Frobenius splitting in characteristic zero
and the quantum Frobenius map

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Introduction

In the representation theory of a semisimple algebraic group $G$, the Schubert varieties $X(w)$, $w$ an element of the Weyl group $W$ of $G$, play a prominent rôle. An important breakthrough in the analysis of the geometry of these subvarieties of the flag variety $G/B$ was the introduction by Mehta-Ramanathan of the notion of a Frobenius split variety and compatibly split subvarieties (for varieties defined over a field of char. $p > 0$). They proved [MR] that $G/B$ (more generally any $G/P$ for a parabolic subgroup $P$) is a Frobenius split variety such that all of the Schubert subvarieties are compatibly split, in particular, one immediately obtains the Kodaira-Kempf vanishing theorem. Using this tool, it was shown for example that Schubert varieties are normal, Cohen-Macaulay and have rational singularities. Moreover, they are projectively normal, projectively Cohen-Macaulay, and are defined by quadratic relations in any embedding given by an ample homogeneous line bundle on $G/B$ (cf. [RR], [Ra1], [Ra2]). Of course, as is well-known, the normality of Schubert varieties is equivalent to the validity of the Demazure’s character formula. Further, the Frobenius splitting was used by Mathieu to give a uniform proof that the category of finite dimensional $G$-representations (over char. $p > 0$) admitting a good filtration is stable under tensor product and more generally under the restriction to the semisimple part of a Levi subgroup (cf. [D], [Ma]).

Earlier, a different way to analyze the geometry of Schubert varieties was suggested by Seshadri and his school. They proposed to construct a standard monomial theory

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for the homogeneous coordinate ring of an embedding \( G/P \hookrightarrow \mathbb{P}(V(\omega)) \) given by the orbit of a highest weight vector in a fundamental representation \( V(\omega) \) (see for example [LS], [LLM] for comments on the development). In this approach, the extremal weight vectors play a prominent role. Using the quantum Frobenius map defined by Lusztig, the second author defined the “\( \ell \)-th root” of a product of extremal weight vectors in the quantum Demazure module \((V_v(\lambda)_w)^*\) at an \( \ell \)-th root of unity \( v \). It turned out that this method presented the perfect tool to develop a standard monomial theory for arbitrary embeddings \( X(w) \hookrightarrow \mathbb{P}(V(\lambda)) \) of Schubert varieties, avoiding all case by case considerations. Many of the results proved by Frobenius splitting methods follow then also naturally by standard monomial theory, see [Li1], [Li2], [LLM].

The aim of this article it to systematically begin unifying these two approaches. For the Borel subgroup \( B \) of \( G \), let \( b \) be its Lie algebra over the complex numbers \( \mathbb{C} \) and \( b^- \) be the opposite Borel subalgebra. We first establish a duality between the algebra \( X \) arbitrary embeddings that this method presented the perfect tool to develop a standard monomial theory for the quantum Demazure module \((V_v(\lambda))\) for comments on the development). In this approach, the extremal weight vectors play a prominent role. Using the quantum Frobenius map defined by Lusztig, the second author defined the “\( \ell \)-th root” of a product of extremal weight vectors in the quantum Demazure module \((V_v(\lambda)_w)^*\) at an \( \ell \)-th root of unity \( v \). It turned out that this method presented the perfect tool to develop a standard monomial theory for arbitrary embeddings \( X(w) \hookrightarrow \mathbb{P}(V(\lambda)) \) of Schubert varieties, avoiding all case by case considerations. Many of the results proved by Frobenius splitting methods follow then also naturally by standard monomial theory, see [Li1], [Li2], [LLM].

