1 := \{\alpha_{j+1},\alpha_{j+2},\dots,\alpha_{k-1},\alpha_k,\alpha_{i+1}\},\quad \Delta_2 := \{\alpha_{k+1},\alpha_{k+2},\dots,\alpha_{i-1},\alpha_i\}$$

Further, for $l = j+1,\dots,k,i+1$ there exists an p, $j \le p < l$ such that $\langle\alpha_l,\alpha_p^\vee\rangle < 0$, and for $k+1 \le l \le i$ there exists an p, $p = j$ or $k+1 \le p < l$, such that $\langle\alpha_l,\alpha_p^\vee\rangle < 0$. Also, we may assume that for $j+1 \le l \le k$ there exists an p, $j < p \le k$ or $p = i+1$, such that $\langle\alpha_l,\alpha_p^\vee\rangle < 0$, and for $k+1 \le l \le i-1$ there exists an q, $l < q \le i$ such that $\langle\alpha_l,\alpha_q^\vee\rangle < 0$.

We will show that β_i and β_{i+1} are obtained from β_j by subtracting certain simple roots. The conditions above imply that one can find two sequences:

$$j := p_0 < \dots < p_{r-1} \le k,\ p_r := i+1;\quad j := q_0,\ k+1 \le q_1 < \dots < q_t := i,$$

such that $\langle\alpha_{p_s},\alpha_{p_{s-1}}^\vee\rangle < 0$ but $\langle\alpha_{p_s},\alpha_l^\vee\rangle = 0$ for $p_{s-1} < l < p_s$, and similar, $\langle\alpha_{q_s},\alpha_{q_{s-1}}^\vee\rangle < 0$, but $\langle\alpha_{q_s},\alpha_l^\vee\rangle = 0$ for $q_{s-1} < l < q_s$.

This implies that in the Dynkin diagram the node of α_j is connected with two other nodes (which are orthogonal to each other), and these two roots are (successively) connected to the nodes of α_i and α_{i+1}. Since these two are of the same length, the roots $\alpha_j,\alpha_i,\alpha_{i+1}$ are all of the same length.

The sets Δ_1,Δ_2 are disjoint, so at least one of the two contains only roots of the same length as α_j. Without loss of generality we may assume that Δ_2 is this set. Note if $\alpha_l \in \Delta_1$ is the only simple root of different length (the case $\mathtt{F}_4$ has been excluded), then we may assume without loss of generality that $\langle\alpha_{l-1},\alpha_l^\vee\rangle < 0$: The conditions on the set Δ_1 imply that there exists an $j \le l' < l$ with this property, by exchanging orthogonal reflections (i.e., "moving s_{α_l} to the left") if necessary we may assume that $l' = l-1$.

Now consider the root β_{q_s}, suppose it is of the same length as β_i,β_{i+1} and β_j. By the definition of β_{q_s} and the choice of q_{s-1} we get

$$\beta_{q_s} = ws_{\alpha_1}\dots s_{\alpha_j}\dots s_{\alpha_{q_{s-1}}}(\alpha_{q_s}).$$

If $\alpha_{q_{s-1}}$ and α_{q_s} are of the same length, then the arguments above proving *ii)* show that $\beta_{q_s} = \beta_{q_{s-1}} - \gamma$ for some simple root. Note that β_{q_s} and $\beta_{q_{s-1}}$ are of the same length. If $\alpha_{q_{s-1}}$ and α_{q_s} are not of the same length, then we know that $q_{s-2} = q_{s-1} - 1$, and necessarily $\alpha_{q_{s-2}} = \alpha_{q_s}$. The arguments above proving *iii), iv)* show that $\beta_{q_s} = \beta_{q_{s-2}} - \gamma$ or $\beta_{q_s} = \beta_{q_{s-2}} - 2\gamma$ for some simple root, and β_{q_s} and $\beta_{q_{s-2}}$ are of the same length.

So we have shown inductively that $\beta_{i+1} = \beta_j - \sum a_\gamma\gamma$ for some simple roots γ and $\beta_i = \beta_j - \sum a_{\gamma'}\gamma'$ for some simple roots γ'.

But remember that the arguments to prove *ii)–iv)* show that the simple roots γ are of the form $ws_{\alpha_1}\dots s_{\alpha_{j-1}}(\delta)$, where δ is a linear combination of α_j and the roots in Δ_1, and the γ' are of the form $ws_{\alpha_1}\dots s_{\alpha_{j-1}}(\delta')$, where δ' is a linear combination of α_j and the roots in Δ_2. Since the part of the linear combination coming from Δ_1 respectively Δ_2 is always nonzero, it follows that the γ' are linearly independent of the γ and vice versa. Since the γ and γ' are simple roots, this implies that β_i and β_{i+1} are not compatible. □

Fix a reduced decomposition $\tau = s_{\alpha_1} \dots s_{\alpha_r}$, and suppose $\mathrm{rk}\,\mathfrak{g} > 2$.

LEMMA 3.4.
i) If α_j, $j \geq 2$, is such that $\alpha_i \neq \alpha_j$ for all $i < j$, then there exists exactly one α_l, $1 \leq l < j$, such that $\langle \alpha_j, \alpha_l^\vee \rangle \neq 0$.
ii) Suppose $\alpha := \alpha_p = \alpha_q$, but $\alpha_i \neq \alpha$ for all $p < i < q$. Then either there exist two (not necessarily different) simple roots α_i, α_j, $p < i < j < q$, which are not orthogonal to α, or there exists only one simple root α_i with this property, in which case the root system spanned by α, α_i is of type $\mathtt{B}_2$.

Proof. It is evident in the situation in *i*) that there exists at least one simple root α_i, $1 \leq i < j$ not orthogonal to α_j, otherwise the decomposition would not be reduced. Now if α_i is not orthogonal to α_j, then $\langle \alpha_i, \alpha_j^\vee \rangle \leq -1$, and hence $\langle s_{\alpha_i} \dots s_{\alpha_1}(\omega), \alpha_j^\vee \rangle > 0$, so $s_{\alpha_1} \dots s_{\alpha_i} s_{\alpha_j}$ is a reduced decomposition in ${}^\omega W$. Since the decomposition of τ is unique up to the exchange of orthogonal reflections, this proves *i*).

Consider the situation in *ii*). There exists at least one α_i which is not orthogonal to α. If there is only one, then, by exchanging orthogonal reflections (i.e., by moving $s_{\alpha_p} = s_\alpha$ to the right and $s_{\alpha_q} = s_\alpha$ to the left), we can assume that the decomposition is of the form $\dots s_\alpha s_{\alpha_i} s_\alpha \dots$, which is only possible if α, α_i form a root system of type $\mathtt{B}_2$.

It remains to prove that there are at most two such roots. Suppose there are at least two such roots, let α_i, α_j, $p < i < j < q$, be such that i, j are minimal with this property. Now

$$(1) \qquad -2 \leq \langle s_{\alpha_p} \dots s_{\alpha_1}(\omega), \alpha_p^\vee \rangle < 0$$

implies (2) : $\langle s_{\alpha_j} \dots s_{\alpha_i} \dots s_{\alpha_p} \dots s_{\alpha_1}(\omega), \alpha_p^\vee \rangle \geq 0$. If the latter is positive, then $s_{\alpha_1} \dots s_{\alpha_i} \dots s_{\alpha_j} s_\alpha$ is a reduced decomposition and hence, by the uniqueness of the decomposition, we have proved the lemma.

If (2) is positive (for example if the lower bound in (1) is -1), then the lemma is proved. It remains to consider the remaining non-minuscule cases. Suppose $\mathfrak{g}$ is of type $\mathtt{B}_n$ and $\omega = \omega_1$. Then $\langle \sigma(\omega), \alpha^\vee \rangle = -2$ for some $\sigma \in W$ only if $\alpha = \alpha_n$ is the short root. Now $\tau = s_1 \dots s_n \dots s_1$ and s_n occurs hence only once in the decomposition.

Suppose $\mathfrak{g}$ is of type $\mathtt{C}_n$ and $\omega = \omega_n$, we may assume that $i = p + 1$ in (2). If (1) $= -2$ and (2) $= 0$, then α is a short root, $\langle s_{\alpha_{p+1}} \dots s_{\alpha_1}(\omega), \alpha_p^\vee \rangle = -1$, and hence $\langle \alpha_{p+1}, \alpha^\vee \rangle = \langle \alpha_j, \alpha^\vee \rangle = -1$. It follows that both are short roots. Since $s_{\alpha_1} \dots s_{\alpha_p} s_{\alpha_{p+1}}$ is a reduced decomposition and ω is braidless, we know $\langle s_{\alpha_{p-1}} \dots s_{\alpha_1}(\omega), \alpha_{p+1}^\vee \rangle = 0$, and hence $\langle s_{\alpha_p} \dots s_{\alpha_1}(\omega), \alpha_{p+1}^\vee \rangle = \langle -2\alpha_p, \alpha_{p+1}^\vee \rangle = 2$. But this would imply $\langle s_{\alpha_{p+1}} \dots s_{\alpha_1}(\omega), \alpha_p^\vee \rangle = 0$, which is a contradiction. □

4. Cones for nice decompositions

Let $\mathfrak{g}$ be semisimple and denote by D its Dynkin diagram. We call a simple root α braidless if the corresponding fundamental weight ω_α is so. We write $D-\{\alpha\}$ for the diagram obtained by removing the node of α from D.

Suppose that the set of simple roots of G admits an enumeration

$$\Delta=\{\alpha_1,\dots,\alpha_n\} \text{ such that } \alpha_i \text{ is braidless for } D-\{\alpha_1,\dots,\alpha_{i-1}\}$$

for all $i=1,\dots,n$. We call this a *good enumeration.*

Remark 4.1. Note that all simple Lie algebras admit such a *good* enumeration except the ones of type $\mathtt{E}_8$ and $\mathtt{F}_4$. We will describe such good enumerations in the later sections for the respective types. The groups of type $\mathtt{E}_8$ and $\mathtt{F}_4$ will be discussed separately.

