JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 11, Number 3, July 1998, Pages 551–567
S 0894-0347(98)00268-9

CONTRACTING MODULES AND STANDARD MONOMIAL THEORY FOR SYMMETRIZABLE KAC-MOODY ALGEBRAS

PETER LITTELMANN

Introduction

Let G be a reductive algebraic group defined over an algebraically closed field k. We fix a Borel subgroup B, and for a dominant weight λ let $\mathcal{L}_\lambda$ be the associated line bundle on the generalized flag variety G/B. In a series of articles, Lakshmibai, Musili and Seshadri initiated a program to construct a basis for the space $H^0(G/B,\mathcal{L}_\lambda)$ with some particularly nice geometric properties. The purpose of the program is to extend the Hodge-Young standard monomial theory for the group $SL(n)$ to the case of any semisimple algebraic group and, more generally, to Kac-Moody algebras. We refer to [3], [7], [10], [14] for a survey of the subject and applications.

We provide a new approach which completes the program and which avoids the case by case considerations of the earlier articles. In fact, the method works for all symmetrizable Kac-Moody algebras. The most important tools we need in our approach are the combinatorial language of the path model of a representation [11], [12], and quantum groups at a root of unity. Let $U_{\mathbf{v}}(\mathfrak{g})$ be the quantum group associated to G at an ℓ-th root of unity $\mathbf{v}$. We use the quantum Frobenius map [15] to "contract" certain $U_{\mathbf{v}}(\mathfrak{g})$-modules so that they become G-modules. The corresponding map between the dual spaces can be seen as a kind of splitting of the power map $H^0(G/B,\mathcal{L}_\lambda)\rightarrow H^0(G/B,\mathcal{L}_{\ell\lambda})$, $s\mapsto s^\ell$.

For simplicity let us assume we are in the simply laced case. Let V_λ be the Weyl module of G of highest weight λ, and let M_λ be the corresponding Weyl module of $U_{\mathbf{v}}(\mathfrak{g})$. There is a canonical way to attach a tensor product $b_\pi := b_{\nu_1}\otimes\ldots\otimes b_{\nu_\ell}$ of extremal weight vectors $b_{\nu_j}\in M_\lambda^*$ to each L-S path π of shape λ [11] for an appropriate ℓ (recall that an L-S path can be characterized by a collection of extremal weights and rational numbers). To construct a basis of $H^0(G/B,\mathcal{L}_\lambda) = V_\lambda^*$, we use the contraction map to embed V_λ into $(M_\lambda)^{\otimes\ell}$. Denote by p_π the image of b_π in V_λ^* under the dual map $(M_\lambda^*)^{\otimes\ell}\rightarrow V_\lambda^*$.

We show that the vectors p_π, π an L-S path of shape λ, form a basis of V_λ^*. Further, the ℓ-th power $p_\pi^\ell\in H^0(G/B,\mathcal{L}_{\ell\lambda})$ is a product of extremal weight vectors $p_{\nu_1}\cdots p_{\nu_\ell}$, $p_{\nu_i}\in H^0(G/B,\mathcal{L}_\lambda)$, plus a linear combination of elements which are "bigger" in some partial order.

The basis given by the p_π is compatible with the restriction map $H^0(G/B,\mathcal{L}_\lambda)\rightarrow H^0(X,\mathcal{L}_\lambda)$ to a Schubert variety X, and it has the "standard monomial property".

Received by the editors July 17, 1997.
1991 *Mathematics Subject Classification.* Primary 17B10, 17B67, 20G05, 14M15.
Key words and phrases. Path model, quantum Frobenius map, standard monomial theory.

I.e., for λ, μ dominant, there exists a simple combinatorial rule to choose out of the set of all monomials $p_\pi p_\eta \in H^0(G/B, \mathcal{L}_{\lambda+\mu})$ of basis elements $p_\pi \in H^0(G/B, \mathcal{L}_\lambda)$ and $p_\eta \in H^0(G/B, \mathcal{L}_\mu)$, a subset, the *standard monomials*, which forms a basis of $H^0(G/B, \mathcal{L}_{\lambda+\mu})$.

Combining this with the combinatorial results in [11], [12], we get a new and short proof of the following facts: The restriction map $H^0(G/B, \mathcal{L}_\lambda) \to H^0(X, \mathcal{L}_\lambda)$ is surjective, the multiplication maps $H^0(X, \mathcal{L}_\lambda) \otimes H^0(X, \mathcal{L}_\mu) \to H^0(X, \mathcal{L}_{\lambda+\mu})$ and $S^m H^0(X, \mathcal{L}_\lambda) \to H^0(X, \mathcal{L}_{m\lambda})$ are surjective, $H^i(X, \mathcal{L}_\lambda) = 0$ for $i > 0$, the Demazure character formula holds, and Schubert varieties are projectively normal. Other applications (good filtration, the defining ideal for Schubert varieties, etc.) will be discussed in a subsequent article.

This construction provides the most direct way so far to attach a basis to a path model given by L-S paths or standard monomials of L-S paths. It seems natural to conjecture that it should be possible to generalize the method such that one can associate in a canonical way a basis to all path models of a representation. The basis constructed in [9] can be viewed as a very special case of the more general construction provided in this article. The relation between the path basis and the canonical basis of Kashiwara [5] and Lusztig [15] or the basis constructed in [8] is not clear. However, the properties of the path basis suggest that the transformation matrix should be upper triangular (with respect to the bijection between the crystal graph and the path graph [4], and the partial order on the paths), with roots of unity on the diagonal.

In the first two sections we recall some facts about the path model and quantum groups at a root of unity. In the next three sections we introduce the path vectors and prove the basis theorem. Standard monomials and some special relations are discussed in the sixth and seventh section, and the application to the geometry of Schubert varieties is presented in the last section.

The author wishes to thank C. De Concini, V. Lakshmibai, O. Mathieu, K. N. Raghavan and A. Ram for helpful and interesting discussions, and the Newton Institute in Cambridge for its hospitality.

1. Some notation

Let X be the weight lattice of a complex symmetrizable Kac-Moody algebra $\mathfrak{g}$. For a dominant weight $\lambda \in X^+$ denote by V_λ the corresponding irreducible complex representation. Recall that the character of V_λ can be combinatorially described by the path model [12]. Denote by Π the set of all piecewise linear paths (modulo reparameterization) in $X_\mathbb{R} := X \otimes_\mathbb{Z} \mathbb{R}$ starting in the origin and ending in an integral weight, and let Π^+ be the subset of paths having its image in the dominant Weyl chamber.

Fix a path $\pi \in \Pi^+$ ending in λ. The corresponding path model $\mathbb{B}$ is the set of paths obtained from π by applying the root operators f_α, e_α. In particular, the character of V_λ is equal to the (formal) sum $\operatorname{Char} V_\lambda = \sum_{\eta\in\mathbb{B}} e^{\eta(1)}$.

Let $A = (a_{i,j})_{1\le i,j\le n}$ be an indecomposable symmetrizable generalized Cartan matrix and denote by $A^t = (\overline{a}_{i,j})$, $\overline{a}_{i,j} := a_{j,i}$, the transposed matrix. Let $\underline{d} = (d_1, \dots, d_n)$, $d_i \in \mathbb{N}$, be minimal such that the matrix $(d_i a_{i,j})$ is symmetric. We denote by d the smallest common multiple of the d_j, and we set $\overline{d} = (\overline{d}_1, \dots, \overline{d}_n)$, where $\overline{d}_i := d/d_i$. We fix a realization $(\mathfrak{H}, \Delta, \Delta^\vee)$ of A, i.e., $\mathfrak{H}$ is a complex vector space, $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{H}^*$, $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{H}$ are linearly independent vectors such that $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$, and $\dim \mathfrak{H} - n = n - \operatorname{rk} A$. Denote by $\mathfrak{H}'$ the span of the $\alpha_i^\vee$ in $\mathfrak{H}$. Once a splitting $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$ is chosen, the symmetric bilinear form $(\cdot,\cdot)$, defined by the properties

$$(\alpha_i^\vee, h) := \langle \alpha_i, h \rangle / d_i, \quad (h_1, h_2) := 0 \ \forall h_1, h_2 \in \mathfrak{H}'',$$

is uniquely determined and non-degenerate. The form allows us to identify $\mathfrak{H}$ with its dual $\mathfrak{H}^*$. Note that $\alpha_i = d_i \alpha_i^\vee$. Let

$$X := \{\lambda \in \mathfrak{H}^* \mid \forall i : \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}\}$$

be the weight lattice of $\mathfrak{g}$, and denote by $Y \subset \mathfrak{H}$ the co-root lattice generated by the $\alpha_i^\vee$. We can view Y as the lattice in $\mathfrak{H}^*$ generated by the α_i/d_i.

The $\overline{d}_i$ are also minimal with the property that $(\overline{d}_j \overline{a}_{i,j})$ is a symmetric matrix. The triple $(\mathfrak{H}, \Gamma, \Gamma^\vee)$ defined by $\gamma_i := \alpha_i/d_i = \alpha_i^\vee \in \mathfrak{H}^*$ and $\gamma_i^\vee := d_i \alpha_i^\vee = \alpha_i \in \mathfrak{H}$ is a realization of A^t. Let $(\cdot,\cdot)^t$ be the corresponding unique symmetric bilinear form on $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$. Denote by $Y^t \subset \mathfrak{H}$ the co-root lattice generated by the $\gamma_j^\vee$, and let X^t be the dual lattice:

$$X^t = \{\lambda \in \mathfrak{H}^* \mid \forall i : \langle \gamma_i^\vee, \lambda \rangle \in \mathbb{Z}\} = \{\lambda \in \mathfrak{H}^* \mid \forall i : d_i \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}\}.$$

It follows immediately from the definition that $dX^t \subset X \subset X^t$.

2. Contracting modules

Let M_λ be the Weyl module for the quantum group $U_{\mathbf{v}}(\mathfrak{g}^t)$ at a 2ℓ-th root of unity. We show that the subspace $\bigoplus_{\mu \in (\ell/d)X} M_\lambda(\mu)$ admits in a natural way a $\mathfrak{g}$-action. The calculation resembles that in [15], section 35.3, but since we consider the Weyl module M_λ and not a simple module, we have to use different arguments to construct a $\mathfrak{g}$-action. We assume throughout the rest of the article that ℓ is divisible by $2d$.

Let $U_q(\mathfrak{g}^t)$ be the quantum group associated to $\mathfrak{g}^t$ over the field $\mathbb{Q}(q)$, with generators $E_{\gamma_i}, F_{\gamma_i}, K_{\gamma_i}$ and $K_{\gamma_i}^{-1}$. We use the usual abbreviations

$$[n]_i := \frac{q^{\overline{d}_i n} - q^{-\overline{d}_i n}}{q^{\overline{d}_i} - q^{-\overline{d}_i}}, \qquad [n]_i! := [1]_i \cdots [n]_i, \qquad \begin{bmatrix} n \\ m \end{bmatrix}_i := \frac{[n]_i!}{[m]_i![n-m]_i!},$$

where we define the latter to be zero for $n < m$. We will sometimes just write $E_i, K_i, \ldots$ for $E_{\gamma_i}, K_{\gamma_i}, \ldots$. In addition, we use the following abbreviations:

$$q_i := q^{\overline{d}_i} = q^{\frac{(\gamma_i,\gamma_i)^t}{2}}, \qquad \begin{bmatrix} K_i; c \\ p \end{bmatrix} := \prod_{s=1}^{p} \frac{K_i q^{\overline{d}_i(c-s+1)} - K_i^{-1} q^{\overline{d}_i(-c+s-1)}}{q^{\overline{d}_i s} - q^{-\overline{d}_i s}}.$$

Let $U_{q,A}$ be the form of U_q defined over the ring of Laurent polynomials $A := \mathbb{Z}[q, q^{-1}]$. We denote by R the ring A/I, where I is the ideal generated by the 2ℓ-th cyclotomic polynomial, and set $U_{q,R} := U_{q,A} \otimes_A R$.