Next we assume that \( \ell \) is a prime integer \( \ell \) coprime to 3 if \( G_2 \) is a factor of \( G \) a certain Frobenius homomorphism \( \text{Fr} : U_v(b^-) \to U_{\mathbb{Z}}(b^-) \) and also a certain splitting of it on the “\( n^-\)-part” (which we shall refer to as Frobenius splitting homomorphism) \( \text{Fr}' : U_{\mathbb{Z}}(n^-) \to U_v(n^-) \), where \( U_{\mathbb{Z}}(b^-) := U_{\mathbb{Z}}(b^-) \otimes_{\mathbb{Z}} \mathbb{Z}, n^- \) is the nilradical of \( b^- \), and \( U_{\mathbb{Z}}(n^-), U_v(n^-) \) have meaning similar to that of the corresponding \( b^- \). We extend the definition of \( \text{Fr}' \) to \( U_{\mathbb{Z}}(b^-) \) (cf. Lemma 3). By using the duality mentioned above, we get maps \( \text{Fr}^* : \bigoplus_{\lambda \in P^+} V_{\mathbb{Z}}(\lambda)^* \to \bigoplus_{\lambda \in P^+} V_v(\lambda)^* \) respectively \( \text{Fr}'^* : \bigoplus_{\lambda \in P^+} V_{\mathbb{Z}}(\lambda)^* \to \bigoplus_{\lambda \in P^+} V_v(\lambda)^* \) (cf. Propositions 4 and 5).

Next assume that \( \ell \) is a prime, and show that, after a base change with an algebraically closed field of char. \( p = \ell \), the map \( \text{Fr}'^* \) has a natural interpretation as the \( p \)-th power map on the space of sections \( H^0(G/B, L_\lambda) \), and \( \text{Fr}'^* \) can naturally be interpreted as a splitting of this map. This is our principal result of the paper (cf. Theorem 1). So on the level of quantum groups, the map \( \text{Fr}'^* \) can be considered as a char. zero lift of the Frobenius splitting of \( G/B \) in char. \( p \), and it is exactly this map which has been used in [Li2] to define the “\( \ell \)-th root” of certain sections.

In a subsequent paper we will use Lusztig’s \( \text{Fr} \) (resp. \( \text{Fr}'^* \)) to define the Frobenius map (resp. Frobenius splitting map) at ‘higher cohomology’ level as well and show that \( \text{Fr}'^* \) after base-change provides a canonical (in the sense of Mathieu) splitting of \( G/B \) compatibly splitting all of the Schubert subvarieties. In particular, this will provide a purely algebraic proof (via the quantum groups at roots of unity) of results of Mehta-Ramanathan mentioned above and also various results on the geometry of Schubert varieties mentioned in the first paragraph.
1. A pairing and its quantum analogue

For a complex semisimple Lie algebra \( \mathfrak{g} \), associated to a Cartan matrix \( C = (c_{i,j})_{1 \leq i,j \leq n} \), fix a Borel subalgebra \( \mathfrak{b} \) and a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{b} \). Let \( \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \) be the associated triangular decomposition and set \( \mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h} \). We write \( U, U^+, U^- \) for the enveloping algebras \( U(\mathfrak{g}), U(\mathfrak{n}^+), U(\mathfrak{n}^-) \) respectively.

Let \( \omega_1, \ldots, \omega_n \) be the set of fundamental weights in the weight lattice \( P \) of \( \mathfrak{g} \). Denote by \( P^+ \) the set of non-negative linear combinations \( \mathbb{N}\omega_1 + \ldots + \mathbb{N}\omega_n \), \( \mathbb{N} \) being the set of non-negative integers. We write \( V(\lambda) \) for the complex irreducible representation of highest weight \( \lambda \in P^+ \). As an \( \mathfrak{h} \)-module, \( V(\lambda) \) decomposes into the direct sum \( \bigoplus_{\mu \in P} V(\lambda)_\mu \) of weight spaces. Here \( V(\lambda)_\mu = \{ v \in V(\lambda) \mid hv = \mu(h)v \ \forall h \in \mathfrak{h} \} \).

Let \( \alpha_1, \ldots, \alpha_n \) be the set of simple roots (corresponding to the Borel subalgebra \( \mathfrak{b} \)). For a simple root \( \alpha_i \) choose \( X_i \in \mathfrak{g}_{\alpha_i}, Y_i \in \mathfrak{g}_{-\alpha_i} \) and \( H_i \in \mathfrak{h} \) such that \( [H_i, X_j] = a_{i,j}X_j, [H_i, Y_j] = -a_{i,j}Y_j, [X_i, Y_j] = \delta_{i,j}H_i \). We denote by \( U_Z \) the Kostant form of \( U \) over the ring of integers \( \mathbb{Z} \), generated by the divided powers \( X_i^{(k)} := X_i^k/k! \) and \( Y_i^{(k)} := Y_i^k/k! \), \( k \geq 0 \). Let \( U_Z^+, U_Z^- \) be the corresponding Kostant forms of \( U^+ \) and \( U^- \), generated respectively by \( \{ X_i^{(k)} \} \) and \( \{ Y_i^{(k)} \} \), and let \( U_Z(\mathfrak{b}^-) \) be the Kostant form of \( U(\mathfrak{b}^-) \) generated by \( \{ Y_i^{(k)}(H_i^j) \} \), where \( (H_i^j) = H_i(H_i-1) \cdots (H_i-k+1)/k! \). Then \( U_Z(\mathfrak{b}^-) \subset U_Z \).