Let W_i be the Weyl group generated by the simple reflections $s_{\alpha_1},\dots,s_{\alpha_i}$, and let ${}^iW_i$ be the set of minimal representatives in W_i of $W_{i-1}\backslash W_i$. We fix a reduced decomposition of the longest word w_0 which is compatible with this enumeration, i.e., the decomposition

$$w_0=\underbrace{s_1}_{\tau_1}\underbrace{s_2s_1\cdots}_{\tau_2}\cdots\underbrace{s_{j_1}\cdots s_{j_r}}_{\tau_j}\cdots\underbrace{s_{n_1}\cdots s_{n_t}}_{\tau_n}$$

is such that τ_j is the longest word in ${}^jW_j$. This decomposition is, up to the exchange of orthogonal reflections, uniquely determined by this condition (Lemma 3.2), we call this the *nice decomposition* associated to the good enumeration. Let $C_0\subset\mathbb{R}^N$ be the cone of strings adapted to this decomposition.

Let $\Phi^+=\{\beta_1,\dots,\beta_N\}$ be the enumeration of the positive roots corresponding to this decomposition, i.e.,

$$\beta_1=\alpha_1,\ \beta_2:=s_1(\alpha_2),\ \beta_3:=s_2s_1(\alpha_2)\dots.$$

The defining inequalities for the cone C_0 are closely related to the usual ordering on the set of roots. Denote by $\langle\beta,\delta\rangle$ the root system spanned by the roots β,δ. For $\mathfrak{g}$ of type $\mathtt{G}_2$ let C be the cone in Corollary 2.1, else let C be the cone of all N-tuples $\underline{a}\in\mathbb{R}^N$ such that $a_i\geq 0$ for all $i=1,\dots,N$, and for all $i<j$ such that β_i is not orthogonal to β_j:

$$\begin{cases} a_i\geq a_j & \text{if } \langle\beta_i,\beta_j\rangle \text{ is of type } \mathtt{A}_2 \text{ and } \beta_i>\beta_j;\\ a_i\geq a_j & \text{if } \langle\beta_i,\beta_j\rangle \text{ is of type } \mathtt{C}_2,\ \beta_i \text{ is long and } \beta_j \text{ is short, and } \beta_i>2\beta_j,\\ & \text{or, } \beta_i \text{ is short and } \beta_j \text{ is long, and } 2\beta_i>\beta_j \text{ but } \beta_i\not\geq\beta_j;\\ a_i\geq 2a_j & \text{if } \langle\beta_i,\beta_j\rangle \text{ is of type } \mathtt{C}_2,\ \beta_i \text{ is short, } \beta_j \text{ is long, and } \beta_i>\beta_j;\\ 2a_i\geq a_j & \text{if } \langle\beta_i,\beta_j\rangle \text{ is of type } \mathtt{C}_2,\ \beta_i \text{ is long and } \beta_j \text{ is short,}\\ & \text{and } \beta_i>\beta_j \text{ but } \beta_i\not\geq 2\beta_j. \end{cases}$$

THEOREM 4.2. *$C=C_0$ is the cone of adapted strings.*

Remark 4.3. Let C' be the cone associated to the reduced decomposition obtained from the given one by exchanging two orthogonal simple reflections (let us say the j-th and $j+1$-st) in the reduced decomposition of one of the τ_p. Note that the map $\mathbb{R}^N\to\mathbb{R}^N$, which exchanges the j-th and $j+1$-st coordinate, induces a bijection $C\to C'$: The inequalities above are defined only for a_k,a_l such that β_k,β_l are not orthogonal to each other. But if $\alpha_{i_j},\alpha_{i_{j+1}}$ are orthogonal to each other, then so are β_j and β_{j+1}.

On the other hand, recall that the root operators commute if the corresponding simple roots are orthogonal to each other. It follows that a tuple $\underline{a} = (a_1, \dots, a_j, a_{j+1}, \dots, a_N)$ is adapted to the first decomposition if and only if $\underline{a}' = (a_1, \dots, a_{j+1}, a_j, \dots, a_N)$ is adapted to the second.

Proof. The theorem is obviously correct for $\mathfrak{g}$ of rank 1, for $\mathfrak{g}$ of rank 2 one checks easily that the cone defined above is the one given in Corollary 2.1.

Suppose now $\mathfrak{g}$ is of rank $n > 2$, set $w = \tau_1 \cdots \tau_{n-1}$, and let $\tau_n = s_{i_1} \cdots s_{i_t}$ be a reduced decomposition. Denote by $\mathfrak{l}$ the Levi subalgebra associated to the roots $\alpha_1, \dots, \alpha_{n-1}$, then $w = \tau_1 \cdots \tau_{n-1}$ provides a nice decomposition of the longest word of the Weyl group of $\mathfrak{l}$. It follows from the Demazure type formula and the restriction rule for crystal graphs (or the path model, see [7] or [13]), that $\underline{n} = (n_1, \dots, n_N)$ is adapted if and only if

$$(n_1, \dots, n_{N-t}) \textit{ is adapted for } \mathfrak{l} \textit{ and } (0, \dots, 0, n_{N-t+1}, \dots, n_N) \in C_0.$$

So to prove the theorem, by induction it is sufficient to prove that an element of the form $\underline{a} = (0, \dots, 0, a_1, \dots, a_t)$ is adapted if and only if $\underline{a} \in C$.

By abuse of notation we will just say in the following that $\underline{a} = (a_1, \dots, a_t)$ is adapted. Also, let $\beta_1, \dots, \beta_t$ be the induced enumeration of the roots in the Lie algebra of the unipotent radical of P_{ω_n}, i.e., $\beta_1 = w(\alpha_{i_1})$, $\beta_2 = ws_{i_1}(\alpha_{i_2}), \dots$. We denote by $C_n \subset C$ the cone such that the first $N - t$ coordinates are equal to 0. We identify C_n with the subcone defined in $\mathbb{R}^t$ by the inequalities above corresponding to the roots $\beta_1, \dots, \beta_t$ in the Lie algebra of the unipotent radical of P_{ω_n}, and the condition that all entries are non-negative. To prove that $\underline{a} \in \mathbb{N}^t$ is adapted, we have to prove that for $\pi \gg 0$:

$$e_\alpha(f_{i_1}^{a_1} \dots f_{i_t}^{a_t}\pi) = 0 \ \forall \alpha \neq \alpha_n; \text{ and } e_{i_{j-1}}(f_{i_j}^{a_j} \dots f_{i_t}^{a_t}\pi) = 0 \text{ for } j = 2, \dots, t.$$

To simplify the notation, we write

$$e_\alpha \underline{a} \text{ or } e_\alpha(a_j, \dots, a_t) \text{ instead of } e_\alpha(f_{i_1}^{a_1} \dots f_{i_t}^{a_t}\pi) \text{ or } e_\alpha(f_{i_j}^{a_j} \dots f_{i_t}^{a_t}\pi).$$

Suppose $\alpha = \alpha_{i_j}$ and $\alpha' = \alpha_{i_{j-1}}$ are not orthogonal to each other. By decreasing induction (the case $k = t-1, t$ being trivial) we may assume that if α, α' are of the same length, then $e_{i_k}(a_{k+1}, \dots, a_t) = 0$ for $k \geq j$ and $\underline{a} \in C_n$. And if α, α' have different lengths and $\alpha' = \alpha_{i_{j+1}}$, then $e_{i_k}(a_{k+1}, \dots, a_t) = 0$ for $k \geq j+1$ and $\underline{a} \in C_n$.

Lemma 4.4. *If α, α' have the same length, then $e_\alpha(a_{j-1}, a_j, \dots) = 0$ for $\underline{a} \in C_n$ only if $e_{\alpha'}(a_{j+1}, \dots) = 0$. If α, α' are of different length and $\alpha' = \alpha_{i_{j+1}}$, then $e_\alpha(a_{j-1}, a_j, \dots) = 0$ for $\underline{a} \in C_n$ only if $e_\alpha(a_{j+2}, \dots, a_t) = 0$.*

Proof. If α, α' have the same length, then so have β_j and β_{j-1}. Since α is not orthogonal to α', the same is true for β_{j-1} and β_j. By Lemma 3.4, we know that $\beta_{j-1} \geq \beta_j$. Since $e_\alpha(a_{j+1}, \dots) = 0$ by assumption, it follows by section 2 that $e_\alpha(a_{j-1}, \dots) = 0$ if and only if $a_{j-1} \geq a_j$ and $e_{\alpha'}(a_{j+1}, \dots, a_t) = 0$.

If α, α' have different lengths, then so have β_j and β_{j-1}. Since ω is braidless and $\alpha_{i_{j-1}} = \alpha_{i_{j+1}}$, we know that $N_j := \langle s_{\alpha_{i_{j-2}}} \dots s_{\alpha_{i_1}}(\omega), \alpha_{i_j}^\vee \rangle = 0$. By induction we can suppose that $e_{\alpha'}(a_{j+2}, \dots) = 0$ and $e_\alpha(a_{j+1}, \dots) = 0$. Section 2 implies: $e_\alpha(a_{j-1}, a_j, \dots, a_t) = 0$ if and only if also $e_\alpha(a_{j+2}, \dots) = 0$, and $a_{j-1} \geq a_j \geq a_{j+1}$ respectively $2a_{j-1} \geq a_j \geq 2a_{j+1}$, depending on whether

β_j or β_{j-1} is a long root. Since $\underline{a} \in C_n$, the conditions on the coefficients are satisfied in both cases, which finishes the proof of the lemma. □

Consider the last condition: $e_\alpha(\ldots) = 0$ respectively $e_{\alpha'}(\ldots) = 0$, of the lemma. If all the simple roots occurring later in the decomposition are different from α respectively α', then this condition is trivially fulfilled. If not, then, after may be exchanging orthogonal reflections, Lemma 3.4 implies that we can transform this condition into one of the following, depending on the length of the roots. Here $\gamma = \alpha$ or α':

i) $e_\gamma(a_k, a_{k+1}, \ldots) = 0$, $k > j$, $\gamma = \alpha_{i_{k+1}}$, and $\alpha_{i_k}, \alpha_{i_{k+1}}$ span a root system of type $\mathtt{A}_2$.

ii) $e_\gamma(a_k, a_{k+1}, a_{k+2}, \ldots) = 0$, $k > j$, and $\alpha_{i_k} = \alpha_{i_{k+2}}$, $\gamma = \alpha_{i_{k+1}}$ span a root system of type $\mathtt{C}_2$.