Similarly, let U_q^+ (respectively U_q^-) be the subalgebra generated by the E_i (F_i), and denote by $U_{q,A}^+$ (respectively $U_{q,A}^-$) the subalgebra of $U_{q,A}$ generated by the divided powers $E_i^{(n)} := \frac{E_i^n}{[n]_i!}$ ($F_i^{(n)} := \frac{F_i^n}{[n]_i!}$). Let $U_{q,R}^+$ be the algebra $U_{q,A}^+ \otimes_A R$, and denote by $U_{q,R}^-$ the algebra $U_{q,A}^- \otimes_A R$.

We use a similar notation for the enveloping algebra $U(\mathfrak{g})$. To distinguish better between the elements of $U(\mathfrak{g})$ and $U_q(\mathfrak{g}^t)$, we denote the generators of $U(\mathfrak{g})$ by $X_\alpha, H_\alpha, Y_\alpha$ or X_i, H_i, Y_i. Let $U = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ defined over

$\mathbb{Q}$, let $U_{\mathbb{Z}}$ be the Kostant-$\mathbb{Z}$-form of U, set $U_R := U_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$, etc. Denote by $\mathbf{v}$ the image of q in R. Set $\ell_i := \frac{\ell d_i}{d}$; then, by the definition of d, ℓ_i is minimal such that

$$\ell_i \frac{(\gamma_i,\gamma_i)^t}{2} = \ell_i \overline{d}_i = \ell_i \frac{d}{d_i} \in \ell\mathbb{Z}.$$

For a dominant weight $\lambda \in X^t$ let M_λ be the simple $U_q(\mathfrak{g}^t)$-module of highest weight λ, and fix an A-lattice $M_{\lambda,A} := U_{q,A} m_\lambda$ in M_λ by choosing a highest weight vector $m_\lambda \in M_\lambda$. Set $M_{\lambda,R} := M_{\lambda,A} \otimes_A R$; then $M_{\lambda,R}$ is a $U_{q,R}$-module such that its character is given by the Weyl-Kac character formula. Consider the weight space decomposition:

$$M_{\lambda,R} = \bigoplus_{\mu \in X^t} M_{\lambda,R}(\mu) \quad \text{and set} \quad M_{\lambda,R}^{\frac{d}{\ell}} := \bigoplus_{\mu \in (\ell/d)X} M_{\lambda,R}(\mu).$$

The subspace $M_{\lambda,R}^{\frac{d}{\ell}}$ is obviously stable under the subalgebra of $U_{q,R}$ generated by the $E_i^{(n\ell_i)}$ and $F_i^{(n\ell_i)}$: If $\mu \in (\ell/d)X$, then so is $\mu \pm n\ell_i\gamma_i = \mu \pm \frac{nd_i\ell}{d}\gamma_i = \mu \pm \frac{n\ell}{d}\alpha_i$.

Theorem 1. *The map*

$$X_i^{(n)} \mapsto E_i^{(n\ell_i)}|_{M_{\lambda,R}^{\frac{d}{\ell}}}, \quad Y_i^{(n)} \mapsto F_i^{(n\ell_i)}|_{M_{\lambda,R}^{\frac{d}{\ell}}}, \quad \binom{H_i+m}{n} \mapsto \begin{bmatrix} K_i; m\ell_i \\ n\ell_i \end{bmatrix}|_{M_{\lambda,R}^{\frac{d}{\ell}}}$$

extends to a representation map $U_R \to \mathrm{End}_R\, M_{\lambda,R}^{\frac{d}{\ell}}$.

Proof. We have to prove that the map is compatible with the Serre relations. For U_R^+ and U_R^-, this is a direct consequence of the higher order quantum Serre relations ([15], Chapter 7). For a detailed proof see [15], section 35.2.3. Since U_R^+ and U_R^- have a presentation by the Serre relations, the assumption 35.1.2 (b) in [15] is not necessary. Remember that ℓ is divisible by $2d$, which implies that 2 divides ℓ_i and hence:

$$\mathbf{v}^{\frac{(\ell_i\gamma_i,\ell_i\gamma_i)^t}{2}} = \mathbf{v}^{\ell_i^2 \frac{(\gamma_i,\gamma_i)^t}{2}} = \mathbf{v}^{\ell_i^2 \overline{d}_i} = \mathbf{v}^{2\ell \frac{\ell_i}{2}} = 1.$$

To prove that also the remaining Serre relations hold, we need the following simple lemma on Gaussian binomial coefficients. I wish to thank Olivier Mathieu who communicated to me this useful way of computing Gaussian binomial coefficients. See also Lemma 34.1.2 in Lusztig's book [15].

Lemma 1. *The following relations hold in R:*

a) $\begin{bmatrix} \ell_i \\ k \end{bmatrix}_i = 0$ *for* $0 < k < \ell_i$.

b) Suppose $m \geq k$, $m = m_1\ell_i + t$ *and* $k = k_1\ell_i + r$, *where* $0 \leq t, r < \ell_i$. *Then*

$$\begin{bmatrix} m \\ k \end{bmatrix}_i = \mathbf{v}^{\overline{d}_i \ell_i (k_1 t - r m_1)} \binom{m_1}{k_1} \begin{bmatrix} t \\ r \end{bmatrix}_i.$$

Proof of Lemma 1. Consider the quantum torus $R[x_{(1,0)}, x_{(0,1)}]$ with multiplication rule $x_{(a,b)}x_{(c,d)} := \mathbf{v}^{\overline{d}_i(ad-bc)} x_{(a+c,b+d)}$. One proves by induction on m:

$$(x_{(1,0)} + x_{(0,1)})^m = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_i x_{(k,m-k)}.$$

Now $a)$ holds because the nominator of $\begin{bmatrix} \ell_i \\ k \end{bmatrix}_i$ is divisible by the 2ℓ-th cyclotomic polynomial, but the denominator is not. As a consequence we get: $(x_{(1,0)} + x_{(0,1)})^{\ell_i} =$

$x_{(\ell_i,0)} + x_{(0,\ell_i)}$. Further, the elements of the form $x_{(a\ell_i,b\ell_i)}$ commute:

$$x_{(a\ell_i,b\ell_i)}x_{(c\ell_i,d\ell_i)} = \mathbf{v}^{\overline{d}_i\ell_i^2(ad-bc)}x_{((a+c)\ell_i,(b+d)\ell_i)} = x_{(c\ell_i,d\ell_i)}x_{(a\ell_i,b\ell_i)},$$

because $\overline{d}_i\ell_i^2 = 2\ell(\ell_i/2)$. Part $b)$ of the lemma is now proved by comparing the coefficients of the expression above with the coefficients of the following expression:

$$\begin{aligned}(x_{(1,0)} + x_{(0,1)})^m &= \big(x_{(\ell_i,0)} + x_{(0,\ell_i)}\big)^{m_1}\big(x_{(1,0)} + x_{(0,1)}\big)^t \\ &= \Big(\sum_{k_1=0}^{m_1}\binom{m_1}{k_i}x_{(k_i\ell_i,(m_1-k_i)\ell_i)}\Big)\Big(\sum_{r=0}^{t}\begin{bmatrix}t\\r\end{bmatrix}_i x_{(r,t-r)}\Big).\end{aligned}$$

□

(*Proof of Theorem 1, continuation*) We get for a weight vector m_μ, $\mu = \frac{\ell}{d}\mu'$, $\mu' \in X$:

$$\begin{bmatrix}K_i;k\ell_i\\n\ell_i\end{bmatrix}_i m_\mu = \begin{bmatrix}\langle\gamma_i^\vee,\mu\rangle + k\ell_i\\n\ell_i\end{bmatrix}_i m_\mu = \begin{bmatrix}\ell_i\langle\alpha_i^\vee,\mu'\rangle + k\ell_i\\n\ell_i\end{bmatrix}_i m_\mu = \binom{\mu'(H_i)+k}{n}m_\mu.$$

The $\binom{H_i+k}{n}$ act hence with the desired eigenvalues. It remains to prove the Serre relations involving the X_i and Y_i. Let m_μ be a weight vector of weight $\mu = \frac{\ell}{d}\mu'$:

$$\begin{aligned}X_i^{(n)}Y_i^{(k)}m_\mu &= E_i^{(n\ell_i)}F_i^{(k\ell_i)}m_\mu \\ &= \sum_{t=0}^{\min\{n\ell_i,k\ell_i\}}\begin{bmatrix}\langle\gamma_i^\vee,\mu\rangle + (k-n)\ell_i\\t\end{bmatrix}_i F_i^{(k\ell_i-t)}E_i^{(n\ell_i-t)}m_\mu \\ &= \sum_{t=0}^{\min\{n\ell_i,k\ell_i\}}\begin{bmatrix}\ell_i\langle\alpha_i^\vee,\mu'\rangle + (k-n)\ell_i\\t\end{bmatrix}_i F_i^{(k\ell_i-t)}E_i^{(n\ell_i-t)}m_\mu.\end{aligned}$$

Since $\begin{bmatrix}\ell_i\langle\alpha_i^\vee,\mu'\rangle+(k-n)\ell_i\\t\end{bmatrix}_i = 0$ if t is not divisible by ℓ_i, this implies:

$$X_i^{(n)}Y_i^{(k)}m_\mu = \sum_{t=0}^{\min\{n,k\}}\binom{\mu'(H_i)+(k-n)}{t}Y_i^{(k-t)}X_i^{(n-t)}m_\mu,$$

and hence $X_i^{(n)}Y_i^{(k)} = \sum_{t=0}^{\min\{n,k\}}\binom{H_i+(k-n)}{t}Y_i^{(k-t)}X_i^{(n-t)}$ in $\mathrm{End}_R\, M_{\lambda,R}^{\frac{d}{\ell}}$. □

Let M be a $U_q(\mathfrak{g}^t)$-module with a weight decomposition $M = \bigoplus_{\mu\in X^t} M(\mu)$. We assume in the following that $\dim M(\mu) < \infty$ for all weights, and the set of weights with $M(\mu) \neq 0$ is bounded above (i.e., there exists a finite number of weights $\lambda_1,\ldots,\lambda_r$ such that $\dim M(\mu) > 0$ only if $\mu < \lambda_j$ for at least one j).

If M admits a $U_{q,A}(\mathfrak{g}^t)$-stable A-lattice M_A such that $M_A = \bigoplus_{\mu\in X^t} M_A(\mu)$ (where $M_A(\mu) := M_A\cap M(\mu)$), then we denote by M_R the $U_{q,R}(\mathfrak{g}^t)$-module $M_A\otimes_A R$ for any A-algebra R. We have a corresponding weight space decomposition $M_R = \bigoplus_{\mu\in X^t} M_R(\mu)$, and for $m \in M_A$ we write $m = \sum_{\nu\in X^t} m_\nu$ with $m_\nu \in M_R(\nu)$.