For \( i = 1, \ldots, n \), fix a highest weight vector \( \bar{v}_{\omega_i} \in V(\omega_i)_{\omega_i} \) and let \( V_Z(\omega_i) := U_Z\bar{v}_{\omega_i} \) be the corresponding \( U_Z \)-stable \( \mathbb{Z} \)-lattice. For \( \lambda = \sum a_i \omega_i \in P^+ \) denote by \( \bar{v}_{\lambda} \) the vector \( \bar{v}_{\omega_1}^{\otimes a_1} \otimes \cdots \otimes \bar{v}_{\omega_n}^{\otimes a_n} \), and let \( V_Z(\lambda) \) be the lattice

\[
U_Z\bar{v}_{\lambda} := U_Z(\bar{v}_{\omega_1}^{\otimes a_1} \otimes \cdots \otimes \bar{v}_{\omega_n}^{\otimes a_n}) \hookrightarrow V_Z(\omega_1)^{\otimes a_1} \otimes \cdots \otimes V_Z(\omega_n)^{\otimes a_n}.
\]

The dual module \( \text{Hom}_Z(V_Z(\lambda), \mathbb{Z}) \) is denoted by \( V_Z(\lambda)^* \). Setting \( R_Z := \bigoplus_{\lambda \in P^+} V_Z(\lambda)^* \), we have a natural pairing

\[
\Phi : U_Z(\mathfrak{b}^-) \times R_Z \rightarrow \mathbb{Z}, \quad \Phi(u, \xi) := \xi(u.\bar{v}_{\lambda}) \text{ for } u \in U_Z(\mathfrak{b}^-), \xi \in V_Z(\lambda)^*.
\]

Denote by \( U(\mathfrak{b}^-)^* \) the Hopf dual of \( U(\mathfrak{b}^-) \) (under the standard Hopf algebra structure on \( U(\mathfrak{b}^-) \)), i.e., \( U(\mathfrak{b}^-)^* \) is the subspace of linear forms \( f \in \text{Hom}(U(\mathfrak{b}^-), \mathbb{C}) \) for which there exists a two-sided ideal \( I \subset U(\mathfrak{b}^-) \) of finite codimension such that \( f(I) = 0 \).

Recall that \( U(\mathfrak{b}^-)^* \) is again a Hopf algebra. Let \( U_Z(\mathfrak{b}^-)^* \) be the \( \mathbb{Z} \)-submodule of forms \( f \in U(\mathfrak{b}^-)^* \) such that \( f(U_Z(\mathfrak{b}^-)) \subset \mathbb{Z} \). Note that \( U_Z(\mathfrak{b}^-)^* \) is a \( \mathbb{Z} \)-subalgebra of \( U(\mathfrak{b}^-)^* \). The following proposition can be viewed as a certain “half” of the Peter-Weyl type theorem. One may also consider it as an algebraic analogue of the result of Bernstein, Gelfand and Gelfand that the ring \( \mathbb{C}[G/U] \) of regular functions on \( G/U \) is isomorphic to \( \bigoplus_{\lambda \in P^+} V(\lambda)^* \), where \( G \) is the corresponding simply connected complex algebraic group with Borel subgroup \( B \) and unipotent radical \( R_u(B) = U \) having \( \mathfrak{n}^+ \) as its Lie algebra.
Proposition 1.