A priori, there could be a third case: $e_\gamma(a_k, a_{k+1}, \ldots) = 0$, $k > j$, α_{i_k} and $\gamma = \alpha_{i_{k+1}}$ span a root system of type $\mathtt{C}_2$, and $\langle s_{i_{k+2}} \ldots s_{i_t}(\omega), \alpha_{i_k}^\vee \rangle = 0$. (Note that the latter being negative would imply we are in case *ii*), and the latter can not be positive because ω is braidless). But this would imply that $s_\gamma s_{i_k} \ldots s_{i_t}$ is reduced in W/W_ω. We will show that this leads to a contradiction:

If we start with the situation $\alpha = \alpha_{i_j}, \alpha' = \alpha_{i_{j-1}}$ span a root system of type $\mathtt{A}_2$, then $\gamma = \alpha'$ and we would have a reduced decomposition of the form $\ldots s_{\alpha'} s_\alpha \ldots s_{\alpha'} s_{i_k} \ldots s_{i_t}$, where all the reflection coming between s_α and $s_{\alpha'}$ are orthogonal to $s_{\alpha'}$, which can not be since ω is braidless.

If we start with the situation $\alpha = \alpha_{i_j}, \alpha' = \alpha_{i_{j-1}} = \alpha_{i_{j+1}}$ span a root system of type $\mathtt{C}_2$, then $\gamma = \alpha$ and we would have a reduced decomposition of the form $\ldots s_{\alpha'} s_\alpha s_{\alpha'} \ldots s_\alpha s_{i_k} \ldots s_{i_t}$, where all the reflection coming between $s_{\alpha'}$ and s_α are orthogonal to s_α, which can not be since ω is braidless.

Now in the cases *i*) and *ii*), we can apply the lemma above again. By repeating the argument we get:

Corollary 1. *If $\alpha = \alpha_{i_j}$, $\alpha' = \alpha_{i_{j-1}}$ span a root system of type $\mathtt{A}_2$, then $e_{\alpha'}(a_{j+1}, \ldots) = e_\alpha(a_{j+1}, \ldots) = 0$ for $\underline{a} \in C_n$. If $\alpha = \alpha_{i_j}$, $\alpha' = \alpha_{i_{j-1}} = \alpha_{i_{j+1}}$ span a root system of type $\mathtt{C}_2$, then $e_{\alpha'}(a_{j+2}, \ldots) = e_\alpha(a_{j+2}, \ldots) = 0$ for $\underline{a} \in C_n$.*

Proof. (of the theorem, continuation) An immediate consequence of the corollary and section 2 is of course: If $\alpha = \alpha_{i_j}, \alpha' = \alpha_{i_{j-1}}$ are of the same length, then $\underline{a} \in C_n$ implies $e_{\alpha'}(a_j, \ldots) = 0$, and if the two roots have different lengths, then $e_{\alpha'}(a_j, \ldots) = 0$ and $e_\alpha(a_{j+1}, \ldots) = 0$. So by induction we get: $e_{i_j}(a_{j+1}, \ldots) = 0$ for $j = 1, \ldots, t-1$. Further, the corollary and Lemma 3.4 also imply that $e_\alpha \underline{a} = 0$ for all $\alpha \neq \alpha_n$, so we have proved:

$$\underline{a} \in C_n \Longrightarrow \underline{a} \text{ is adapted.}$$

It remains to prove that the converse is true, we may suppose that $\operatorname{rk} \mathfrak{g} \geq 3$. Suppose first $\alpha = \alpha_{i_j}$, $\gamma = \alpha_{i_{j+1}}$, α is not orthogonal to γ and

$$\tau_n = \ldots s_\alpha s_\gamma \ldots \text{ or } \tau_n = \ldots s_\alpha s_\gamma s_\alpha \ldots,$$

where α, γ are of different lengths in the latter case. Lemma 3.3 implies for C_n the inequalities: $a_j \geq a_{j+1}$ respectively $a_j \geq a_{j+1} \geq a_{j+2}$ or $2a_j \geq a_{j+1} \geq 2a_{j+2}$, depending on which of the roots is the long one. Since ω is braidless, we know that if we insert a s_γ, then, using the braid relations, we get:

$$\dots s_\gamma s_\alpha s_\gamma \dots = s_\delta \tau_n \text{ respectively } \dots s_\gamma s_\alpha s_\gamma s_\alpha \dots = s_\delta \tau_n$$

for some simple root $\delta \neq \alpha_n$. Using the "braid relations" for the root operators (section 2), we know that $e_\gamma(a_j \dots) \neq 0$ would imply $e_\delta \underline{a} \neq 0$. So if $\underline{a}$ is adapted, then we know that $e_\gamma(a_j \dots) = 0$, which in turn implies that the inequalities $a_j \geq a_{j+1}$ respectively $a_j \geq a_{j+1} \geq a_{j+2}$ or $2a_j \geq a_{j+1} \geq 2a_{j+2}$ (depending on which of the roots is the long one) are necessary conditions.

Suppose now β_j, β_k, $j < k$, span a root system of type $\mathtt{A}_2$ and $\beta_j > \beta_k$. If $k = j+1$ (modulo the exchange of orthogonal reflections), then we are in the situation above, so we know already that the condition $a_j \geq a_{j+1}$ is a necessary condition for $\underline{a}$ to be adapted. If $k \neq j+1$ (modulo exchange), then (modulo exchange) there exists a sequence $i_1 = j < i_2 < \dots < i_r = k$ such that β_{i_p} is not orthogonal to $\beta_{i_{p+1}}$ but orthogonal to all β_l for $i_p < l < i_{p+1}$. If all the roots are of the same length, then Lemma 3.3 implies $\beta_{i_1} > \beta_{i_2} > \dots > \beta_{i_r}$ and, by the discussion above, we know that $a_j = a_{i_1} \geq \dots \geq a_{i_r} = a_k$. In particular, $a_j \geq a_k$, is a necessary condition for $\underline{a}$ to be adapted.

Suppose β_{i_p} and $\beta_{i_{p+1}}$ are not of the same length, let p be minimal with this property. The arguments above show that $a_{i_1} \geq \dots \geq a_{i_p}$. Note that there is only one simple root which is not of the same length as all the other simple roots, and the corresponding node is an extremal node of the Dynkin diagram. Since the long roots in the $\mathtt{C}_n$-case respectively the short roots in the $\mathtt{B}_n$-case are orthogonal to each other, we know that $\alpha_{i_1}, \alpha_{i_r} \neq \alpha_n$. By the minimality of p we know also that $\alpha_{i_p} \neq \alpha_n$, but $\alpha_{i_{p+1}} = \alpha_n$ and hence $\alpha_{i_p} = \alpha_{n-1}$. Note that $p+1 < r$ because $\alpha_{i_{p+1}}$, and hence $\beta_{i_{p+1}}$, has not the right length. Since all other simple roots are orthogonal to α_n, it follows that $\alpha_{i_{p+2}} = \alpha_{n-1}$. The discussion above shows that $a_{i_p} \geq a_{i_{p+2}}$ is a necessary condition for $\underline{a}$ to be adapted. Repeating the arguments, we see that $a_j \geq a_k$ is a necessary condition for $\underline{a}$ to be adapted.

Suppose now β_j, β_k, $j < k$, span a root system of type $\mathtt{C}_2$ and β_k is not orthogonal to β_j. If $k = j+1$ (modulo the exchange of orthogonal reflections), then we can assume that α_{i_j} and $\alpha_{i_{j+1}}$ form a root system of type $\mathtt{C}_2$, and we are in one of the cases discussed at the beginning. If $k \neq j+1$ (modulo exchange), then (modulo exchange) there exists a sequence $i_1 = j < i_2 < \dots < i_r = k$ such that β_{i_p} is not orthogonal to $\beta_{i_{p+1}}$ but orthogonal to all β_l for $i_p < l < i_{p+1}$. Since the long roots in the $\mathtt{C}_n$-case and the short roots in the $\mathtt{B}_n$-case are orthogonal to each other, we may further assume that either $\beta_j, \beta_{i_{r-1}}$ or β_{i_2}, β_k are of the same length. It follows then easily from the discussion above that, depending on the length of the roots, $a_j \geq a_k$, $2a_j \geq a_k$ respectively $a_j \geq 2a_k$, is a necessary condition for $\underline{a}$ to be adapted. □

5. Example: Sl_{n+1} and the Gelfand-Tsetlin Patterns

We consider in this section the case $\mathfrak{g} = \mathfrak{sl}_{n+1}$. The following description of the cone and its connection with the classical Gelfand-Tsetlin patterns can be found in [4], we include this section for completeness. Let ϵ_i be the projection of a diagonal matrix onto its i-th entry. We fix as Borel subalgebra the upper triangular matrices and as Cartan subalgebra the diagonal matrices. The dominant weights are then of the form:

$$\lambda_1\epsilon_1 + \ldots + \lambda_n\epsilon_n, \quad \text{where} \quad \lambda_1 \geq \ldots \geq \lambda_n \geq 0.$$

The enumeration of the simple roots $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_n = \epsilon_n - \epsilon_{n+1}$ is a good enumeration, the corresponding nice decomposition is

$$w_0 = s_1(s_2s_1)(s_3s_2s_1)(\ldots)(s_ns_{n-1}\ldots s_2s_1),$$

and the corresponding enumeration of the positive roots is:

$$\beta_1 = \epsilon_1 - \epsilon_2;\ \beta_2 = \epsilon_1 - \epsilon_3, \beta_3 = \epsilon_2 - \epsilon_3;\ \beta_4 = \epsilon_1 - \epsilon_4, \ldots, \beta_6 = \epsilon_3 - \epsilon_4;$$
$$\ldots;\ \beta_{N-n+1} = \epsilon_1 - \epsilon_{n+1},\ \ldots\ , \beta_N = \epsilon_n - \epsilon_{n+1};$$

where $N := \frac{1}{2}n(n+1)$. We write $f_i,\ e_i$ for the operators $f_{\alpha_i},\ e_{\alpha_i}$. Let Δ be the triangle consisting of left justified rows of boxes, having n-boxes in the first row, $(n-1)$ in the second, ... For $\underline{a} \in \mathbb{R}^{\frac{1}{2}n(n+1)}$ denote by $\Delta(\underline{a})$ the filling of Δ with the coefficients of $\underline{a}$: row-wise, from the bottom to the top, and in each row from the right to left.