For such a module let M_R^* be the direct sum $\bigoplus_{\mu\in X^t}\mathrm{Hom}_R(M_R(\mu), R)$. Denote by $p_\mu : M_R \to M_R(\mu)$ the projection $\sum_{\nu\in X^t} m_\nu \to m_\mu$. We can view M_R^* as a submodule of the dual space of M_R by the definition $f(m) := f(p_\mu(m))$ for $f \in \mathrm{Hom}_R(M_R(\mu), R)$ and $m \in M_R$. Further, M_R^* is a $U_{q,R}(\mathfrak{g}^t)$-module, the dual module: Let S be the antipode; the action of $U_{q,R}(\mathfrak{g}^t)$ on M_R^* is defined by ($m \in M_R$):

$$(uf)(m) := f(p_\mu(S(u)(m))) \quad \text{for all } u \in U_{q,R}(\mathfrak{g}^t),\ f \in \mathrm{Hom}_R(M_R(\mu), R).$$

Proposition 1. *The map $U_\mathbb{Z} \otimes_\mathbb{Z} R \to \mathrm{End}_R\left(M_{\lambda,R}^{\frac{d}{\ell}}\right)^*$ defined by*

$$X_i^{(n)} f(m) := f(S(E_i^{(n\ell_i)})m), \quad Y_i^{(n)} f(m) := f(S(F_i^{(n\ell_i)})m),$$

and $\binom{H_i+k}{n} f(m) := f(S(\left[\begin{smallmatrix}K_i;k\ell_i\\ n\ell_i\end{smallmatrix}\right])m)$, is the representation map corresponding to the dual representation of the representation of $U(\mathfrak{g})$ on $M_{\lambda,R}^{\frac{d}{\ell}}$.

Proof. By Theorem 1 we know that the action of the $X_i^{(n)}, Y_i^{(n)}$ and $\binom{H_i+k}{n}$ on $\left(M_{\lambda,R}^{\frac{d}{\ell}}\right)^*$ is well-defined. Since S is an anti-homomorphism, it is easy to check that the map extends to an algebra homomorphism $U_\mathbb{Z} \otimes_\mathbb{Z} R \to \mathrm{End}_R\left(M_{\lambda,R}^{\frac{d}{\ell}}\right)^*$.

To see that this is the dual representation of $M_{\lambda,R}^{\frac{d}{\ell}}$, it suffices to check the action of the generators. Note that

$$S(E_i^{(n\ell_i)}) = (-1)^{n\ell_i} \mathbf{v}^{\overline{d}_i(n^2\ell_i^2 - n\ell_i)} K_i^{n\ell_i} E_i^{(n\ell_i)} = (-1)^n K_i^{n\ell_i} E_i^{(n\ell_i)}.$$

Since $\mu(K_i^{n\ell_i}) = 1$ for $\mu \in (\ell/d)X$, we get $X_i^{(n)} f(m) = f((-1)^n X_i^{(n)} m)$. Similarly one sees that $Y_i^{(n)} f(m) = f((-1)^n Y_i^{(n)} m)$. Finally,

$$S(\begin{bmatrix}K_i;k\ell_i\\ n\ell_i\end{bmatrix}) m_\mu = \begin{bmatrix}-\langle\gamma_i^\vee,\mu\rangle + k\ell_i\\ n\ell_i\end{bmatrix}_i m_\mu = \binom{-\mu'(H_i)+k}{n} m_\mu$$

for a weight vector of weight $\mu = \frac{\ell}{d}\mu'$, $\mu' \in X$, which finishes the proof. □

3. Path vectors

Let $\lambda \in X^+$ be a dominant weight for $\mathfrak{g}$ and fix $\ell \in \mathbb{N}$ such that $2d$ divides ℓ. Let $\tilde{R}$ be the ring obtained by adjoining all roots of unity to $\mathbb{Z}$. We fix an embedding $R \hookrightarrow \tilde{R}$.

If k is an algebraically closed field and $\mathrm{Char}\, k = 0$, then we consider k as an R-module by the inclusion $R \hookrightarrow \tilde{R} \subset k$. If $\mathrm{Char}\, k = p > 0$, then we consider k as an R-module by extending the canonical map $\mathbb{Z} \to k$ to a map $\tilde{R} \to k$ (where the first map is given by the projection $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ and the inclusion $\mathbb{Z}/p\mathbb{Z} \subset k$). Denote by $V_{\lambda,\tilde{R}} = V_{\lambda,\mathbb{Z}} \otimes_\mathbb{Z} \tilde{R}$ the corresponding Weyl module over the ring $\tilde{R}$.

By section 2 we know that we have the following sequence of inclusions of vector spaces (the top row is an inclusion of $U_{q,\tilde{R}}(\mathfrak{g}^t)$-modules, the bottom row is an inclusion of $\mathfrak{g}$-modules):

$$\begin{array}{ccccc} & & M_{\frac{\ell}{d}\lambda,\tilde{R}} & \hookrightarrow & \underbrace{M_{\lambda,\tilde{R}} \otimes \ldots \otimes M_{\lambda,\tilde{R}}}_{\ell/d} \\ & & \uparrow & & \uparrow \\ V_{\lambda,\tilde{R}} & \hookrightarrow & \left(M_{\frac{\ell}{d}\lambda,\tilde{R}}\right)^{\frac{d}{\ell}} & \hookrightarrow & \big(\underbrace{M_{\lambda,\tilde{R}} \otimes \ldots \otimes M_{\lambda,\tilde{R}}}_{\ell/d}\big)^{\frac{d}{\ell}} \end{array}$$

Here $\left(M\right)^{\frac{d}{\ell}}$ is the direct sum of $U_{q,\tilde{R}}(\mathfrak{g}^t)$-weight spaces in M of weight $\mu \in (\ell/d)X$. The same arguments as above prove that this subspace admits a $U_{\tilde{R}}(\mathfrak{g})$-module structure. The inclusions induce restriction maps for the corresponding dual modules (the dual V_λ^* of V_λ is defined in the infinite dimensional case in the same way as the dual of M_λ):

$$\begin{array}{ccccc} \big(\underbrace{M_{\lambda,\tilde R}\otimes\ldots\otimes M_{\lambda,\tilde R}}_{\ell/d}\big)^* & \to & M^*_{\frac{\ell}{d}\lambda,\tilde R} & & \\ \downarrow & & \downarrow & & \\ \big((\underbrace{M_{\lambda,\tilde R}\otimes\ldots\otimes M_{\lambda,\tilde R}}_{\ell/d})^{\frac{d}{\ell}}\big)^* & \to & \big((M_{\frac{\ell}{d}\lambda,\tilde R})^{\frac{d}{\ell}}\big)^* & \to & V^*_{\lambda,\tilde R} \end{array}$$

We use these maps to define some special vectors in $V^*_{\lambda,\tilde R}$. Fix a highest weight vector m_λ in M_λ such that $M_{\lambda,A} := U_{q,A}(\mathfrak{g}^t)m_\lambda$ and $M_{\lambda,\tilde R} = M_{\lambda,A}\otimes_A \tilde R$. We define for $\tau\in W/W_\lambda$ a canonical vector m_τ of weight $\tau(\lambda)$ as follows: Fix a reduced decomposition $\tau = s_{i_1}\cdots s_{i_r}$. According to this decomposition let $n_1,\ldots,n_r$ be defined by

$$n_r := \langle\gamma_{i_r}^\vee,\lambda\rangle,\ n_{r-1} := \langle\gamma_{i_{r-1}}^\vee, s_{i_r}(\lambda)\rangle,\quad \ldots,\quad n_1 := \langle\gamma_{i_1}^\vee, s_{i_2}\cdots s_{i_r}(\lambda)\rangle.$$

We set $m_\tau := F_{i_1}^{(n_1)}\ldots F_{i_r}^{(n_r)}m_\lambda$. The fact that m_τ is independent of the choice of the decomposition follows from the quantum Verma identities. Denote by b_τ the corresponding dual vector in $M^*_{\lambda,\tilde R}(\tau(\lambda))$. We define in the same way vectors $v_\tau\in V_{\lambda,\mathbb{Z}}$ and $p_\tau\in V^*_{\lambda,\mathbb{Z}}$.

Let $\pi = (\tau_1,\ldots,\tau_s;0,a_1,\ldots,1)$ be an L-S path of shape λ [11]. Suppose ℓ is minimal with the property that $2d$ divides ℓ and $\frac{\ell}{d}a_i\in\mathbb{Z}$ for all $i=1,\ldots,s$. Then we can associate to π the vector

$$b_\pi := \underbrace{b_{\tau_s}\otimes\ldots\otimes b_{\tau_s}}_{\frac{\ell}{d}(1-a_{s-1})}\otimes\ldots\otimes\underbrace{b_{\tau_2}\otimes\ldots\otimes b_{\tau_2}}_{\frac{\ell}{d}(a_2-a_1)}\otimes\underbrace{b_{\tau_1}\otimes\ldots\otimes b_{\tau_1}}_{\frac{\ell}{d}a_1}\in (M^*_{\lambda,\tilde R})^{\otimes\frac{\ell}{d}}.$$

Definition 1. We call the image of b_π in $V^*_{\lambda,\tilde R}$ the *path vector* associated to π, and we denote it by p_π. By abuse of notation, we denote by p_π as well its image in $V^*_{\lambda,k} = V^*_{\lambda,\tilde R}\otimes_{\tilde R} k$ for any algebraically closed field k.

4. A basis of $V_{\lambda,\mathbb{Z}}$

The vectors p_π defined above have the nice property that they depend only on the path π (and the choice of $m_\lambda\in M_{\lambda,A}$). To prove that they form a basis of $V^*_{\lambda,k}$ for any algebraically closed field k, we construct now a basis which is, up to a triangular transformation, a dual basis of the p_π. We suppose that $2d$ divides ℓ, and we set $\overline{\ell} = \frac{\ell}{d}$. Let $\pi = (\tau_1,\ldots;0,a_1,\ldots,1), \eta = (\kappa_1,\ldots;0,b_1,\ldots,1)$ be two L-S paths of shape λ.

Definition 2. We say $\pi\geq\eta$ if $\tau_1>\kappa_1$, or $\tau_1=\kappa_1$ and $a_1>b_1$, or $\tau_1=\kappa_1$, $a_1=b_1$ and $\tau_2>\kappa_2$, etc.

For $\nu\in X^t$ let $m_\nu\in M_{\lambda,\tilde R}(\nu)$ be a weight vector. Denote by "$\succ$" the usual partial order on the set of weights. We say $m_{\nu_1}\otimes\ldots\otimes m_{\nu_{\overline{\ell}}} < m_{\lambda_1}\otimes\ldots\otimes m_{\lambda_{\overline{\ell}}}$ if there exists a j such that $\nu_i=\lambda_i$ for all $i<j$ and $\nu_j\succ\lambda_j$. If π is such that $\overline{\ell}a_i\in\mathbb{Z}$ for all $i=1,\ldots,s$, then we associate to π the vector

$$m^\pi := \underbrace{m_{\tau_1}\otimes\ldots\otimes m_{\tau_1}}_{\overline{\ell}a_1}\otimes\underbrace{m_{\tau_2}\otimes\ldots\otimes m_{\tau_2}}_{\overline{\ell}(a_2-a_1)}\otimes\ldots\otimes\underbrace{m_{\tau_s}\otimes\ldots\otimes m_{\tau_s}}_{\overline{\ell}(1-a_{s-1})}\in (M_{\lambda,\tilde R})^{\otimes\overline{\ell}}.$$

Note if $\pi > \eta$ and $\overline{\ell} b_i \in \mathbb{Z}$ for $\eta = (\kappa_1, \dots, \kappa_t, 0, b_1, \dots, 1)$, then $m^\pi > m^\eta$.