Φ is a non-degenerate pairing, identifying $R_Z$ as a subalgebra of $U_Z(b^-)^*$:

$$
\xi^\lambda : \xi^\mu(u) := (\xi^\lambda \otimes \xi^\mu)|_{V_Z(\lambda + \mu)}(u, \bar{\nu}_{\lambda + \mu})
$$

for $\xi^\lambda \in V_Z(\lambda)^*$, $\xi^\mu \in V_Z(\mu)^*$ and $u \in U_Z(b^-)$.

Proof. Denote by $Q$ the root lattice and set $Q^+ = N\alpha_1 + \ldots + N\alpha_n$. The Kostant form $U_Z^0$ of the enveloping algebra $U(\mathfrak{h})$ has as basis the monomials $(H_1^k) \cdots (H_n^k)$, $k_i \in \mathbb{N}$. Fix a $\mathbb{Z}$-basis $B$ of $U_Z^-$ such that for any $\beta \in Q^+$ the elements in $B_{\beta} := U_{Z,-\beta} \cap B$ form a basis of the weight space $V_Z(\lambda)_{\lambda-\beta}$.

Suppose now that $u \in U_Z(b^-)$ is such that $\Phi(u, R_Z) = 0$. We can write $u = \sum_{\beta \in Q^+} \sum_{b \in B_{\beta}} bh_b$, where $h_b \in U_Z^0$. To prove $u = 0$, it is sufficient to show that for any $u \neq 0$, $u \bar{\nu}_{\lambda} \neq 0$ for some $\lambda \in P^+$. It is well known that if $\beta$ is fixed, then for $\lambda \gg 0$ (i.e. $a_i \gg 0$ for all $i = 1, \ldots, n$), the vectors $b \bar{\nu}_{\lambda}$, $b \in B_{\beta}$, form a basis of the weight space $V_Z(\lambda)_{\lambda-\beta}$. So if we choose $\lambda$ big enough, $u \bar{\nu}_{\lambda} = 0$ implies $\lambda(h_b) = 0$ for all $b$ and all $\lambda \gg 0$. But this is possible only if $h_b = 0$.

Consider $f = \sum \xi^\lambda$. Among the $\xi^\lambda \neq 0$ fix $\xi^{\lambda_o}$ such that $\lambda_o$ is maximal in the lexicographic ordering, i.e., if $\lambda_o = \sum a_i \omega_i$ and $\lambda = \sum b_i \omega_i$ is such that $\xi^\lambda \neq 0$, then there exists a $j \leq n$ such that $a_i = b_i$ for $i < j$ and $a_j > b_j$. Set $H_{\lambda_o} := \prod_{i=1}^n (H_i^{a_i})$. Note that $H_{\lambda_o} \bar{\nu}_{\lambda_o} = \bar{\nu}_{\lambda_o}$ and $H_{\lambda_o} \bar{\nu}_{\lambda} = 0$ for all $\lambda$ such that $\xi^\lambda \neq 0$, $\lambda \neq \lambda_o$. Since $V_Z(\lambda_o) = U_Z^- \bar{\nu}_{\lambda_o}$, we can find $u \in U_Z^-$ such that $\xi^{\lambda_o}(u \bar{\nu}_{\lambda_o}) \neq 0$. It follows that $\Phi(uH_{\lambda_o}, f) = \sum \xi^\lambda(uH_{\lambda_o} \bar{\nu}_{\lambda}) = \xi^{\lambda_o}(u \bar{\nu}_{\lambda_o}) \neq 0$. This proves that $\Phi$ is non-degenerate.

To see that $R_Z$ forms a subalgebra of $U_Z(b^-)$, note that the co-product $\Delta$ induces a natural $U_Z(b^-)$-module structure on $V_Z(\lambda) \otimes V_Z(\mu)$. By the definition of the product, we have $\xi^\lambda \cdot \xi^\mu(u) = \xi^\lambda \otimes \xi^\mu(\Delta(u))$. For $\Delta(u) = \sum u_1 \otimes u_2$ we have $\xi^\lambda \otimes \xi^\mu(\Delta(u)) = \sum \xi^\lambda(u_1 \bar{\nu}_{\lambda}) \cdot \xi^\mu(u_2 \bar{\nu}_{\mu})$.