Example. Suppose $n = 4$, for $\underline{a} = (2, 3, 2, 5, 4, 1, 3, 3, 1, 1)$ we write

$$\Delta(\underline{a}) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 1 & 4 & 5 \\ \cline{1-3} 2 & 3 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array}\ .$$

We identify $\underline{a} \in \mathbb{R}^N$ with its triangle $\Delta(\underline{a}) = (a_{i,j})$, where $a_{i,j}$ denotes the j-th entry in the i-th row. Theorem 4.2 implies:

THEOREM 5.1. *C is the cone of triangles $\Delta = (a_{i,j})$ such that the entries are non-negative and weakly increasing in the rows.*

Denote by $s(a_{i,j})$ the sum of the entries above $a_{i,j}$: $s(a_{i,j}) := \sum_{k=1}^{i} a_{k,j}$. We set $a_{i,j} := 0$ for $i + j > n + 1$ or $i = 0$ or $j = 0$, and set $a_{n+1} := 0$. A small weight calculation shows that the polytope C_λ (Proposition 1.5 *b*)) for a dominant weight $\lambda = \lambda_1\epsilon_1 + \ldots + \lambda_n\epsilon_n$ can be described in terms of the triangles as follows:

COROLLARY 1. *C_λ is the convex polytope of all triangles $\Delta(a_{i,j})$ such that the entries in the rows are non-negative and weakly increasing, and*

$$a_{i,j} \leq \lambda_j - \lambda_{j+1} + s(a_{i,j-1}) - 2s(a_{i-1,j}) + s(a_{i-1,j+1})\ \forall 1 \leq i \leq n,\ 1 \leq j \leq n+1-j.$$

Another way to describe the convex polytope C_λ is the following. Recall that a triangular pattern $P = (g_{i,j}) \in \mathbb{R}^{N+n+1}$, $0 \leq i \leq n$, $1 \leq j \leq n+1-i$, is called a *Gelfand-Tsetlin pattern* of type λ if the entries are non-negative and weakly decreasing in the rows, the first row is given by the coordinates of λ (i.e., $\lambda_1 = g_{0,1}, \lambda_2 = g_{0,2}, \ldots$), and the entries satisfy the "betweenness" condition:

$$g_{i,j} \geq g_{i+1,j} \geq g_{i,j+1} \quad \forall 0 \leq i \leq n,\ 1 \leq j \leq n-i+1.$$

Example. A Gelfand-Tsetlin pattern of type $\lambda = 9\epsilon_1 + 7\epsilon_2 + 5\epsilon_3 + 2\epsilon_4 + \epsilon_5$:

$$P = \begin{array}{ccccccccc} 9 & & 7 & & 5 & & 2 & & 1 \\ & 8 & & 7 & & 3 & & 2 & \\ & & 7 & & 4 & & 2 & & \\ & & & 5 & & 3 & & & \\ & & & & 3 & & & & \end{array} .$$

Let $GT(\lambda)$ be the convex polytope of Gelfand-Tsetlin patterns of type λ. Let $\mathfrak{l}_j$ be the Levi subalgebra associated to the simple roots $\alpha_1, \ldots, \alpha_j$, and let $\mathfrak{l}'_j$ be its derived algebra. For $\underline{a} \in C_\lambda$ and $\Delta := \Delta(\underline{a})$ set

$$\underline{a}^j = (\underbrace{0}_{1}, \underbrace{0, 0}_{2}, \ \ldots \ \underbrace{0, 0, \ldots, 0}_{j}, \underbrace{a_{j+1,1}, \ldots, a_{j+1,j+1}}_{j+1}, \ \ldots \ , \underbrace{a_{n,1}, \ldots, a_{n,n}}_{n})$$

The defining equations of C_λ show immediately that $\underline{a}^j \in C_\lambda$. Remember that integral elements of this form correspond to those paths $f^{\underline{a}^j}\pi_\lambda$ such that $e_{\alpha_r}(f^{\underline{a}^j}\pi_\lambda) = 0$ for $j = 1, \ldots, j$. It follows that the integral points in C_λ of this form are in one-to-one correspondence with the simple $\mathfrak{l}_j$-submodules of V_λ, the highest weight of the corresponding module is the endpoint of the path $f^{\underline{a}^j}\pi_\lambda$. Fix $\underline{a} \in C_\lambda$ and let $\Delta(\underline{a}) = (a_{i,j})$ be the associated triangle. Denote by $P(\underline{a})$ the triangle defined in the following way. Let $g_{i,j} \in \mathbb{R}$ be given by:

$$\begin{array}{lcl} \lambda & = & g_{0,1}\epsilon_1 + \ldots + g_{0,n}\epsilon_{n+1} \\ \lambda - \sum_{l=1}^{n} a_{1,l}\alpha_l & = & g_{1,1}\epsilon_1 + \ldots + g_{1,n}\epsilon_{n+1} \\ \lambda - \sum_{l=1}^{n} a_{1,l}\alpha_l - \sum_{l=1}^{n-1} a_{2,l}\alpha_l & = & g_{2,1}\epsilon_1 + \ldots + g_{2,n}\epsilon_{n+1} \\ \quad \ldots & = & \quad \ldots \\ \lambda - \sum_{k=1}^{n}\sum_{l=1}^{n+1-k} a_{k,l}\alpha_l & = & g_{n,1}\epsilon_1 + \ldots + g_{n,n}\epsilon_{n+1} \end{array}$$

then $P(\underline{a})$ is the triangle $(g_{i,j})$, where $0 \leq i \leq n$, $1 \leq j \leq n+1-i$. A simple inductive procedure shows that the inequalities in Corollary 5.1 transform under this map into the inequalities defining the Gelfand-Tsetlin patterns:

COROLLARY 2. *The map $\Delta(\underline{a}) \to P(\underline{a})$ induces a bijection between the convex polytope C_λ and the convex polytope $GT(\lambda)$. Further, this map induces a bijection between the integral points in C_λ and the integral points in $GT(\lambda)$.*

Example. Suppose $n = 5$, and let λ be the dominant weight $\lambda = 9\epsilon_1 + 7\epsilon_2 + 5\epsilon_3 + 2\epsilon_4 + \epsilon_5$. The triangle $\Delta(\underline{a})$ associated to $\underline{a} = (2, 3, 2, 5, 4, 1, 3, 3, 1, 1)$ has been presented above. It is easy to see that $\Delta(\underline{a}) \in C_\lambda$. The associated GT-pattern $P(\underline{a})$ is the pattern in the example above.

6. Example: The symplectic and the odd orthogonal Lie algebra

Let $\mathfrak{g} = \mathfrak{sp}_{2m}$ or $\mathfrak{g} = \mathfrak{so}_{2m+1}$. The simple roots are $\alpha_m = \epsilon_1 - \epsilon_2, \ldots, \alpha_2 = \epsilon_{m-1} - \epsilon_m, \alpha_1 = 2\epsilon_m$ respectively $\alpha_1 = \epsilon_m$. (This is different from the usual enumeration in [1] !). The enumeration is *good*, the corresponding nice decomposition is:

$$w_0 = s_1(s_2s_1s_2)(\ldots)(s_{m-1}\ldots s_1\ldots s_{m-1})(s_ms_{m-1}\ldots s_1\ldots s_{m-1}s_m).$$

The corresponding enumeration of the positive roots for $\mathfrak{sp}_{2m}$ is:

$$\beta_1 = 2\epsilon_m;\ \beta_2 = \epsilon_{m-1} + \epsilon_m,\ \beta_3 = 2\epsilon_{m-1},\ \beta_4 = \epsilon_{m-1} - \epsilon_m;\ \ldots;$$
$$\beta_{N-2m+2} = \epsilon_1 + \epsilon_2,\ \ldots, \beta_{N-m+1} = 2\epsilon_1,\ \ldots,\ \beta_N = \epsilon_1 - \epsilon_2.$$

where $N = m^2$, and for $\mathfrak{so}_{2m+1}$ we get the following enumeration:

$$\beta_1 = \epsilon_m;\ \beta_2 = \epsilon_{m-1} + \epsilon_m,\ \beta_3 = \epsilon_{m-1},\ \beta_4 = \epsilon_{m-1} - \epsilon_m;\ \ldots;$$
$$\beta_{N-2m+2} = \epsilon_1 + \epsilon_2,\ \ldots, \beta_{N-m+1} = \epsilon_1,\ \ldots,\ \beta_N = \epsilon_1 - \epsilon_2.$$

We write just $f_i,\ e_i$ for the operators $f_{\alpha_i},\ e_{\alpha_i}$. Let Δ be the triangle consisting of centered rows of boxes, having $(2m-1)$-boxes in the first row, $(2m-3)$ in the second, ..., and 1 box in the bottom row. For $\underline{a} \in \mathbb{R}^{m^2}$ denote by $\Delta(\underline{a})$ the triangle obtained by filling the coefficients of $\underline{a}$ in the boxes of Δ row-wise, from the bottom to the top, and in each row from the left to the right.