To construct a basis of $V_{\lambda,k}$, we provide first an inductive procedure that associates to π a sequence of integers $s(\pi) = (n_1, \dots, n_r)$. Fix a reduced decomposition $\tau_1 = s_{i_1} \dots s_{i_r}$; the sequence $s(\pi)$ will depend on the chosen reduced decomposition. The set up for this procedure has been inspired by the article [20] of K. N. Raghavan and P. Sankaran. Fix j minimal such that $s_{i_1}\tau_j > \tau_j$, and set $j = r+1$ if $s_{i_1}\tau_j \le \tau_j$ for all j. Consider the path $\eta = (s_{i_1}\tau_1, \dots, s_{i_1}\tau_{j-1}, \tau_j, \dots, \tau_r; 0, a_1, \dots, 1)$ (it is understood that we omit a_{j-1} if $s_{i_1}\tau_{j-1} = \tau_j$).

Lemma 2. *η is an L-S path of shape λ.*

Proof. If $j = r+1$, then η is equal to $e_\alpha^n(\pi)$, $n := -\langle \alpha^\vee, \pi(1)\rangle$ (see [11]), and η is hence an L-S path. Suppose now $j \le r$. To prove that η is an L-S path, it suffices to prove that $\eta' = (s_{i_1}\tau_1, \dots, s_{i_1}\tau_{j-1}, \tau_j; 0, a_1 \dots, a_{j-1}, 1)$ is an L-S path of shape λ. Since $\pi' = (\tau_1, \dots, \tau_{j-1}, \tau_j; 0, a_1 \dots, a_{j-1}, 1)$ is an L-S path of shape λ ([11], Lemma 3.1), and $\eta' = e_\alpha^n(\pi')$, $n := -\langle \alpha^\vee, \pi'(a_{j-1})\rangle$ (n is an integer by [11], Lemma 3.5), we conclude that η', and hence also η, are L-S paths of shape λ. □

It follows that $\eta(1) - \pi(1)$ is an integral multiple of the simple root α_{i_1}. Let $n_1 \in \mathbb{N}$ be such that $\eta(1) - \pi(1) = n_1 \alpha_{i_1}$. Note that $s_{i_1}\tau_1 = s_{i_2} \dots s_{i_r}$ is a reduced decomposition, and $s_{i_1}\tau_1 < \tau_1$. Suppose we have already defined $s(\eta) = (n_2, \dots, n_r)$ (where $s(id; 0, 1)$ is the empty sequence). We define the sequence for π to be the one obtained by adding n_1 to the sequence for η:

Definition 3. We denote by $s(\pi)$ the sequence $(n_1, n_2, \dots, n_r)$, and we associate to π the vector $v_\pi := Y_{\alpha_{i_1}}^{(n_1)} \cdots Y_{\alpha_{i_r}}^{(n_r)} v_\lambda \in V_{\lambda,\mathbb{Z}}$.

The vector v_π depends on the choice of the reduced decomposition. By construction, we know that v_π is a weight vector of weight $\pi(1)$. For $\tau \in W/W_\lambda$ denote by $V_{\lambda,\mathbb{Z}}(\tau) \subset V_{\lambda,\mathbb{Z}}$ the submodule $U_\mathbb{Z}^+(\mathfrak{g}) v_\tau$, and by $M_{\lambda,A}(\tau) \subset M_{\lambda,A}$ the submodule $U_{q,A}^+(\mathfrak{g}^t) m_\tau$. For $\pi = (\tau_1, \dots; \dots, 1)$ denote by $i(\pi) := \tau_1$ the initial element.

Theorem 2. *The set $\{v_\pi \mid \pi$ an L-S path of shape $\lambda, \tau_1 \le \tau\}$ is a basis of $V_{\lambda,\mathbb{Z}}(\tau)$.*

Denote by Λ_α the Demazure operator on the group ring $\mathbb{Z}[X]$:

$$\Lambda_\alpha(e^\mu) := \frac{e^{\mu+\rho} - e^{s_\alpha(\mu+\rho)}}{1 - e^{-\alpha}} e^{-\rho}.$$

Together with Theorem 5.1, [11], we get as an immediate consequence:

Corollary 1 (Demazure character formula)**.** *$V_{\lambda,\mathbb{Z}}(\tau)$ is a direct summand in $V_{\lambda,\mathbb{Z}}$, and for any reduced decomposition $\tau = s_{i_1} \dots s_{i_r}$, the character* $\operatorname{Char} V_{\lambda,\mathbb{Z}}(\tau)$ *is given by the Demazure character formula* $\operatorname{Char} V_{\lambda,\mathbb{Z}}(\tau) = \Lambda_{i_1} \dots \Lambda_{i_r} e^\lambda$.

Example. For $U = U(\mathfrak{sl}_3)$, we can always choose the reduced decomposition of τ_1 such that $v_\pi = Y_i^{(n_1)} Y_j^{(n_2)} Y_i^{(n_3)} v_\lambda$, $\{i, j\} = \{1, 2\}$, where $n_2 \ge n_1 + n_3$. It follows that in this case we may assume that the basis of $V_{\lambda,\mathbb{Z}}$ given by the v_π is the canonical basis [15].

To prove the theorem, we consider again the inclusion $V_{\lambda,R} \hookrightarrow M_{\lambda,R}^{\otimes \overline{\ell}}$. We write $\mathbf{v}^\bullet$ as an abbreviation for "an appropriate power of $\mathbf{v}$".

Lemma 3. *i) If $\overline{\ell} a_i \in \mathbb{Z}$ for all i, then in the expression for $v_\pi \in M_{\lambda,R}^{\otimes \overline{\ell}}$:*

$$v_\pi = Y_{\alpha_{i_1}}^{(n_1)} \cdots Y_{\alpha_{i_r}}^{(n_r)} (m_\lambda^{\otimes \overline{\ell}}) = \sum \mathbf{v}^\bullet (F_{i_1}^{(h_1)} \dots F_{i_r}^{(h_r)} m_\lambda) \otimes \dots \otimes (F_{i_1}^{(s_1)} \dots F_{i_r}^{(s_r)} m_\lambda)$$

there is only one summand which is a non-zero multiple of m^π, and the other summands either vanish or $m^\pi > (F_{i_1}^{(h_1)}\dots F_{i_r}^{(h_r)}m_\lambda)\otimes\dots\otimes(F_{i_1}^{(s_1)}\dots F_{i_r}^{(s_r)}m_\lambda)$. In particular, one of these summands is a multiple of m^η only if $\eta<\pi$.

ii) If $\overline{\ell}a_i\in\mathbb{Z}$ for all $i=1,\dots,s$, then there exists an $h\in\mathbb{N}$ such that

$$v_\pi = \mathbf{v}^h m^\pi + \sum m_{\tau_1}^{\otimes\overline{\ell}a_1}\otimes\dots\otimes m_{\tau_p}^{\otimes\overline{\ell}(a_p-a_{p-1})}\otimes m_{\tau_{p+1}}^{\otimes a'}\otimes m_\nu\otimes\dots\otimes m_{\nu'}, \tag{1}$$

where, for each summand on the right side of (1)*, there exists a $0\le p<r$ such that $a'<\overline{\ell}(a_{p+1}-a_p)$ and m_ν is a vector in $M_{\lambda,R}(\tau_{p+1})$ of weight $\nu\neq\tau_{p+1}(\lambda)$.*

Proof of Theorem 2. We may choose ℓ such that for all L-S paths of shape λ, ending in $\pi(1)$ (there are only a finite number of such paths), the conditions of the lemma above are satisfied. Since the m^π are linearly independent over R for the various paths, so are the v_π by Lemma 3 *ii*). The Weyl character formula for the module $V_{\lambda,R}$ and the corresponding formula for L-S paths (see [12]) implies that the R-module spanned by the v_π has the same rank as $V_{\lambda,R}(\mu)$.

The tensor product of the R-lattices $M_{\lambda,R}$ is $U_{q,R}(\mathfrak{g}^t)$-stable, $m_\lambda^{\otimes\overline{\ell}}$ is an element of this lattice, and the m^π form a subset of a basis of the free R-module $M_{\lambda,R}^{\otimes\overline{\ell}}$. Since the m^π are the "leading terms" of the v_π, it follows that the v_π form a basis of the R-lattice $V_{\lambda,R}(\mu)$. The v_π are by definition in $V_{\lambda,\mathbb{Z}}(\mu)$, so they form in fact a basis of the $\mathbb{Z}$-lattice $V_{\lambda,\mathbb{Z}}(\mu)$.

Fix a reduced decomposition $\tau=s_{i_1}\dots s_{i_t}$. It is well-known that $V_{\lambda,\mathbb{Z}}(\tau)$ is spanned by the vectors of the form $Y_{\alpha_{i_1}}^{(a_1)}\cdots Y_{\alpha_{i_t}}^{(a_t)}v_\lambda$, $a_1,\dots,a_t\ge 0$. As a consequence one sees easily that $V_{\lambda,\mathbb{Z}}(\kappa)\subset V_{\lambda,\mathbb{Z}}(\tau)$ whenever $\kappa<\tau$ in the Bruhat order. It follows that the vectors v_π with $i(\pi)\le\tau$ span a free summand $N_\tau\subset V_{\lambda,\mathbb{Z}}$ and $N_\tau\subset V_{\lambda,\mathbb{Z}}(\tau)$. Since $v_\tau=v_{(\tau;0,1)}$ is an element of N_τ, to prove the proposition it suffices to show that N_τ is $U_\mathbb{Z}^+(\mathfrak{g})$-stable. Let π be an L-S path of shape λ such that $i(\pi)\le\tau$. We have to show that if α is simple and $X_\alpha^{(n)}v_\pi=\sum a_\eta v_\eta$, then $a_\eta=0$ if $i(\eta)\not\le\tau$.