Now the map $u \bar{\nu}_{\lambda + \mu} \mapsto u(\bar{\nu}_{\lambda} \otimes \bar{\nu}_{\mu})$, $u \in U_Z(b^-)$, induces an isomorphism between $V_Z(\lambda + \mu)$ and the $U_Z$-submodule $U_Z(b^-)(\bar{\nu}_{\lambda} \otimes \bar{\nu}_{\mu})$ of $V_Z(\lambda) \otimes V_Z(\mu)$. The restriction map induces hence a map $\text{res} : V_Z(\lambda)^* \otimes V_Z(\mu)^* \rightarrow V_Z(\lambda + \mu)^*$. It follows that $\xi^\lambda \cdot \xi^\mu(u) = \text{res}(\xi^\lambda \otimes \xi^\mu)(u \bar{\nu}_{\lambda + \mu})$.

Remark 1. By using the Peter-Weyl theorem, we get an isomorphism $\mathbb{C}[G/U] \simeq \bigoplus_{\lambda \in P^+} V(\lambda)^*$. Let $B^-$ be the opposite Borel subgroup (Lie $B^- = b^-$). Since $B^-$ is open and dense in $G/U$, we have an inclusion $\mathbb{C}[G/U] \hookrightarrow \mathbb{C}[B^-]$. But by [J, Part I, §§7.10 and 7.18], we have $\mathbb{C}[B^-] \simeq U(b^-)^*$, and hence we have $\bigoplus_{\lambda \in P^+} V(\lambda)^* \hookrightarrow U(b^-)^*$. So this gives an alternative geometric derivation of the above proposition over $\mathbb{C}$.

We have a similar construction for the quantum group $U_q := U_q(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$. Let $d_1, \ldots, d_n \geq 1$ be minimal integers such that $(d_i c_{i,j})$ is a symmetric matrix, and let $\bar{Z}$ be the ring obtained from $Z$ by adjoining all roots of unities.
We fix a positive integer \( \ell \). If \( \ell \) is odd, then let \( \phi \) be the \( \ell \)-cyclotomic polynomial, and if \( \ell \) is even, then let \( \phi \) be the \( 2\ell \)-cyclotomic polynomial. Let \( A = \mathbb{Z}[q, q^{-1}] \) be the ring of Laurent polynomials and fix a homomorphism \( A/\langle \phi \rangle \twoheadrightarrow \tilde{\mathbb{Z}} \), where \( \langle \phi \rangle \) is the ideal in \( A \) generated by \( \phi \). Denote by \( v \) the image of \( q \) in \( \tilde{\mathbb{Z}} \).

We denote the generators of the quantum group \( U_q \) over \( \mathbb{C}(q) \) by \( E_i, F_i, K_i^\pm \), and let \( U_A \) be the Lusztig form of \( U_q \) over \( A \) generated by the divided powers \( E_i^{(m)} := E_i^m/[m]! \) and \( F_i^{(m)} := F_i^m/[m]! \) and the \( K_i^{\pm 1} \) \( [Lu3, \S 1] \). Recall that the Gaussian numbers \([m]_i := (q^{d_i} - q^{-d_i})/(q^{d_i} - q^{-d_i})\), and \([m]_i := [1]_i \ldots [m]_i \).

We denote by \( U_A^+, U_A^-, U_A^0 \) the subalgebras of \( U_A \) generated by the \( E_i^{(m)} \), the \( F_i^{(m)} \), and the \{\( K_i^\pm, [K_{i/0}] \)\} respectively for \( 1 \leq i \leq n \) and \( m \in \mathbb{N} \). Recall that the latter is defined by

\[
\left[ K_i ; c \right] \left[ m \right] := \prod_{s=1}^{m} K_i q^{(c-s+1)d_i} - K_i^{-1} q^{(c+s-1)d_i} q^{s d_i} - q^{-s d_i} , \text{ for } c \in \mathbb{Z} \text{ and } m \in \mathbb{N},
\]

and \( U_A^0 \) has as a basis the monomials of the form \( \prod_{s=1}^{m} \left[ K_{i/0} \right] \), where the \( m_i \) are non-negative integers and \( e_1, \ldots, e_n \in \{0, 1\} \) \( [Lu3, \mathrm{Theorem} \ 6.7 \ (c)] \). Let \( U_A(b^-) \) be the subalgebra generated by the \( F_i^{(m)} \), \( K_i^\pm \) and the \( [K_{i/0}] \). The following statements can be found in \([Lu2] \) or \([CP] \), or can be easily deduced from \([Lu3, \S 6.4] \).