Example. Suppose $m = 3$: If $(\underline{a}) = (2, 1, 1, 0, 7, 7, 4, 3, 3)$, then set:

$$\Delta(\underline{a}) = \begin{array}{|c|c|c|c|c|} \hline 7 & 7 & 4 & 3 & 3 \\ \hline & 1 & 1 & 0 & \\ \cline{2-4} & & 2 & & \\ \cline{3-3} \end{array}$$

We will use the following conventions for the triangles:

We identify $\underline{a} \in \mathbb{R}^N$ with its triangle $\Delta(\underline{a}) = (a_{i,j})$, where $a_{i,j}$ denotes the j-th entry in the i-th row. If $1 \leq j \leq m$, then we write sometimes $\overline{j}$ instead of $2m - j$, and we write $\overline{a}_j$ and $\overline{a}_{i,j}$ for a_{2m-j} respectively $a_{i,2m-j}$. If we write $\Delta = (a_{i,j})$, then it is understood that $1 \leq i \leq m$ and $i \leq j \leq \overline{i}$. If an element $a_{i,j}$ occurs in a formula where the pair (i, j) does not satisfy these inequalities, then put $a_{i,j} = 0$. Theorem 4.2 implies:

Theorem 6.1. *For $\mathfrak{g} = \mathfrak{sp}_{2m}$, the cone C of adapted strings is the cone of triangles $\Delta(\underline{a}) = (a_{i,j})$ such that the entries are non-negative and weakly decreasing in the rows. For $\mathfrak{g} = \mathfrak{so}_{2m+1}$, the cone C of adapted strings is the cone of triangles $\Delta(\underline{a}) = (a_{i,j})$ such that the entries are non-negative and*

$$2a_{i,i} \geq \ldots \geq 2a_{i,m-1} \geq a_{i,m} \geq 2a_{i,m+1} \geq \ldots \geq 2a_{i,2m-i}\ \forall\ i = 1, \ldots, m-1.$$

We use the following abbreviations for $j < m$ and $c = 2$ for $\mathfrak{g} = \mathfrak{sp}_{2m}$, $c = 1$ for $\mathfrak{g} = \mathfrak{so}_{2m+1}$):

$$s(\overline{a}_{i,j}) := \overline{a}_{i,j} + \sum_{k=1}^{i-1}(a_{k,j} + \overline{a}_{k,j}),\ s(a_{i,j}) := \sum_{k=1}^{i}(a_{k,j} + \overline{a}_{k,j}),\ s(a_{i,m}) := \sum_{k=1}^{i} c a_{k,m}.$$

As in the case of $\mathfrak{sl}_n$, a simple weight calculations shows (recall that the enumeration of the fundamental weights is different from the one in [1]):

Corollary 1. *Fix a dominant weight $\lambda = \lambda_1\omega_1 + \ldots + \lambda_m\omega_m$. Then C_λ is the convex polytope of all triangles $\Delta(a_{i,j}) \in C$ such that*

$$\begin{aligned} \overline{a}_{i,j} \leq& \lambda_{m-j+1} + s(\overline{a}_{i,j-1}) - 2s(a_{i-1,j}) + s(a_{i-1,j+1}) \\ a_{i,j} \leq& \lambda_{m-j+1} + s(\overline{a}_{i,j-1}) - 2s(\overline{a}_{i,j}) + s(a_{i,j+1}) \\ a_{i,m} \leq& \lambda_1 + ds(\overline{a}_{i,m-1}) - ds(a_{i-1,m}) \end{aligned}$$

where $d = 1$ *for* $\mathfrak{g} = \mathfrak{sp}_{2m}$ *and* $d = 2$ *for* $\mathfrak{g} = \mathfrak{so}_{2m+1}$.

The triangles above can be translated into Gelfand-Tsetlin patterns as in the case of $\mathfrak{sl}_n$. Only, one has to "insert" extra rows. These patterns consist of pairs of rows of decreasing length, and they are usually written in a shape that looks like half a pattern for $\mathfrak{sl}_{2m}$.

Example. Suppose $m = 3$:

9	5	1
6	5	0
5	3	
5	2	
3		
1		

We use the following convention: If we write $P = (y_{i,j}, z_{i,j})$, then it is understood that $1 \leq i \leq m$, $i \leq j \leq m$, and we put $y_{i,j} = z_{i,j} = 0$ else.

Definition. ([2]) A pattern $P = (y_{i,j}, z_{i,j})$ is called a BC–*Gelfand-Tsetlin pattern* if the entries satisfy the "betweenness" condition:

$$y_{i,j} \geq z_{i,j}, z_{i-1,j}, \qquad z_{i,j} \geq y_{i,j+1}, y_{i+1,j+1} \quad \text{and } z_{i,m} \geq 0.$$

The pattern is called of *highest weight* λ if $\lambda = y_{1,1}\epsilon_1 + \ldots + y_{1,m}\epsilon_m$ is a dominant weight for $\mathfrak{g}$. Denote by $GT(\lambda)$ the set of all BC–Gelfand-Tsetlin patterns of highest weight λ.

We associate to an element $\underline{a} \in \mathbb{R}^{m^2}$ a pattern $P(\underline{a}) = (y_{i,j}, z_{i,j})$ of highest weight $\lambda = y_{1,1}\epsilon_1 + \ldots + y_{1,m}\epsilon_m$ as follows: Denote by $\Delta(\underline{a})$ the associated triangle. Then let the $y_{i,j}$ be defined by:

$$y_{i,1}\epsilon_1 + \ldots + y_{i,m}\epsilon_m = \lambda - \sum_{k=1}^{i-1} \big(a_{k,m}\alpha_1 + \sum_{j=k}^{m-1} (a_{k,j} + \overline{a}_{k,j})\alpha_{m-k+1}\big).$$

and let the $z_{i,j}$ be defined by: Set $\lambda(i) := y_{i,i}\epsilon_1 + \ldots + y_{i,m}\epsilon_m$, then

$$z_{i-1,1}\epsilon_1 + \ldots + z_{i,m}\epsilon_m = \lambda(i) - \big(\frac{a_{i,m}}{2}\alpha_1 + \sum_{j=i}^{m-1} \overline{a}_{i,j}\alpha_{m-j+1}\big).$$

Remember that for the pattern $P(\underline{a}) = (y_{i,j}, z_{i,j})$ we need only those with $1 \leq i \leq m$, $i \leq j \leq m$, because the $y_{i+1,j}, z_{i,j}$ are obtained from the $y_{i-1,j}$ by subtracting a linear combination of the α_k, $1 \leq k \leq m - i + 1$. In fact, it is easy to see that one recover $\Delta(\underline{a})$ (and hence $\underline{a}$) from $P(\underline{a})$.

A simple inductive procedure shows that the inequalities in Corollary 6.1 transform under this map into the inequalities defining the G-T patterns:

Corollary 2. *The map* $\Delta(\underline{a}) \rightarrow P(\underline{a})$ *induces a bijection between the convex polytope* C_λ *and the convex polytope* $GT(\lambda)$*. Further, for* $\mathfrak{g} = \mathfrak{sp}_{2m}$*, this map induces a bijection between the integral points in* C_λ *and the integral points in* $GT(\lambda)$*. For* $\mathfrak{g} = \mathfrak{so}_{2m+1}$*, the map induces a bijection between the integral points in* C_λ *and the points in* $GT(\lambda)$ *with the property that all coefficients* *are in* $\mathbb{Z}_{\geq 0} + \frac{1}{2}\mathbb{Z}_{\geq 0}$*, and, except for the* $z_{1,m}, z_{2,m}, \ldots, z_{m,m}$*, all coefficients are either integral or in* $\frac{1}{2} + \mathbb{Z}_{\geq 0}$*.*

Example. Suppose $m = 3$, $\mathfrak{g} = \mathfrak{sp}_{2m}$, and let λ be the dominant weight $\lambda = 9\epsilon_1 + 5\epsilon_2 + \epsilon_3$. The triangle $\Delta(\underline{a})$ associated to $\underline{a} = (2,1,1,0,7,7,4,3,3)$ has been presented above, it is an element of C_λ. The associated GT-pattern $P(\underline{a})$ is the pattern in the example above.

7. Example: The even orthogonal Lie algebra

In this section we consider the orthogonal algebra $\mathfrak{g} = \mathfrak{so}_{2m}$ The simple roots are $\alpha_m = \epsilon_1 - \epsilon_2, \ldots, \alpha_2 = \epsilon_{m-1} - \epsilon_m$ and $\alpha_1 = \epsilon_{m-1} + \epsilon_m$. (This is different from the usual enumeration in [1] !). The enumeration is *good*, the corresponding nice decomposition is:

$$w_0 = s_1 s_2 (s_3 s_1 s_2 s_3)(s_4 s_3 s_1 s_2 s_3 s_4)(\ldots)(s_m \ldots s_4 s_3 s_1 s_2 s_3 s_4 \ldots s_m)$$

The corresponding enumeration of the positive roots for $\mathfrak{so}_{2m}$ is:

$$\begin{array}{l}
\beta_1 = \epsilon_{m-1} + \epsilon_m,\ \beta_2 = \epsilon_{m-1} - \epsilon_m;\ \beta_3 = \epsilon_{m-2} + \epsilon_{m-1},\ \beta_4 = \epsilon_{m-2} - \epsilon_m \\
\beta_5 = \epsilon_{m-2} + \epsilon_m,\ \beta_6 = \epsilon_{m-2} - \epsilon_{m-1}; \quad \ldots \\
\beta_{N-2m+3} = \epsilon_1 + \epsilon_2,\ \ldots, \beta_{N-m} = \epsilon_1 + \epsilon_{m-1},\ \beta_{N-m+1} = \epsilon_1 - \epsilon_m \\
\quad \beta_{N-m+2} = \epsilon_1 + \epsilon_m,\ \beta_{N-m+3} = \epsilon_1 - \epsilon_{m-1}, \ldots, \beta_N = \epsilon_1 - \epsilon_2;
\end{array}$$

where $N = m^2 - m$. We write f_i, e_i for the operators f_{α_i}, e_{α_i}. Let Δ be the triangle consisting of centered rows of boxes, having $(2m-2)$-boxes in the first row, $(2m-4)$ in the second, ..., and 2 boxes in the bottom row. For $\underline{a} \in \mathbb{R}^{m^2-m}$ denote by $\Delta(\underline{a})$ the triangle obtained by filling the coefficients of $\underline{a}$ in the boxes of Δ row-wise, from the bottom to the top, and from the left to the right. Example ($m = 4$): If $(\underline{a}) = (4,2,5,4,3,2,7,5,4,4,3,2)$, then

$$\Delta(\underline{a}) := \begin{array}{|c|c|c|c|c|c|} \hline 7 & 5 & 4 & 4 & 3 & 2 \\ \hline & 5 & 4 & 3 & 2 & \\ \cline{2-5} & & 4 & 2 & & \\ \cline{3-4} \end{array} .$$