We consider again an embedding $V_{\lambda,R}\hookrightarrow M_{\lambda,R}^{\otimes\overline{\ell}}$, where we assume that ℓ is such that $\overline{\ell}b_j\in\mathbb{Z}$ for all $\eta=(\kappa_1,\dots;0,b_1,\dots)$ with $a_\eta\neq 0$ and all η with $i(\eta)\le\tau$. Fix an η_0 such that $a_{\eta_0}\neq 0$ and let η_0 be a maximal element with this property. If we express $X_\alpha^{(n)}v_\pi$ as a linear combination of tensor products of weight vectors in $M_{\lambda,R}^{\otimes\overline{\ell}}$, then the coefficient of m^{η_0} is different from zero:

$$X_\alpha^{(n)}v_\pi=\sum m_{\nu_1}\otimes\dots\otimes m_{\nu_{\overline{\ell}}},\quad m_\nu\in M_{\lambda,R}(\nu).$$

By Lemma 3 we know that $v_\pi=\mathbf{v}^h m^\pi+$ terms strictly smaller in the partial order. It is now easy to see that $X_\alpha^{(n)}v_\pi$ is a sum of tensor products of weight vectors which are smaller then m^π in the partial order. In particular, $m^{\eta_0}<m^\pi$ and hence $\eta_0<\pi$, which implies $i(\eta_0)\le\tau$. Since η_0 was chosen to be maximal in the partial ordering, this implies $a_\eta\neq 0$ only if $i(\eta)\le\tau$. ☐

Proof of Lemma 3. The statement *i*) is an easy consequence of *ii*), we will give the proof for *ii*) only. Let $m_{\nu_1},\dots,m_{\nu_{\overline{\ell}}}\in M_{\lambda,R}$ be weight vectors such that the tensor product is an element of $\big(M_{\lambda,R}^{\otimes\overline{\ell}}\big)^{\frac{1}{\overline{\ell}}}$. Note that

$$Y_j^{(k)}(m_{\nu_1}\otimes\dots\otimes m_{\nu_{\overline{\ell}}})=\sum_{h_1+\dots+h_{\overline{\ell}}=\ell_j k}\mathbf{v}^\bullet(F_j^{(h_1)}m_{\nu_1})\otimes\dots\otimes(F_j^{(h_{\overline{\ell}})}m_{\nu_{\overline{\ell}}}) \tag{2}$$

for some appropriate powers $\mathbf{v}^\bullet$ of $\mathbf{v}$. Suppose now by induction that (1) holds for the path $\eta = (s_{i_1}\tau_1, \ldots, s_{i_1}\tau_{j-1}, \tau_j, \ldots, \tau_r; 0, a_1 \ldots, 1)$ (see above). The leading term (i.e., the maximal summand) is then up to multiplication by a power of $\mathbf{v}$:

$$(m_{s_{i_1}\tau_1})^{\otimes \overline{\ell} a_1} \otimes \ldots \otimes (m_{s_{i_1}\tau_{j-1}})^{\otimes \overline{\ell}(a_{j-1}-a_{j-2})} \otimes (m_{\tau_j})^{\otimes \overline{\ell}(a_j - a_{j-1})} \otimes \ldots \otimes (m_{\tau_r})^{\otimes \overline{\ell}(1-a_{r-1})}.$$

If we apply $Y_{i_1}^{(n_1)}$ to this term, then we get a sum that runs over all $\overline{\ell}$-tuples:

$$\sum \mathbf{v}^\bullet (F_{i_1}^{(h_1)} m_{s_{i_1}\tau_1}) \otimes \ldots \otimes (F_{i_1}^{(h_{\overline{\ell}})} m_{\tau_r}),$$

such that $\ell_{i_1} n_1 = h_1 + \ldots + h_{\overline{\ell}}$. If one of the h_t is too big, then such a term is zero. With respect to the chosen ordering, a maximal term in this sum will be one where $h_1 = \langle \gamma_{i_1}^\vee, s_{i_1}\tau_1(\lambda)\rangle$, i.e., one where the first term is equal to m_{τ_1}. Note that if h_1 is smaller, then we get a weight vector in $M_{\lambda,R}(\tau_1)$ which is of weight $\mu \succ \tau_1(\lambda)$. The same arguments apply also to h_2, etc., so applying $Y_{i_1}^{(n_1)}$ to the leading term gives the desired maximal term plus terms of the form:

$$m_{\tau_1}^{\otimes \overline{\ell} a_1} \otimes \ldots \otimes m_{\tau_p}^{\otimes \overline{\ell}(a_p - a_{p-1})} \otimes m_{\tau_{p+1}}^{\otimes a'} \otimes m_\nu \otimes \ldots \otimes m_{\nu'},$$

where $0 \le p < r$, $a' < \overline{\ell}(a_{p+1} - a_p)$, and $m_\nu \in M_{\lambda,R}(\tau_{p+1})$ is of weight $\nu \succ \tau_{p+1}(\lambda)$.

It remains to discuss the terms we get by applying $Y_{i_1}^{(n_1)}$ to the other summands in the expression of v_η. Let $m_{\nu_1} \otimes \ldots \otimes m_{\nu_{\overline{\ell}}}$ be such a summand. If $\nu_1 \neq s_{i_1}\tau_1(\lambda)$, then we know that m_{ν_1} is a weight vector in $M_{\lambda,R}(s_{i_1}\tau_1)$ of weight $\nu_1 \succ s_{i_1}\tau_1(\lambda)$. Note that $F_{i_1}^{(h_1)} m_{\nu_1}$ is for any h_1 a weight vector of weight ν' in $M_{\lambda,R}(\tau_1)$, where $\nu' \succ \tau_1(\lambda)$. If $\nu_1 = s_{i_1}\tau_1(\lambda)$, then the term is zero for $h_1 > \langle \gamma_{j_1}^\vee, s_{i_1}\tau_1(\lambda)\rangle$. A maximal term in this sum will be one where $h_1 = \langle \gamma_{j_1}^\vee, s_{i_1}\tau_1(\lambda)\rangle$, i.e., the first term is equal to m_{τ_1}. If h_1 is smaller, then we get a weight vector in $M_{\lambda,R}(\tau_1)$ which is of weight $\succ \tau_1(\lambda)$. The same arguments apply also to h_2, h_3, etc. As a consequence we conclude that applying $Y_{i_1}^{(n_1)}$ to the other summands gives only terms of the form

$$m_{s_{i_1}\tau_1}^{\otimes \overline{\ell} a_1} \otimes \ldots \otimes m_{\tau_p}^{\otimes \overline{\ell}(a_p - a_{p-1})} \otimes m_{\tau_{p+1}}^{\otimes a'} \otimes m_\nu \otimes \ldots \otimes m_{\nu'},$$

where $0 \le p < r$, $a' < \overline{\ell}(a_{p+1} - a_p)$, and $m_\nu \in M_{\lambda,R}(\tau_{p+1})$ is a weight vector of weight $\nu \succ \tau_{p+1}(\lambda)$. It follows that these summands are strictly smaller than m^π in the partial order. □

Denote by $\mathbb{B}_\lambda(\tau)$ the set of L-S paths of shape λ such that $i(\pi) \le \tau$. Let $\pi_1, \ldots, \pi_N$ be a numeration of the paths such that $\pi_i > \pi_j$ implies $i > j$, and let $V_{\lambda,\mathbb{Z}}(\tau, j)$ be the $\mathbb{Z}$-submodule spanned by the v_{π_i}, $1 \le i \le j$. To prove that the span of the v_π, $\pi \in \mathbb{B}_\lambda(\tau)$, is equal to $V_{\lambda,\mathbb{Z}}(\tau)$, we proved that $X_\alpha^{(n)} v_\pi$ can be expressed as a linear combination of the v_η with $\eta \le \pi$. Hence we get as a corollary of the proof of the proposition:

Corollary 2. *The complete flag $\mathbb{V}_{\lambda,\mathbb{Z}}(\tau)$ is $U_\mathbb{Z}(\mathfrak{g})^+$-stable:*

$$\mathbb{V}_{\lambda,\mathbb{Z}}(\tau):\ 0 \subset V_{\lambda,\mathbb{Z}}(\tau,1) \subset V_{\lambda,\mathbb{Z}}(\tau,2) \subset \ldots \subset V_{\lambda,\mathbb{Z}}(\tau,N) = V_{\lambda,\mathbb{Z}}(\tau).$$

5. A basis of $V^*_{\lambda,k}$

The basis of $V_{\lambda,\mathbb{Z}}$ constructed above enables us now to prove that the path vectors introduced in section 3 form a basis of $V^*_{\lambda,k}$, k an arbitrary algebraically closed field.

Theorem 3. *The set of path vectors p_π, π an L-S path of shape λ, forms a basis of $V^*_{\lambda,k}$ of $\mathfrak{H}$-eigenvectors of weight $-\pi(1)$. Further, let $\pi_1, \pi_2, \dots$ be a numeration of the L-S paths such that $\pi_i > \pi_j$ implies $i > j$, and denote by $V^*_{\lambda,k}(j)$ the subspace spanned by the p_{π_i}, $i \geq j$. The flag $\mathbb{V}^*_{\lambda,k}$ is $U_k(\mathfrak{g})^+$-stable:*

$$\mathbb{V}^*_{\lambda,k}: \ V^*_{\lambda,k} = V^*_{\lambda,k}(1) \supset V^*_{\lambda,k}(2) \supset V^*_{\lambda,k}(3) \supset \dots .$$

Proof. Let $\pi = (\tau_1, \dots, \tau_r; 0, a_1, \dots, 1)$ be an L-S path of shape λ, and fix ℓ minimal such that $2d$ divides ℓ and $\overline{\ell} a_j \in \mathbb{Z}$ for all j. We consider again the embedding $V_{\lambda,R} \hookrightarrow M^{\otimes \overline{\ell}}_{\lambda,R}$. Let η be an L-S path of shape λ and suppose that $p_\pi(v_\eta) \neq 0$. By Lemma 3, this is only possible if $\eta \geq \pi$, and $p_\pi(v_\pi)$ is a root of unity. Since the v_η form a basis of $V_{\lambda,k}$, it follows that the p_π form a basis of $V^*_{\lambda,k}$.

By construction, the path vectors p_π are $\mathfrak{H}$-eigenvectors of weight $-\pi(1)$. Further, by the choice of the numeration, the subspace $V^*_{\lambda,k}(j)$ is the subspace of $V^*_{\lambda,k}$ of all vectors vanishing on the subspace of $V_{\lambda,k}$ spanned by all v_{π_i} with $i < j$. Since this is a $U_k(\mathfrak{g})^+$-stable subspace (see Corollary 2 of Theorem 2), it follows that $V^*_{\lambda,k}(j)$ is a $U_k(\mathfrak{g})^+$-stable subspace. □

For the definition of the path vector p_π we have chosen ℓ to be minimal such that $2d$ divides ℓ and $\overline{\ell} a_j \in \mathbb{Z}$. Of course, one can define in the same way for any ℓ (which satisfies these properties) a vector $p_{\pi,\ell} \in V^*_{\lambda,R}$. The arguments above show:

Corollary 1. $p_{\pi,\ell} = c_\pi p_\pi + \sum_{\eta > \pi} c_\eta p_\eta$, *where c_π is a root of unity.*

The property: $p_\pi(v_\eta) \neq 0$ only if $\pi \leq \eta$, implies that p_π vanishes on $V_{\lambda,k}(\tau)$ unless $i(\pi) \leq \tau$. We get as an immediate consequence (by abuse of notation we write p_π also for the image of the linear form in $V^*_{\lambda,k}(\tau)$):

Corollary 2. *The restrictions $\{p_\pi \mid i(\pi) \leq \tau\}$ form a basis of $V^*_{\lambda,k}(\tau)$, and the set $\{p_\pi \mid i(\pi) \not\leq \tau\}$ is a basis of the kernel of the restriction map $V^*_{\lambda,k} \to V^*_{\lambda,k}(\tau)$.*

Example. For $\mathfrak{g} = V = \mathfrak{sl}_3$, the basis given by the path vectors is, up to sign, the dual basis of the Chevalley basis of $\mathfrak{g}$.