**Lemma 1.**

a) \( \left[ K_{i/0} \right] \left[ K_{i/0}^{-m} \right] = \left[ m+t \right] \left[ K_{i/0} \right] \).

b) \( \left[ K_{i/0} \right] \left[ K_{t/0} \right] = \sum_{j=0}^t \left[ j \right]_{i} \cdot q^{t(t-c)} K_{j/0}^{t-j} \left[ K_{i/0}^{-t} \right] \), for any \( t \leq t' \).

c) \( \left[ K_{i/0} \right] \left[ K_{t/0} \right] = \sum_{s=0}^t \left[ s \right]_{i} q^{s(t-c)} K_{s/0}^{t-s} \left[ K_{i/0}^{-s} \right] \), for \( c > 0 \).

d) \( \left[ K_{i/0} \right] \left[ K_{t/0} \right] = \sum_{s=0}^t \left[ s \right]_{i} q^{s(t-c)} K_{s/0}^{t-s} \left[ K_{i/0}^{-s} \right] \), for \( c > 0 \).

e) \( \left[ K_{i/0} \right] \left[ K_{t/0} \right] = \sum_{j=0}^t \left[ j \right]_{i} \cdot q^{t(t-s)} K_{j/0}^{s} \left[ K_{t-s/0} \right] \), for \( t \in \mathbb{Z} \) and \( j \in \mathbb{N} \).

We denote by \( U_v, U_v^+, U_v^-, U_v(b^-) \) the algebras over \( \tilde{\mathbb{Z}} \) obtained from the corresponding forms defined over \( A \) by base change \( A \rightarrow A/\langle \phi \rangle \twoheadrightarrow \tilde{\mathbb{Z}} \). For \( \lambda \in P^+ \) let \( V_q(\lambda) \) be the irreducible representation of \( U_q \) over \( \mathbb{C}(q) \) with highest weight \( \lambda \). As in the classical case, we fix for \( i = 1, \ldots, n \) a highest weight vector \( v_{\omega_i} \in V_q(\omega_i)_{\omega_i} \), and let \( V_A(\omega_i) = U_A v_{\omega_i} \) be the corresponding \( A \)-lattice. For \( \lambda = \sum a_i \omega_i \in P^+ \) denote by \( v_\lambda \) the vector \( v_\lambda^a = \otimes v_{\omega_i}^a \), and let \( V_A(\lambda) \) be the lattice

\[
U_A v_\lambda := U_A(\otimes v_{\omega_i}^a) \twoheadrightarrow V_A(\omega_i)^{v_{\lambda}} \otimes \cdots \otimes V_A(\omega_n)^{v_{\lambda}}.
\]

Then \( U_A v_\lambda \) is indeed \( A \)-free. We denote by \( V_v(\lambda) \) the corresponding representation \( V_A(\lambda) \otimes_A \tilde{\mathbb{Z}} \) of \( U_v \). Recall that \( K_i \) acts on a weight vector \( v_\mu \in V_v(\lambda)_{\mu} \) by multiplication with \( v_\mu^{d_i} \mu(\ell_i) \). As in the classical case, let \( R_v \) denote the direct sum \( \bigoplus_{\lambda \in P^+} V_v(\lambda) \), where \( V_v(\lambda)^* := \text{Hom}_{\tilde{\mathbb{Z}}}(V_v(\lambda), \tilde{\mathbb{Z}}) \).
Let \( \ell_i \in \mathbb{N} \) be minimal such that \( d_i \ell_i \equiv 0 \mod \ell \) (recall: \( d_i \in \{1, 2, 3\} \)). Then \( v^{d_i} \) is a primitive \( \ell_i \)-th root of unity if \( \ell \) is odd and a primitive \( 2\ell_i \)-th root of unity if \( \ell \) is even. Note that in either case \( K_i^{\ell_i} = 1 \) in \( U_v \) (cf. [Lu2, Lemma 4.4 (a)]), as can be easily seen from the following relation in \( U_0^A \):

\[
\prod_{j=1}^{\ell_i} (q^{d_{i,j}} - q^{-d_{i,j}}) = \prod_{j=1}^{\ell_i} (K_i q^{d_i(j+1)} - K_i^{-1} q^{d_i(j-1)}).
\]