We identify $\underline{a} \in \mathbb{R}^N$ with its triangle $\Delta(\underline{a}) = (a_{i,j})$, where $a_{i,j}$ denotes the j-th entry in the i-th row. If we write $\Delta = (a_{i,j})$, then it is understood that $1 \leq i \leq m-1$ and $i \leq j \leq 2m-2$. If $1 \leq j \leq m-1$, then we write sometimes $\bar{j}$ for $2m-1-j$. Theorem 4.2 implies:

THEOREM 7.1. *For* $\mathfrak{g} = \mathfrak{so}_{2m}$*, the cone* C *of adapted strings is the cone of triangles* $\Delta(\underline{a}) = (a_{i,j})$ *such that the* $a_{i,j} \geq 0$ *and for all* $1 \leq i \leq m-1$*:*

$$a_{i,i} \geq \ldots \geq a_{i,m-2} \geq \left\{ \begin{array}{c} a_{i,m-1} \\ a_{i,m} \end{array} \right\} \geq a_{i,m+1} \geq \ldots \geq a_{i,2m-i-1}.$$

With the following abbreviations ($j < m-1$)we get as before:

$$s(\overline{a}_{i,j}) := \overline{a}_{i,j} + \sum_{k=1}^{i-1}(a_{k,j}+\overline{a}_{k,j}),\ \ s(a_{i,j}) := \sum_{k=1}^{i}(a_{k,j}+\overline{a}_{k,j}),$$

$$s(a_{i,m-1}) = s(\overline{a}_{i,m-1}) := \sum_{k=1}^{i}(a_{k,m-1}+a_{k,m})$$

$$t(a_{i,m-1}) := \sum_{k=1}^{i} a_{k,m-1},\ \ t(a_{i,m}) := \sum_{k=1}^{i} a_{k,m}.$$

Corollary 1. *Fix a dominant weight $\lambda = \lambda_1\omega_1 + \ldots + \lambda_m\omega_m$. Then C_λ is the convex polytope of all triangles $\Delta(a_{i,j}) \in C$ such that*

$$\begin{aligned}
\overline{a}_{i,j} &\le \lambda_{m-j+1} + s(\overline{a}_{i,j-1}) - 2s(a_{i-1,j}) + s(a_{i-1,j+1}) \quad \text{for } \ j \le m-2;\\
a_{i,j} &\le \lambda_{m-j+1} + s(a_{i,j+1}) - 2s(\overline{a}_{i,j}) + s(\overline{a}_{i,j-1}) \quad \text{for } \ j \le m-2;\\
a_{i,m-1} &\le \lambda_2 + s(\overline{a}_{i,m-2}) - 2t(a_{i-1,m-1})\\
a_{i,m} &\le \lambda_1 + s(\overline{a}_{i,m-2}) - 2t(a_{i-1,m})
\end{aligned}$$

Definition. ([2]) A pattern $P = (y_{i,j}, z_{k,l})$, $1 \le i \le m, i \le j \le m$ and $1 \le k \le m-1, i \le l \le m-1$ is called a D*–Gelfand-Tsetlin pattern* if the entries satisfy the "betweenness" condition: $y_{1,j} \ge z_{1,j}$ and $y_{i,j} \ge z_{i,j}, z_{i-1,j}$ for $2 \le i \le m-1$, $z_{i,j} \ge y_{i,j+1}, y_{i+1,j+1}$ for $1 \le i \le m-1$, and

$$\begin{cases} y_{i,m} + y_{i+1,m} + \min\{y_{i,m-1}, y_{i+1,m-1}\} \ge z_{i,m-1} & \text{if } i < m-1,\\ y_{m-1,m} + y_{m,m} + y_{m-1,m-1} \ge z_{m-1,m-1}. \end{cases}$$

The pattern is called of *highest weight* λ if $\lambda = y_{1,1}\epsilon_1 + \ldots + y_{1,m}\epsilon_m$ is a dominant weight for $\mathfrak{g}$. Denote by $GT(\lambda)$ the set of all D–Gelfand-Tsetlin patterns of highest weight λ.

We associate to an element $\underline{a} \in \mathbb{R}^{m^2-m}$ a pattern $P(\underline{a}) = (y_{i,j}, z_{i,j})$ of highest weight $\lambda = y_{1,1}\epsilon_1 + \ldots + y_{1,m}\epsilon_m$ as follows: Denote by $\Delta(\underline{a})$ the associated triangle. Then let the $y_{i,j}$ be defined by:

$$y_{i,1}\epsilon_1 + \ldots + y_{i,m}\epsilon_m = \lambda - \sum_{k=1}^{i-1}\big(a_{k,m}\alpha_1 + a_{k,m-1}\alpha_2 + \sum_{j=k}^{m-2}(a_{k,j}+\overline{a}_{k,j})\alpha_{m-k+1}\big),$$

and let the $z_{i,j}$ be defined by: $z_{i,i} = y_{i,i} - \overline{a}_{i,i}$, $z_{i,j} = y_{i,j} + \overline{a}_{i,j-1} - \overline{a}_{i,j}$ for $i+1 \le j \le m-2$, and $z_{i,m-1} = y_{i,m} + \min\{a_{i,m-2} - a_{i,m-1}, a_{i,m} - a_{i,m+1}\}$ for $i < m-1$.

It is easy to see that one can recover $\Delta(\underline{a})$ (and hence $\underline{a}$) from $P(\underline{a})$. A simple inductive procedure shows that the inequalities in Corollary 7.1 transform under this map into the inequalities defining the patterns:

Corollary 2. *The map $\Delta(\underline{a}) \to P(\underline{a})$ induces a bijection between the convex polytope C_λ and the convex polytope $GT(\lambda)$. Further, the map induces a bijection between the integral points in C_λ and the points in $GT(\lambda)$ with the property that either all coefficients are integral, or all are in $\frac{1}{2} + \mathbb{Z}$.*

Example. Suppose $m = 4$, let λ be the dominant weight $\lambda = 9\epsilon_1 + 7\epsilon_2 + 5\epsilon_3 + 2\epsilon_4$. The triangle $\Delta(\underline{a})$ associated to $\underline{a} = (1,1,3,1,2,1,3,2,2,1,1,1)$ is an element of C_λ, the associated GT-pattern $P(\underline{a})$ is:

$$\Delta(\underline{a}) = \begin{array}{cccccc} 3 & 2 & 2 & 1 & 1 & 1 \\ & 3 & 1 & 2 & 1 & \\ & & 1 & 1 & & \end{array}, \quad P(\underline{a}) = \begin{array}{cccc} 9 & 7 & 5 & 2 \\ & 8 & 7 & 6 \\ & 8 & 5 & 1 \\ & & 7 & 2 \\ & & 6 & 2 \\ & & & 5 \\ & & & 2 \end{array}$$

Remark 7.2. The bijection above induces a canonical bijection between the Lakshmibai-Seshadri tableaux for classical groups and the Gelfand-Tsetlin patterns for classical groups. In particular, it defines for $\mathfrak{g} = \mathfrak{sp}_{2m}$ a canonical bijection between the Lakshmibai-Seshadri tableaux and the King tableaux (a tableau oriented version of the Gelfand-Tsetlin patterns). (See [14] for the connection of the Lakshmibai-Seshadri tableaux with the crystal base, or, more generally, combine [16] with the isomorphism of graphs proved in [9].)

8. Example: $\mathtt{E}_6$ and $\mathtt{E}_7$

The following enumerations of the roots for the root system of type $\mathtt{E}_6$ respectively $\mathtt{E}_7$ are a "good" enumeration:

$$\begin{array}{ccccccccc} & & & & \alpha_1 & & & & \\ & & & & | & & & & \\ \alpha_5 & - & \alpha_4 & - & \alpha_3 & - & \alpha_2 & - & \alpha_6 \end{array} \qquad \begin{array}{ccccccccccc} & & & & \alpha_1 & & & & & & \\ & & & & | & & & & & & \\ \alpha_5 & - & \alpha_4 & - & \alpha_3 & - & \alpha_2 & - & \alpha_6 & - & \alpha_7. \end{array}$$

The corresponding nice decompositions of w_0 are:

$$\mathtt{E}_6: \quad \tau \cdot s_6 s_2 s_3 s_1 s_4 s_5 s_3 s_4 s_2 s_3 s_1 s_6 s_2 s_3 s_4 s_5,$$
$$\mathtt{E}_7: \quad \sigma \cdot s_7 s_6 s_2 s_3 s_1 s_4 s_5 s_3 s_4 s_2 s_3 s_1 s_6 s_2 s_3 s_4 s_5 s_7 s_6 s_2 s_3 s_1 s_4 s_3 s_2 s_6 s_7,$$

here τ respectively σ are the longest words in the Weyl group of the Levi subgroups of type $\mathtt{D}_5$ respectively $\mathtt{E}_6$. We describe in the following only the cone C_n, to get the full cone one has just to add the corresponding cone for $\mathtt{D}_5$ respectively $\mathtt{E}_6$ (see the proof of Theorem 4.2). A small calculation provides by Theorem 4.2 the following descriptions of the cones of adapted strings:

THEOREM 8.1. *For $\mathfrak{g} = \mathtt{E}_6$, the cone $C_6 \subset \mathbb{R}^{16}$ is given by the following inequalities:*

$$a_1 \geq a_2 \geq a_3 \geq \left\{ \begin{array}{c} a_4 \\ a_5 \end{array} \right\} \geq a_7 \geq \left\{ \begin{array}{c} a_8 \\ a_9 \end{array} \right\} \geq a_{10} \geq \left\{ \begin{array}{c} a_{11} \\ a_{13} \end{array} \right\} \geq a_{14} \geq a_{15} \geq a_{16};$$
$$a_5 \geq a_6 \geq a_8; \;\; a_9 \geq a_{12} \geq a_{13}.$$

THEOREM 8.2. *For $\mathfrak{g} = \mathtt{E}_7$, the cone $C_7 \subset \mathbb{R}^{27}$ is given by the following inequalities:*