6. Standard monomials

Let $\lambda_1, \dots, \lambda_r$ be some dominant weights, set $\lambda = \sum \lambda_i$, and fix $\tau \in W/W_\lambda$. For each i let τ_i be the image of τ in W/W_{λ_i}. A module V_λ (without specifying the underlying ring) is always meant to be the Weyl module of highest weight λ over an algebraically closed field. The inclusion $V_\lambda \hookrightarrow V_{\lambda_1} \otimes \dots \otimes V_{\lambda_r}$ induces a map $V_\lambda(\tau) \hookrightarrow V_{\lambda_1}(\tau_1) \otimes \dots \otimes V_{\lambda_r}(\tau_r)$, and hence in turn a map $V^*_{\lambda_1}(\tau_1) \otimes \dots \otimes V^*_{\lambda_r}(\tau_r) \to V^*_\lambda(\tau)$.

We write π_i and π_λ for the paths $t \mapsto t\lambda_i$ and $t \mapsto t\lambda$, respectively. Denote by $\mathbb{B}_i$ the set of L-S paths of shape λ_i, and by $\mathbb{B}_\lambda$ the set of paths of shape λ. Recall that the associated graph $G(\pi_\lambda)$ has as vertices the set $\mathbb{B}_\lambda$, and we put an arrow $\eta \xrightarrow{\alpha} \eta'$ with colour a simple root α if $f_\alpha(\eta) = \eta'$.

Denote by $\mathbb{B}_1 * \dots * \mathbb{B}_r$ the set of concatenations of all paths in $\mathbb{B}_1, \dots, \mathbb{B}_r$. Remember [12] that the set of paths is stable under the root operators, and the associated graph decomposes into the disjoint union of irreducible components. Denote by $G(\pi_1 * \ldots * \pi_r)$ the irreducible component containing $\pi_1 * \ldots * \pi_r$. Recall that the map $\pi_1 * \ldots * \pi_r \mapsto \pi_\lambda$ extends to an isomorphism of graphs $\phi : G(\pi_1 * \ldots * \pi_r) \to G(\pi_\lambda)$ [12]. A monomial $\eta_1 * \ldots * \eta_r \in \mathbb{B}_1 * \ldots * \mathbb{B}_r$ is called standard if it is in the irreducible component $G(\pi_1 * \ldots * \pi_r)$, and in this case we define: $i(\eta_1 * \ldots * \eta_r) := i(\phi(\eta_1 * \ldots * \eta_r))$ [13].

Definition 4. Let $\eta_1, \ldots, \eta_r$ be L-S paths of shape $\lambda_1, \ldots, \lambda_r$. A monomial of path vectors $p_{\eta_1} \cdots p_{\eta_r}$ is called *standard* if the concatenation $\eta_1 * \ldots * \eta_r$ is standard. The standard monomial is called *standard with respect to* τ if $i(\eta_1 * \ldots * \eta_r) \leq \tau$.

Theorem 4. *The set of standard monomials forms a basis of V_λ^*, and the set of monomials standard with respect to τ forms a basis of $V_\lambda^*(\tau)$.*

Proof. To simplify the notation, we give a proof only for the case $r = 2$, the proof for $r > 2$ is similar. Suppose η_1, η_2 are such that $\eta_1 * \eta_2$ is standard:

$$\eta_1 = (\kappa_1, \ldots, \kappa_s; 0, a_1, \ldots, 1) \in \mathbb{B}_1, \quad \eta_2 = (\kappa_{s+1}, \ldots, \kappa_t; 0, a_{s+1}, \ldots, 1) \in \mathbb{B}_2.$$

Fix ℓ such that $2d$ divides ℓ and $\overline{\ell} a_j \in \mathbb{Z}$ for all j. We consider the embedding

$$V_{\lambda,R} \hookrightarrow V_{\lambda_1,R} \otimes V_{\lambda_2,R} \hookrightarrow M_{\lambda_1,R}^{\otimes \overline{\ell}} \otimes M_{\lambda_2,R}^{\otimes \overline{\ell}}.$$

We associate to $\eta_1 * \eta_2$ a sequence of integers using a procedure similar to the one in section 4 (Lemma 2, Definition 3): Let $(w_1, \ldots, w_s; w_{s+1}, \ldots, w_t)$, $w_i \in W/W_\lambda$, be the minimal defining chain [13], so $w_1 = \kappa_1 \mod W_{\lambda_1}$.

Note that $i(\eta_1 * \eta_2) = w_1$ [13]. Fix a reduced decomposition $w_1 = s_{i_1} \cdots s_{i_k}$ such that $\kappa_1 = s_{i_1} \cdots s_{i_q}$ for some $q \leq k$. Fix j minimal such that $s_{i_1} \kappa_j > \kappa_j$.

Let η_1' be obtained from η_1 as in section 4 (by replacing the κ_i by $s_{i_1} \kappa_i$ for $i < j$), and, if $j > s+1$, then let η_2' be obtained from η_2 as in section 4. If $j \leq s+1$, then we set $\eta_2' := \eta_2$. The concatenation $\eta_1' * \eta_2'$ is again standard, and $i(\eta_1' * \eta_2') = s_{i_1} w_1$. We fix n_1 to be such that $\eta_1' * \eta_2'(1) - \eta_1 * \eta_2(1) = n_1 \alpha_{i_1}$. As in Definition 3, let $s(\eta_1 * \eta_2) = (n_1, \ldots, n_k)$ be the sequence obtained from $s(\eta_1' * \eta_2')$ by adding n_1, and denote by $v_{\eta_1 * \eta_2}$ the vector:

$$v_{\eta_1 * \eta_2} = Y_{\alpha_{i_1}}^{(n_1)} \ldots Y_{\alpha_{i_k}}^{(n_k)} (v_{\lambda_1} \otimes v_{\lambda_2}) \in V_{\lambda,\mathbb{Z}} \subset V_{\lambda_1,\mathbb{Z}} \otimes V_{\lambda_2,\mathbb{Z}} \subset M_{\lambda_1,R}^{\otimes \overline{\ell}} \otimes M_{\lambda_2,R}^{\otimes \overline{\ell}}.$$

Let "$<$" be the induced lexicographic partial order on tensor products of weight vectors. The same arguments as in the proof of Lemma 3 show that if we express $v_{\eta_1 * \eta_2}$ as a linear combination of tensor products of weight vectors in $M_{\lambda_1,R}^{\otimes \overline{\ell}} \otimes M_{\lambda_2,R}^{\otimes \overline{\ell}}$, then we get:

$$(3) \qquad v_{\eta_1 * \eta_2} = \mathbf{v}^h m^{\eta_1} \otimes m^{\eta_2} + \text{terms strictly smaller in the partial order.}$$

The same arguments as in the proof of Theorem 2 show: The $v_{\eta_1 * \eta_2}$, $\eta_1 * \eta_2$ standard, form a basis for $V_{\lambda,\mathbb{Z}}$. And the same arguments as in the proof of Theorem 3 and Corollary 1 show that the standard monomials $p_{\eta_1} p_{\eta_2}$ form a basis of $V_{\lambda,k}^*$.

The vectors of the form $v_{\eta_1 * \eta_2}$, $\eta_1 * \eta_2$ standard with respect to τ, are elements of $V_{\lambda,\mathbb{Z}}(\tau)$ (by construction). The graph isomorphism $\phi : G(\pi_1 * \ldots * \pi_r) \to G(\pi_\lambda)$ and Corollary 2 of Theorem 3 imply that they span a free $\mathbb{Z}$-submodule of $V_{\lambda,\mathbb{Z}}(\tau)$ which is of the same rank as $V_{\lambda,\mathbb{Z}}(\tau)$. But since they form a subset of a basis of $V_{\lambda,\mathbb{Z}}$, this implies that these vectors form a $\mathbb{Z}$-basis of $V_{\lambda,\mathbb{Z}}(\tau)$. Now (3) implies that the restrictions of the $p_{\eta_1} p_{\eta_2}$, standard with respect to τ, form a basis of $V_{\lambda,k}^*(\tau)$. □

Remark 1. The monomials $\{p_{\eta_1}\cdots p_{\eta_r}\mid \eta_1*\ldots*\eta_r$ *not* standard w.r.t. $\tau\}$, do *not* form in general a basis of the kernel of the restriction map $V^*_{\lambda,k}\to V^*_{\lambda,k}(\tau)$ for $r\geq 2$. The reason for this is that the partial order on the tensor products of weight vectors is no longer related to $i(\eta_1*\ldots*\eta_2)$. For example, set $\mathfrak{g}=\mathfrak{sl}_4$, let $\mathfrak{H}$ be the Cartan subalgebra of diagonal matrices of trace zero, let ϵ_i be the character that projects a diagonal matrix onto its i-th entry, and let $\omega_i=\epsilon_1+\ldots+\epsilon_i$ be the i-th fundamental weight. Set $\lambda_1:=\omega_1$, $\eta_1:=(s_2s_1;0,1)$, $\pi_1:=(s_1;0,1)$, and set $\lambda_2:=\omega_3$, $\eta_2:=(s_3;0,1)$, $\pi_2:=(s_2s_3;0,1)$. The concatenations $\eta_1*\eta_2$ and $\pi_1*\pi_2$ are standard, $\eta_1*\eta_2>\pi_1*\pi_2$, but $i(\eta_1*\eta_2)=s_2s_1s_3$, whereas $i(\pi_1*\pi_2)=s_1s_2s_3$. These two are not compatible. It is easy to check that $p_{\pi_1}p_{\pi_2}(v_{\eta_1*\eta_2})\neq 0$, so the restriction of $p_{\pi_1}p_{\pi_2}$ to $V_{\omega_1+\omega_3,k}(s_1s_2s_3)$ does not vanish though $i(\pi_1*\pi_2)\not\leq s_1s_2s_3$.

7. Some special relations

Let λ be a dominant weight, and let η_1 and η_2 be L-S paths of type $p\lambda$ and $q\lambda$, respectively, for some $p,q\in\mathbb{N}$: $\pi=(\kappa_1,\ldots,\kappa_s;0,a_1,\ldots,1)$, $\pi'=(\tau_1,\ldots,\tau_t;0,b_1,\ldots,1)$. We say that the paths π,π' have the *same support* if there exists a chain of linearly ordered elements $w_j\in W/W_\lambda$: $C=\{w_1>w_2>\ldots>w_l\}$ such that $C=\{\kappa_1,\ldots,\kappa_s,\tau_1,\ldots\tau_t\}$. We associate to (π,π') a new L-S path of shape $(p+q)\lambda$

$$\eta:=(w_1,w_2,\ldots,w_l;0,c_1,c_2,\ldots,1),$$

where the c_j are defined inductively as follows (set $c_0,a_0,b_0:=0$): If $w_i=\kappa_j$ and is not equal to one of the τ_m, then $c_i:=c_{i-1}+p(a_j-a_{j-1})/(p+q)$; if $w_i=\tau_j$, and is not equal to one of the κ_m, then $c_i:=c_{i-1}+q(b_j-b_{j-1})/(p+q)$; and if $w_i=\kappa_j=\tau_m$, then $c_i:=c_{i-1}+(p(a_j-a_{j-1})+q(b_m-b_{m-1}))/(p+q)$. Note that η is an L-S path; this follows easily from [1], Theorem 2.3.

Theorem 5. *a) If π,π' have the same support, then there exists a root of unity a_η such that in $V^*_{(p+q)\lambda,k}$ we have:*

$$p_\pi p_{\pi'}=a_\eta p_\eta+\sum_{\eta'>\eta}a_{\eta'}p_{\eta'}.$$

b) Let t be such that $2d|t$ and $ta_j\in\mathbb{Z}$ for all $j=1,\ldots,s$. For $\kappa\in W/W_\lambda$ denote by p_κ the extremal weight vector $p_\kappa\in V^*_{\lambda,k}$ of weight $-\kappa(\lambda)$. There exists a root of unity a_π such that for $\pi=(\kappa_1,\ldots,\kappa_s;0,a_1,a_2\ldots,a_{s-1},1)$ we have in $V^*_{t\lambda,k}$:

$$p_\pi^t=a_\pi p_{\kappa_1}^{ta_1}p_{\kappa_2}^{t(a_2-a_1)}\cdots p_{\kappa_s}^{t(1-a_{s-1})}+\sum_{\eta>t\pi}a_\eta p_\eta.$$

Proof. Fix ℓ_1,ℓ_2 minimal such that $2d|\ell_1,\ell_2$ and $\overline{\ell}_1a_j,\overline{\ell}_2b_i\in\mathbb{Z}$ for all i,j, and let ℓ be the smallest common multiple of ℓ_1,ℓ_2. Denote by R the corresponding ring. Consider the embeddings of $U_R(\mathfrak{g})$-modules:

$$V_{(p+q)\lambda,R}\hookrightarrow V_{p\lambda,R}\otimes V_{q\lambda,R}\hookrightarrow (M_{p\lambda,R}^{\otimes\overline{\ell}_1})^{1/\overline{\ell}_1}\otimes(M_{q\lambda,R}^{\otimes\overline{\ell}_2})^{1/\overline{\ell}_2}.$$

Let $\sigma=(\xi_1,\ldots;0,d_1,\ldots)$ be an L-S path of shape $(p+q)\lambda$. Express the corresponding vector $v_\sigma\in V_{(p+q)\lambda,\mathbb{Z}}$, $v_\sigma=Y_{\alpha_{i_1}}^{(n_1)}\cdots Y_{\alpha_{i_r}}^{(n_r)}(v_{p\lambda}^{\otimes\overline{\ell}_1}\otimes v_{q\lambda}^{\otimes\overline{\ell}_2})$ as a linear combination of tensor products of weight vectors in $(M_{p\lambda,R}^{\otimes\overline{\ell}_1})^{1/\overline{\ell}_1}\otimes(M_{q\lambda,R}^{\otimes\overline{\ell}_2})^{1/\overline{\ell}_2}$:

$$v_\sigma=\sum\mathbf{v}^\bullet(F_{i_1}^{(h_1)}\ldots F_{i_r}^{(h_r)}v_{p\lambda})\otimes\ldots\otimes(F_{i_1}^{(s_1)}\ldots F_{i_r}^{(s_r)}v_{q\lambda}).\tag{4}$$

(Note that we have here two different quantum groups involved, one at the $2\ell_1$-th root of unity, the other at the $2\ell_2$-th root of unity. To make the notation not too unreadable, we used the same letter F for the generators of the two algebras.)

We have $p_\pi p_{\pi'}(v_\sigma) \neq 0$ obviously only if one of the summands above is a non-zero multiple of $m^\pi \otimes m^{\pi'}$, where $m^\pi \in M^{\otimes \overline{\ell}_1}_{p\lambda,R}$ and $m^{\pi'} \in M^{\otimes \overline{\ell}_2}_{q\lambda,R}$ are the vectors defined in section 4. Now it is easy to see that if we find such a summand, then, after replacing ℓ_1 and ℓ_2 by a common multiple, we still find a summand which is a multiple of $m^\pi \otimes m^{\pi'} \in M^{\otimes \overline{\ell}}_{p\lambda,R} \otimes M^{\otimes \overline{\ell}}_{q\lambda,R}$. So from now on we may assume that $\ell_1 = \ell_2 = \ell$. But this implies that we find in the expression of $v_\sigma \in M^{\otimes(p\overline{\ell}+q\overline{\ell})}_{\lambda,R}$,

$$v_\sigma = \sum \mathbf{v}^\bullet (F^{(h_1)}_{i_1} \dots F^{(h_r)}_{i_r} v^{\otimes p}_\lambda) \otimes \dots \otimes (F^{(s_1)}_{i_1} \dots F^{(s_r)}_{i_r} v^{\otimes q}_\lambda),$$

a summand which is a non-zero multiple of $m^{\pi,p} \otimes m^{\pi',q}$, where

$$m^{\pi,p} := \underbrace{v_{\kappa_1} \otimes \dots \otimes v_{\kappa_1}}_{p\overline{\ell}a_1} \otimes \underbrace{v_{\kappa_2} \otimes \dots \otimes v_{\kappa_2}}_{p\overline{\ell}(a_2-a_1)} \otimes \dots \otimes \underbrace{v_{\kappa_s} \otimes \dots \otimes v_{\kappa_s}}_{p\overline{\ell}(1-a_{s-1})} \in (M_{\lambda,R})^{\otimes p\overline{\ell}},$$

and $m^{\pi',q}$ is defined accordingly. Since

$$m^{\eta,(p+q)} := \underbrace{v_{w_1} \otimes \dots \otimes v_{w_1}}_{(p+q)\overline{\ell}c_1} \otimes \dots \otimes \underbrace{v_{w_l} \otimes \dots \otimes v_{w_l}}_{(p+q)\overline{\ell}(1-c_{l-1})} \in (M_{\lambda,R})^{\otimes(p+q)\overline{\ell}}$$

is obtained from $m^{\pi,p} \otimes m^{\pi',q}$ just by permuting the vectors, we find also a non-zero multiple of $m^{\eta,(p+q)}$ as a summand in the expression of $v_\sigma \in M^{\otimes(p\overline{\ell}+q\overline{\ell})}_{\lambda,R}$.

But now the same arguments as in the proof of Lemma 3 show that this is only possible if $\sigma \geq \eta$, which proves that $p_\pi p_{\pi'}(v_\sigma) \neq 0$ only if $\sigma \geq \eta$. It remains to prove that $p_\pi p_{\pi'}(v_\eta)$ is a root of unity. The coefficient of $m^{\eta,(p+q)}$ as a summand in the expression of $v_\eta \in M^{\otimes(p\overline{\ell}+q\overline{\ell})}_{\lambda,R}$ is a root of unity (it is the leading term), and since $m^{\pi,p} \otimes m^{\pi',q}$ is obtained from the latter by a permutation, the coefficient of the corresponding summand is also a root of unity. During the step of going from $M^{\otimes \overline{\ell}}_{p\lambda,R} \otimes M^{\otimes \overline{\ell}}_{q\lambda,R}$ to $M^{\otimes \overline{\ell}(p+q)}_{\lambda,R}$, the coefficient of the summand in the expression for v_η corresponding to $m^{\pi,p} \otimes m^{\pi',q}$ (respectively $m^\pi \otimes m^{\pi'}$) changes only by a root of unity. And, finally, during the step of going from $M^{\otimes \overline{\ell}_1}_{p\lambda,R}$ to $M^{\otimes \overline{\ell}}_{p\lambda,R}$ (respectively $M^{\otimes \overline{\ell}_2}_{q\lambda,R}$ to $M^{\otimes \overline{\ell}}_{q\lambda,R}$), the coefficients of m^π and $m^{\pi'}$ change only by a root of unity.

The statement *b*) of the theorem is a consequence of *a*). We apply *a*) successively to $p^2_{\kappa_1}, p^3_{\kappa_1}, \dots, p^{ta_1}_{\kappa_1} p_{\kappa_2}, \dots$ to see that

$$p^{ta_1}_{\kappa_1} p^{t(a_2-a_1)}_{\kappa_2} \cdots p^{t(1-a_{s-1})}_{\kappa_s} = a_{t\pi} p_{t\pi} + \sum_{\eta > t\pi} a_\eta p_\eta,$$

where a_π is a root of unity and $t\pi = (\kappa_1, \dots, \kappa_s; 0, a_1, \dots, 1)$ (same as π, but considered as an L-S path of shape $t\lambda$). It remains to compare $p_{t\pi}$ and p^t_π.

Let ℓ be minimal such that $2d|\ell$ and $\overline{\ell}a_i \in \mathbb{Z}$ for all i. Consider the embeddings:

$$\begin{array}{ccc} M^{\otimes \overline{\ell}}_{t\lambda,R} & \hookrightarrow & M^{t\overline{\ell}}_{\lambda,R} \\ \uparrow & & \uparrow \\ V_{t\lambda,R} & \hookrightarrow & V^{\otimes t\overline{\ell}}_{\lambda,R} \end{array}$$

We know that $p_{t\pi}(v_\sigma) \neq 0$ for an L-S path σ of shape $t\lambda$ only if $\sigma \geq t\pi$, and $p_{t\pi}(v_{t\pi})$ is a root of unity. Now $p^t_\pi(v_\sigma) \neq 0$ only if in the expression of v_σ in $M^{\otimes t\overline{\ell}}_\lambda$ there is a summand which is a multiple of $(m^\pi)^{\otimes t}$. Now $(m^\pi)^{\otimes t}$ and $m^{t\pi} \in M_\lambda^{\otimes t\bar{\ell}}$ are obtained from each other by a permutation, which implies that v_σ has a summand which is a multiple of $m^{t\pi}$. But this is only possible if $\sigma \geq t\pi$. It is again easily seen that $p_\pi^t(v_{t\pi})$ is a root of unity, so $p_\pi^t = b_{t\pi}p_{t\pi} + \sum_{\sigma > t\pi} b_\sigma p_\sigma$ for some root of unity $b_{t\pi}$. After replacing $p_{t\pi}$ above by $b_{t\pi}^{-1}p_\pi^t - \sum_{\sigma>t\pi} b_{t\pi}^{-1} b_\sigma p_\sigma$, we get the desired relation between p_π^t and the product of extremal weight vectors. □

8. Schubert varieties and standard monomial theory

Let k be an algebraically closed field; we will omit the subscript k whenever there is no confusion possible. Let G be the Kac-Moody group corresponding to $\mathfrak{g}$, and, according to the choice of the triangular decomposition of $\mathfrak{g}$, let $B \subset G$ be a Borel subgroup. Fix a dominant weight λ and let $P \supset B$ be the parabolic subgroup of G associated to λ. We identify the dual space V_λ^* with the space of global sections $\Gamma(G/P, \mathcal{L}_\lambda)$ of the line bundle $\mathcal{L}_\lambda := G \times_P k_\lambda$. Let $\phi : G/P \hookrightarrow \mathbb{P}(V_\lambda)$ be the corresponding embedding.

For $\tau \in W/W_\lambda$ denote by $X(\tau) \subset G/P$ the Schubert variety. By abuse of notation, we denote by $\mathcal{L}_\lambda$ and p_π also the restrictions $\mathcal{L}_\lambda|_{X(\tau)}$ and $p_\pi|_{X(\tau)}$. Recall that the linear span of the affine cone over $X(\tau)$ in V_λ is the submodule $V_\lambda(\tau)$. The restriction map $\Gamma(G/P, \mathcal{L}_\lambda) \to \Gamma(X(\tau), \mathcal{L}_\lambda)$ hence induces an injection $V_\lambda^*(\tau) \hookrightarrow \Gamma(X(\tau), \mathcal{L}_\lambda)$.