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq \left\{ \begin{matrix} a_5 \\ a_6 \end{matrix} \right\} \geq a_8 \geq \left\{ \begin{matrix} a_9 \\ a_{10} \end{matrix} \right\} \geq a_{11} \geq \left\{ \begin{matrix} a_{12} \\ a_{14} \end{matrix} \right\} \geq$$

$$\geq a_{15} \geq \left\{ \begin{matrix} a_{16} \\ a_{20} \end{matrix} \right\} \geq a_{21} \geq \left\{ \begin{matrix} a_{22} \\ a_{23} \end{matrix} \right\} \geq a_{24} \geq a_{25} \geq a_{26} \geq a_{27};$$

$$a_6 \geq a_7 \geq a_8; \quad a_{10} \geq a_{13} \geq \left\{ \begin{matrix} a_{14} \\ a_{18} \end{matrix} \right\} \geq a_{19} \geq a_{20}; \quad a_{16} \geq a_{17} \geq a_{23}.$$

9. Example: A cone for $\mathtt{F}_4$ and $\mathtt{E}_8$

We fix the following enumerations of the roots for the root system of type $\mathtt{F}_4$ respectively $\mathtt{E}_8$:

$$\alpha_3 - \alpha_2 => \alpha_1 - \alpha_4 \qquad\qquad \begin{matrix} & & & \alpha_1 & & & \\ & & & | & & & \\ \alpha_5 - & \alpha_4 - & \alpha_3 - & \alpha_2 & - \alpha_6 & - \alpha_7 & - \alpha_8. \end{matrix}$$

and we fix the following decompositions of w_0:

$\mathtt{F}_4 :\ \tau \cdot s_4 s_1 s_2 s_3 s_1 s_2 s_1 s_4 s_1 s_2 s_3 s_1 s_2 s_1 s_4,$

$\mathtt{E}_8 :\ \sigma \cdot s_8 s_7 s_6 s_2 s_3 s_1 s_4 s_5 s_3 s_4 s_2 s_3 s_1 s_6 s_2 s_3 s_4 s_5 s_7 s_6 s_2 s_3 s_1 s_4 s_3 s_2 s_6 s_7 s_8$
$s_7 s_6 s_2 s_3 s_1 s_4 s_5 s_3 s_4 s_2 s_3 s_1 s_6 s_2 s_3 s_4 s_5 s_7 s_6 s_2 s_3 s_1 s_4 s_3 s_2 s_6 s_7 s_8.$

Here τ respectively σ are the longest words in the Weyl group of the Levi subgroups of type $\mathtt{B}_3$ respectively $\mathtt{E}_7$. We describe in the following only the cone C_n, to get the full cone one has just to add the corresponding cone for $\mathtt{B}_3$ respectively $\mathtt{E}_7$. Note that the enumeration of the simple roots for the subdiagrams of the Levi subalgebras is a good enumeration.

THEOREM 9.1. *For $\mathfrak{g} = \mathtt{F}_4$, the cone $C_4 \subset \mathbb{R}^{15}$ is given by the following inequalities: $a_i \geq 0$ for $i = 1, \ldots, 15$, and $a_5 \geq a_9$; $a_7 \geq a_{12}$;*

$$a_1 \geq a_2 \geq a_3 \geq \left\{ \begin{matrix} a_4 \\ a_5 \end{matrix} \right\} \geq a_6 \geq a_7; \quad a_9 \geq a_{10} \geq \left\{ \begin{matrix} a_{11} \\ a_{12} \end{matrix} \right\} \geq a_{13} \geq a_{14} \geq a_{15};$$

$$a_5 + a_7 \geq a_8 \geq a_9 + a_{12}; \quad 2a_6 \geq a_7 + a_9 \geq 2a_{10}.$$

Remark 9.2. Though the decomposition above is not "nice", the algorithm developed in section 4 yields the first inequalities. To get the full set of linear inequalities defining the cone one has just to add the last 4 inequalities.

Proof. Note first that the decomposition is such that $w_0 = \tau s_4 \xi s_4 \xi s_4$, where ξ is such that $s_3 s_2 s_3 \xi$ is a nice decomposition of the longest word of the Weyl group of $\mathtt{B}_3$ (the nice decomposition is the one associated to the enumeration $\alpha_1 - \alpha_2 => \alpha_3$!). If $\underline{a} \in C_4$, then we have obviously $e_1(a_8, \ldots) = 0$. Using the "braid relations" (Propositions 2.3, 2.4), it is easy to see that we have also $e_2(a_8, \ldots) = e_3(a_8, \ldots) = 0$ as necessary conditions. Since α_2, α_3 commute with α_4, the latter conditions are equivalent to $e_2(a_7,\ldots) = e_3(a_7,\ldots) = 0$. Now we can apply Theorem 4.2 for the case $\mathfrak{g} = \mathtt{B}_3$ to get the necessary and sufficient conditions for $a_7,\ldots,a_{14}$. Again by Theorem 4.2, the conditions

$$e_1(a_8,\ldots) = e_2(a_8,\ldots) = e_3(a_8,\ldots) = 0 \text{ and } e_1\underline{a} = e_2\underline{a} = e_3\underline{a} = 0$$

imply that the same inequalities have to be satisfied by $a_2,\ldots,a_7$.

It remains to get the necessary and sufficient conditions for the remaining coefficients. The first necessary condition is obviously $e_4(a_9,\ldots) = 0$, we will come back to this one later. The next condition is $e_1(a_8,\ldots) = 0$. Since $e_4(a_{10},\ldots) = e_1(a_{10},\ldots) = 0$, this is (by section 2) equivalent to $a_8 \geq a_9$.

Consider the conditions $e_4(a_2,\ldots) = 0$ and $e_1(a_1,\ldots) = 0$. The latter is equivalent (section 2) to $a_1 \geq a_2$ and $e_4(a_3,\ldots) = 0$, so it is sufficient to consider the condition $e_4(a_3,\ldots) = 0$. Now α_4 is orthogonal to α_3,α_2, so the latter is equivalent to $e_4(a_5,\ldots) = 0$. If we set (braid relations, section 2)

$$\underline{n} = (a_1,a_2,a_3,a_4,a_5,\max\{a_9,a_8-a_7\},a_6,a_7+a_9,\min\{a_8-a_9,a_7\},\ldots)$$

to be the sequence associated to $s_4s_1s_2s_3s_1s_4s_2s_1s_4s_2s_3s_1s_2s_1s_4$, then we see that $e_4(a_5,\ldots) = 0$ is equivalent to $a_5 \geq \max\{a_9,a_8-a_7\}$ and

$$e_1(n_6,\ldots) = e_1(a_6,a_7+a_9,\min\{a_8-a_9,a_7\},a_{10},a_{11},a_{12},a_{13},a_{14},a_{15}) = 0.$$

After exchanging s_4 and s_2 (the 9th and 10th reflection) in the expression above, this condition is (by section 2) equivalent to $2a_6 \geq a_7 + a_9 \geq 2a_{10}$ and

$$e_1(\min\{a_8-a_9,a_7\},a_{11},a_{12},a_{13},a_{14},a_{15}) = 0$$

(the corresponding reduced expression is $s_4s_3s_1s_2s_1s_4$). After exchanging the commuting reflections s_3 and s_1, we see (by section 2) that this condition is equivalent to $\min\{a_8-a_9,a_7\} \geq a_{12}$ and $e_4(a_{11},a_{13},a_{14},a_{15}) = 0$ (the corresponding reduced expression is $s_3s_2s_1s_4$). Now the latter condition is equivalent to $a_{14} \geq a_{15}$.

Since $e_4(a_{11},a_{13},a_{14},a_{15}) = 0$ implies $e_4(a_9,\ldots) = 0$, the first condition is also satisfied if $a_{14} \geq a_{15}$. Hence the inequalities are necessary and sufficient conditions for $\underline{a}$ to be an adapted string. □

The same arguments (only this time ξ is part of a nice decomposition of the longest element for W of type $\mathtt{E}_7$, see the section before) give the following description of the cone C_8 for $\mathfrak{g}$ of type $\mathtt{E}_8$:

Theorem 9.3. *For $\mathfrak{g} = \mathtt{E}_8$, the cone $C_8 \subset \mathbb{R}^{57}$ is given by the following inequalities: The elements $(a_2,\ldots,a_{28})$ and $(a_{30},\ldots,a_{56})$ satisfy the inequalities for $\mathtt{E}_7$ in Theorem 8.2 and $a_1 \geq a_2$; $a_{29} \geq a_{30}$; $a_{56} \geq a_{57}$; and*

$$\begin{array}{ll}
\alpha_{19} \geq \max\{a_{30},a_{29}-a_{28}\}; & \min\{a_{23},a_{25}+a_{33}-a_{34}\} \geq a_{35} \\
\alpha_{20} \geq \max\{a_{31},a_{28}+a_{30}-a_{27}\}; & \min\{a_{25},a_{26}+a_{32}-a_{33}\} \geq a_{37}; \\
\alpha_{21} \geq \max\{a_{32},a_{27}+a_{31}-a_{26}\}; & \min\{a_{26},a_{27}+a_{31}-a_{32}\} \geq \alpha_{39}; \\
\alpha_{22} \geq \max\{a_{33},a_{26}+a_{32}-a_{25}\}; & \min\{a_{27},a_{28}+a_{30}-a_{31}\} \geq \alpha_{42}; \\
\alpha_{24} \geq \max\{a_{34},a_{25}+a_{33}-a_{23}\}; & \min\{a_{28},a_{29}-a_{30}\} \geq \alpha_{47}.
\end{array}$$

Remark 9.4. Again, the decomposition above is not "nice". But if one just applies the algorithm developed in section 4, one sees that the list of inequalities obtained by the algorithm coincides nearly with the list given above, one has just to add the last ten inequalities to get the full list. This second coincidence suggests that for any reduced decomposition of w_0, which is compatible with a maximal decreasing sequence of Levi subgroups, the set of inequalities given by the algorithm in section 4 should be a subset of the set of inequalities defining the cone of adapted strings.

10. Cones and bases for representations

Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the usual triangular decomposition of a symmetrizable Kac-Moody algebra $\mathfrak{g}$ such that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Fix a reduced decomposition $w = s_{i_1} \dots s_{i_p}$ and let $\mathcal{S}_w$ be the corresponding set of adapted strings. In this section we associate to the points in $\mathcal{S}_w^\lambda$ a basis of the submodule $V_\lambda(w)$.