The following results are well-known, mostly proved using the machinery of Frobenius splitting (Andersen, Kumar, Mathieu, Mehta, Ramanan, Ramanathan), in some special cases proofs had been given before using standard monomial theory (Lakshmibai, Musili, Rajeswari, Seshadri), see for example [6], [16], [17], [18], [19] and [10] for a description of the development.

We provide in the following a sketch of an alternative proof using the path vectors. The proof is in the spirit of standard monomial theory. But since the construction of the basis is no longer part of the inductive machinery, the arguments, used for example in [2] or in [8] for the classical groups, can now be applied in a straightforward way to the general case.

Theorem 6. *Let π be an L-S path of shape λ. The restriction of the section p_π to $X(\tau)$ vanishes if and only if $i(\pi) \not\leq \tau$. Further, the set of path vectors $\{p_\eta \mid i(\eta) \leq \tau\}$ of shape λ forms a basis of $\Gamma(X(\tau), \mathcal{L}_\lambda)$.*

The proof will be by induction on the length $l(\tau)$ of the element. The main point is to show that the injective map $V_\lambda^*(\tau) \hookrightarrow \Gamma(X(\tau), \mathcal{L}_\lambda)$ is in fact an isomorphism. The induction procedure also yields the following results:

Theorem 7. *i) $X(\tau)$ is a normal variety.*

ii) The restriction map $V_\lambda^* = \Gamma(G/B, \mathcal{L}_\lambda) \to \Gamma(X(\tau), \mathcal{L}_\lambda)$ is surjective and induces an isomorphism $V_\lambda^*(\tau) \to \Gamma(X(\tau), \mathcal{L}_\lambda)$. Further, $H^i(X(\tau), \mathcal{L}_\lambda) = 0$ for $i \geq 1$.

iii) For any reduced decomposition $\tau = s_{i_1} \dots s_{i_r}$, the character $\mathrm{Char}\,\Gamma(X(\tau), \mathcal{L}_\lambda)^*$ is given by the Demazure character formula $\mathrm{Char}\,\Gamma(X(\tau), \mathcal{L}_\lambda)^* = \Lambda_{i_1} \dots \Lambda_{i_r} e^\lambda$.

As an immediate consequence we get by Theorem 4:

Corollary 1. *Let λ, ν be dominant weights which are characters of P.*

i) The multiplication map $\Gamma(X(\tau), \mathcal{L}_\lambda) \otimes \Gamma(X(\tau), \mathcal{L}_\nu) \to \Gamma(X(\tau), \mathcal{L}_{\lambda+\nu})$ is surjective.

ii) The multiplication map $S^n\Gamma(X(\tau), \mathcal{L}_\lambda) \to \Gamma(X(\tau), \mathcal{L}_{n\lambda})$ is surjective.

iii) The linear system on $X(\tau)$ given by an ample line bundle on G/P embeds $X(\tau)$ as a projectively normal variety.

Proofs of the theorems. The theorems hold obviously if $X(\tau)$ is a point, i.e., $l(\tau) = 0$. Assume now $l(\tau) \geq 1$, and let α be a simple root such that $\tau > \kappa := s_\alpha \tau$. Denote by $Sl_2(\alpha)$ the corresponding subgroup of G with Borel subgroup $B_\alpha = B \cap Sl_2(\alpha)$. The canonical map $\Psi : Z_\alpha := Sl_2(\alpha) \times_{B_\alpha} X(\kappa) \to X(\tau)$ is birational and has connected fibres. The map induces an injection $\Gamma(X(\tau), \mathcal{L}_\lambda) \hookrightarrow \Gamma(Z_\alpha, \Psi^* \mathcal{L}_\lambda)$.

By the induction hypothesis, we know that $H^i(X(\kappa), \mathcal{L}_\lambda) = 0$ for $i \geq 1$. Since the restriction of $\Psi^* \mathcal{L}_\lambda$ to $X(\kappa)$ is again $\mathcal{L}_\lambda$, the bundle map $Z_\alpha \to \mathbb{P}^1 = Sl_2(\alpha)/B_\alpha$ induces isomorphisms $H^i(Z_\alpha, \Psi^* \mathcal{L}_\lambda) \to H^i(\mathbb{P}^1, \tilde{\Gamma}(X(\kappa), \mathcal{L}_\lambda))$. (Here $\tilde{\Gamma}(X(\kappa), \mathcal{L}_\lambda)$ denotes the vector bundle associated to the B_α-module $\Gamma(X(\kappa), \mathcal{L}_\lambda)$.)

The short exact sequence $0 \to K \to V_\lambda^*(\tau) \to V_\lambda^*(\kappa) = \Gamma(X(\kappa), \mathcal{L}_\lambda) \to 0$ of B_α-modules induces a long exact sequence in cohomology:

$$\ldots \to H^i(\mathbb{P}^1, \tilde{K}) \to H^i(\mathbb{P}^1, \tilde{V}_\lambda^*(\tau)) \to H^i(\mathbb{P}^1, \tilde{\Gamma}(X(\kappa), \mathcal{L}_\lambda)) \to \ldots .$$

Since $V_\lambda^*(\tau)$ is an $Sl_2(\alpha)$-module, the higher cohomology groups vanish for $\tilde{V}_\lambda^*(\tau)$ and hence also for $\tilde{\Gamma}(X(\kappa), \mathcal{L}_\lambda)$. It follows that $H^i(Z_\alpha, \Psi^* \mathcal{L}_\lambda) = 0$ for $i > 0$. Recall that if M is a B_α-module and $\tilde{M}$ the associated vector bundle on $\mathbb{P}^1$, then

$$\Lambda_\alpha \operatorname{Char} M = \operatorname{Char} \Gamma(\mathbb{P}_1, \tilde{M}) - \operatorname{Char} H^1(\mathbb{P}^1, \tilde{M}).$$

Since $H^1(\mathbb{P}^1, \tilde{\Gamma}(X(\kappa), \mathcal{L}_\lambda)) = 0$, it follows that

$$\operatorname{Char} \Gamma(Z_\alpha, \Psi^* \mathcal{L}_\lambda) = \Lambda_\alpha \operatorname{Char} \Gamma(X(\kappa), \mathcal{L}_\lambda).$$

By induction, the character of $\Gamma(Z_\alpha, \Psi^*, \mathcal{L}_\lambda)$ is hence given by the Demazure character formula. Since the same is true for $V_\lambda^*(\tau)$ by Corollary 1 of Theorem 2, the inclusions $V_\lambda^*(\tau) \hookrightarrow \Gamma(X(\tau), \mathcal{L}_\lambda) \hookrightarrow \Gamma(Z_\alpha, \Psi^* \mathcal{L}_\lambda)$ have to be isomorphisms.

Let $\tilde{X}$ be the normalization of $X(\tau)$. Since Z_α is normal, the map Ψ factors into $\tilde{\Psi} : Z_\alpha \to \tilde{X}$ and $f : \tilde{X} \to X(\tau)$. Since $\Gamma(X(\tau), \mathcal{L}_\lambda) \to \Gamma(Z_\alpha, \Psi^* \mathcal{L}_\lambda)$ is an isomorphism, we know that $\Gamma(X(\tau), \mathcal{L}_\lambda) \simeq \Gamma(\tilde{X}, f^* \mathcal{L}_\lambda)$ (for arbitrary ample $\mathcal{L}_\lambda$ on G/P), which implies that $X(\tau)$ is normal.

The normality of $X(\tau)$ has as consequence that $\Psi_*(\mathcal{O}_{Z_\alpha}) = \mathcal{O}_{X(\tau)}$. Further, if $\mathcal{L}$ is an ample line bundle, then the Leray spectral sequence

$$H^p(X(\tau), R^q \Psi_*(\mathcal{O}_{Z_\alpha}) \otimes \mathcal{L}^n) = H^p(X(\tau), R^q \Psi_*(\Psi^* \mathcal{L}^n)) \Rightarrow H^{p+q}(Z_\alpha, \Psi^* \mathcal{L}^n)$$

degenerates for $n \gg 0$, so $R^q \Psi_*(\mathcal{O}_{Z_\alpha}) = 0$ for $q > 0$. It follows that we get isomorphisms in cohomology:

$$H^i(X(\tau), \mathcal{L}_\lambda) \simeq H^i(Z_\alpha, \Psi^* \mathcal{L}_\lambda),$$

which finishes the proof because: $H^i(Z_\alpha, \Psi^* \mathcal{L}_\lambda) = 0$ for all $i > 0$. □

We reformulate Theorem 5 *ii)*: For $\pi = (\kappa_1, \ldots, \kappa_s; 0, a_1, a_2, \ldots, a_{s-1}, 1)$, an L-S path of shape λ, let ℓ be such that $2d | \ell$ and $\ell a_j \in \mathbb{Z}$ for all $j = 1, \ldots, s$. For $\kappa \in W/W_\lambda$ denote by p_κ the extremal weight vector $p_\kappa \in \Gamma(G/B, \mathcal{L}_\lambda)$ of weight $-\kappa(\lambda)$, and denote by $\delta X(\kappa)$ the union of all Schubert varieties $X(\kappa') \subset X(\kappa)$, $\kappa' \neq \kappa$. Theorem 5 *i)+ii)* implies that the restriction of p_π^ℓ to $X(\kappa_1)$ is divisible by $p_{\kappa_1}^{\ell a_1}$ and can be written as $p_{\kappa_1}^{\ell a_1} q_1$. Further, there exists a root of unity a_π such that $q_1 = a_\pi p_{\kappa_2}^{\ell(a_2 - a_1)} \cdots p_{\kappa_s}^{\ell(1 - a_{s-1})}$ plus terms which are higher in the partial ordering. In particular, the restriction of q_1 to $\delta X(\tau_1)$ does not identically vanish. The same arguments prove inductively:

Corollary 2. *There exists a root of unity a_π such that in $\Gamma(X(\kappa_1), \mathcal{L}_{\ell\lambda})$ we have $p_\pi^\ell = p_{\kappa_1}^{\ell a_1} q_1$, where $q_1 \in \Gamma(X(\kappa_1), \mathcal{L}_{(1-a_1)\ell\lambda})$ is such that $q_1|_{\delta X(\kappa_1)} \not\equiv 0$, and $q_1 = p_{\kappa_2}^{\ell(a_2-a_1)} q_2$ in $\Gamma(X(\kappa_2), \mathcal{L}_{(1-a_1)\ell\lambda})$, where $q_2 \in \Gamma(X(\kappa_2), \mathcal{L}_{(1-a_2)\ell\lambda})$ is such that $q_2|_{\delta X(\kappa_2)} \not\equiv 0$, ..., and $q_{r-1} = a_\pi p_{\kappa_r}^{\ell(1-a_{r-1})}$ in $\Gamma(X(\kappa_r), \mathcal{L}_{(1-a_{r-1})\ell\lambda})$.*

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Université Louis Pasteur et Institut Universitaire de France, Institut de Recherche Mathématique Avancée 7, rue René Descartes, F-67084 Strasbourg Cedex, France

E-mail address: `littelma@math.u-strasbg.fr`