The following theorem and its proof is a variation of a the theorem of Lakshmibai [11]. Note that if the root system is simply laced and the decomposition of w_0 is "nice" in the sense of section 4, then the same arguments as in [17] give an elementary proof of the theorem below.

We fix a basis of $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ and $\alpha^\vee \in \mathfrak{h}$ for α simple such that $[E_\alpha, F_\alpha] = \alpha^\vee$. Let $U_\mathbb{Z}, U_\mathbb{Z}^+$ and $U_\mathbb{Z}^-$ be the corresponding $\mathbb{Z}$-forms of the enveloping algebras $U(\mathfrak{g})$, $U(\mathfrak{n}^+)$ respectively $U(\mathfrak{n}^-)$.

For a simple root α_j we write sometimes just F_j instead of F_{α_j}. For $\underline{a} \in \mathcal{S}_w$ denote by $F^{\underline{a}}$ the element

$$F^{\underline{a}} := \frac{F_{i_1}^{a_1}}{a_1!} \cdots \frac{F_{i_p}^{a_p}}{a_p!} \in U(\mathfrak{n}^-).$$

Corresponding to the chosen decomposition let $U_\mathbb{Z}^-(w) \subset U_\mathbb{Z}^-$ be the submodule spanned by these monomials of the form $F^{\underline{a}}$. Note that $U_\mathbb{Z}^-(w_0) = U_\mathbb{Z}^-$ in the finite dimensional case for the longest word w_0 in the Weyl group.

Theorem 10.1.
i) The elements $\{F^{\underline{a}} \mid \underline{a} \in \mathcal{S}_w\}$ form a basis of $U_\mathbb{Z}^-(w)$.
ii) Let $v_\lambda \in V_\lambda$ be a highest weight vector and denote by $V_{\mathbb{Z},\lambda}$ the Kostant lattice $U_\mathbb{Z}^- v_\lambda$. The vectors $\{F^{\underline{n}} v_\lambda \mid \underline{n} \in \mathcal{S}_w^\lambda\}$ form a basis of the Demazure module $V_{\mathbb{Z},\lambda}(w) := V_\lambda(w) \cap V_{\mathbb{Z},\lambda}$.

The theorem follows by specialization at $q = 1$ from the theorem below, which is a "quantum version" of the statement above. Let U_q be the corresponding q-analog of U, by abuse of notation we use the same notation: F_j, E_j, K_j, K_j^{-1} for the generators. Let A be the ring $\mathbb{Z}[q, q^{-1}]$ of Laurent polynomials, and let U_A, U_A^+ and U_A^- be the corresponding A-forms. For a dominant weight $\lambda \in X^+$ denote by V_λ the corresponding irreducible representation of U_q (over $\mathbb{Q}(q)$). Fix a highest weight vector v_λ, and denote by $V_{A,\lambda}$ the lattice $U_A^- v_\lambda \subset V_\lambda$.

As above, for $\underline{a} \in \mathcal{S}_w$ denote by $F^{\underline{a}}$ the monomial

$$F^{\underline{a}} := \frac{F_{i_1}^{a_1}}{[a_1]!} \cdots \frac{F_{i_p}^{a_p}}{[a_p]!} \in U_\mathcal{A}^-.$$

where $[a] := (q^a - q^{-a})/(q - q^{-1})$. The notion of a Demazure module etc. generalizes in the obvious way.

THEOREM 10.2.
i) The elements $\{F^{\underline{a}} \mid \underline{a} \in \mathcal{S}_w\}$ *form an A-basis of* $U_A^-(w)$.
ii) The elements $\{F^{\underline{a}} v_\lambda \mid \underline{a} \in \mathcal{S}_w^\lambda\}$ *form an A-basis of the Demazure module* $V_{A,\lambda}(w) := V_\lambda(w) \cap V_{A,\lambda}$.

The theorem is an easy consequence of the following proposition. For $\underline{a} \in \mathcal{S}_w$ let $b_{\underline{a}}$ be the corresponding element in the crystal base $\mathbb{CB}$ of U_q^-, and let $G(b_{\underline{a}}) \in U_A^-$ be the corresponding element of the global crystal basis [8]. Or, in Lusztig's language, let it be the corresponding element of the good basis of U_A^- [20]. Recall that $G(b_{\underline{a}})v_\lambda = 0$ for $\underline{a} \notin \mathcal{S}_w^\lambda$. So if $\underline{a} \in \mathcal{S}_w^\lambda$, then we write also $G(b_{\underline{a}}) \in V_\lambda$ for the corresponding element $G(b_{\underline{a}})v_\lambda$ in the representation space. Let "$\geq$" be the lexicographic ordering on $\mathbb{N}^p$

PROPOSITION 10.3. *Suppose* $\underline{a} \in \mathcal{S}_w^\lambda$. *Then*

$$F^{\underline{a}} v_\lambda = G(b_{\underline{a}}) + \sum_{\substack{\underline{a} < \underline{a}' \\ \underline{a}' \in \mathcal{S}_w^\lambda}} c(\underline{a}, \underline{a}') G(b_{\underline{a}'}),$$

where the coefficients are elements in A.

Proof of the theorem. Kashiwara [7] has proved that the elements $G(b_{\underline{a}})$, $\underline{a} \in \mathcal{S}_w^\lambda$, form a $\mathbb{Q}(q)$-basis of $V_\lambda(w)$, and Hansen [5] has proved that they form in fact an A-basis. The proposition above implies that if we express the $F^{\underline{a}} v_\lambda$ as linear combinations in the $G(b_{\underline{a}})$, then the corresponding matrix is unipotent, and the entries off the diagonal are in A. It follows that the elements $F^{\underline{a}} v_\lambda$, $\underline{a} \in \mathcal{S}_w^\lambda$, form an A-basis of $V_{A,\lambda}(w)$, which proves part *ii)*. The first part is now simple consequence of this by passing to the limit "$\lambda \to \infty$". □

Proof of the proposition. We proceed by induction on $l(w)$. If $l(w) = 0$, nothing is to prove. Suppose now $l(w) \geq 1$, and assume the proposition is true for $w' = s_{i_2} \dots s_{i_p}$. Choose $\underline{a} \in \mathcal{S}_w^\lambda$, $\underline{a} = (a_1, \dots, a_p)$. Note that $\underline{a}' := (a_2, \dots, a_p) \in \mathcal{S}_{w'}^\lambda$. So by induction we can write:

$$F^{\underline{a}'} v_\lambda = G(b_{\underline{a}'}) + \sum_{\substack{\underline{a}' < \underline{m}' \\ \underline{m} \in \mathcal{S}_{w'}^\lambda}} c(\underline{a}', \underline{m}') G(b_{\underline{m}'}).$$

Suppose first $a_1 = 0$, so $F^{\underline{a}} v_\lambda = F^{\underline{a}'} v_\lambda$. Let $\underline{m}' = (m_2, \dots, m_p) \in \mathcal{S}_{w'}^\lambda$ be such that $c(\underline{a}', \underline{m}') \neq 0$. Note that $\underline{m}' \in \mathcal{S}_{w'}^\lambda$ does not imply $(0, m_2, \dots, m_p) \in \mathcal{S}_w^\lambda$. Set $\eta := f_{i_2}^{m_2} \dots f_{i_p}^{m_p} \pi_\lambda$.

If $\underline{m} := (0, m_2, \dots, m_p) \in \mathcal{S}_w^\lambda$, then $G(b_{\underline{m}'}) = G(b_{\underline{m}})$ and $\underline{m} > \underline{a}$ (because $\underline{m}' > \underline{a}'$).

Suppose $(0, m_2, \dots, m_p) \notin \mathcal{S}_w^\lambda$, then the adapted string of η (with respect to the decomposition of w) is of the form $\underline{m} = (m_1, k_2, \dots, k_p)$ with $m_1 > 0$. In particular, $\underline{m} > \underline{a}$. Since $G(b_{\underline{m}'}) = G(b_{\underline{m}})$, the equation above can be rewritten as

$$F^{\underline{a}} v_\lambda = G(b_{\underline{a}}) + \sum_{\substack{\underline{m} > \underline{a} \\ \underline{m} \in \mathcal{S}_w^\lambda}} c(\underline{a}, \underline{m}) G(b_{\underline{m}}).$$

Suppose now $a_1 > 0$, set $\underline{a}' = (0, a_2, \ldots, a_p)$. By the discussion above we get:

$$F^{\underline{a}} v_\lambda = \frac{F_{i_1}^{a_1}}{[a_1]!} F^{\underline{a}'} v_\lambda = \frac{F_{i_1}^{a_1}}{[a_1]!} G(b_{\underline{a}'}) + \sum_{\substack{\underline{m}' > \underline{a}' \\ \underline{m}' \in \mathcal{S}_w^\lambda}} c(\underline{a}', \underline{m}') \frac{F_{i_1}^{a_1}}{[a_1]!} G(b_{\underline{m}'})$$

By [8] we know that

$$\frac{F_{i_1}^{a_1}}{[a_1]!} G(b_{\underline{m}'}) = c(a_1, \underline{m}') G(b_{(m_1'+a_1, m_2', \ldots, m_p')}) + \sum_{\substack{\underline{p} \in \mathcal{S}_w^\lambda \\ p_1 > m_1' + a_1}} c(a_1, \underline{p}) G(b_{\underline{p}}),$$

where the coefficient $c(a_1, \underline{m}') = 1$ if $m_1' = 0$, and all coefficients are elements of A [8], [5]. Now $p_1 > a_1$ implies $\underline{p} > \underline{a}$, and if $m_1' \geq 1$, then $m_1' + a_1 > a_1$ and all terms in the sum above are $> \underline{a}$. If $m_1' = 0$, then recall that $m' > \underline{a}'$, which implies $(a_1, m_2', \ldots, m_p') > \underline{a}$. So the equation above yields:

$$F^{\underline{a}} v_\lambda = G(b_{\underline{a}}) + \sum_{\substack{\underline{p} > \underline{a} \\ \underline{p} \in \mathcal{S}_w^\lambda}} c(\underline{a}, \underline{p}) G(b_{\underline{p}}).$$

□

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