If \( \ell \) is odd, then \( K_i^{\ell_i} v_{\mu} = v_{\mu} \) for all \( \mu \in P \) and all weight vectors \( v_{\mu} \), in particular, \( K_i^{\ell_i} \) are in the center of \( U_v(b^-) \). Denote by \( J' \) the ideal of \( U_v(b^-) \otimes \mathbb{Z} C \) generated by \( (K_i^{\ell_i} - 1), i = 1, \ldots, n \), and let \( J \) be the ideal \( J' \cap U_v(b^-) \) in \( U_v(b^-) \). Observe that \( U_v(b^-) \) embeds inside \( U_v(b^-) \otimes \mathbb{Z} C \) since \( U_v(b^-) \) is \( \mathbb{Z} \)-free.

Define the pairing \( \Phi_v : U_v(b^-) \times R_v \to \mathbb{Z} \) by \( (u, \xi^\lambda) \mapsto \xi^\lambda(u v_\lambda) \) for \( u \in U_v(b^-) \) and \( \xi^\lambda \in V_v(\lambda)^* \).

**Proposition 2.**

a) If \( \ell \) is odd, then the pairing \( \Phi_v \) has radical precisely equal to \( (J, 0) \). The induced pairing \( \Phi'_v : U_v(b^-)/J \times R_v \to \mathbb{Z} \) is hence non-degenerate.

b) If \( \ell \) is even, then the pairing \( \Phi_v \) is non-degenerate.

c) The induced map \( \psi_v : R_v \to U_v(b^-)^* \) is injective, and the image is a subalgebra of \( U_v(b^-)^* \), where the multiplication of \( \xi^\lambda \in V_v(\lambda)^* \) and \( \xi^\mu \in V_v(\mu)^* \) is given by:

\[
(\xi^\lambda \cdot \xi^\mu)(u) := (\xi^\lambda \otimes \xi^\mu)(u v_{\lambda + \mu}) \quad \text{for } u \in U_v(b^-).
\]

**Proof.** Let \( v_\lambda \in V_v(\lambda) \) be the fixed highest weight vector. Recall that \([K_i; \ell_i, m_0] v_\lambda = [\lambda(H_i)] v_\lambda\); in particular, \([K_i; \ell_i, m_0] v_\lambda = 0 \) if \( m > \lambda(H_i) \). The same argument as above in the classical case shows that the map \( \psi_v : R_v \to U_v(b^-)^* \) is injective.

Suppose now \( u \in U_v(b^-) \) is such that \( \Phi_v(u, R_v) = 0 \). We can find linearly independent \( u_1, \ldots, u_t \in U_v^- \) and some \( h_1, \ldots, h_t \in U_v^0 \) such that \( u = \sum_{i=1}^t u_i h_i \). To say that \( u \) is in the radical of the pairing is equivalent to saying that \( u v_\lambda = 0 \) for all highest weight vectors \( v_\lambda \in V_v(\lambda), \lambda \in P^+ \). Since the \( u_i \) are linearly independent, the vectors \( u_i v_\lambda \) are linearly independent for \( \lambda \gg 0 \). So \( u v_\lambda = 0 \) for all \( \lambda \gg 0 \) is equivalent to \( h_i v_\lambda = 0 \) for all \( i = 1, \ldots, t \) and all \( \lambda \gg 0 \). In the next lemma we determine a basis of \( U_v^0 \otimes \mathbb{Z} C \) to find those \( h \in U_v^0 \otimes \mathbb{Z} C \) satisfying this property.

**Lemma 2.**

The complex algebra \( U_v^0 \otimes \mathbb{Z} C \) has as basis the monomials \( \prod_{i=1}^n ([K_i; 0] K_i^{e_i}) \), where \( m_i \in \mathbb{N} \) and \( 0 \leq e_i < 2\ell_i \).

**Proof.** If \( \ell_i = 1 \) (which can of course only happen if \( \ell = 2, d_i = 2 \) or \( \ell = 3, d_i = 3 \) or \( \ell = 1 \)), then the corresponding \( i \)-part of the monomial is exactly of the form \([K_i; 0] \) or
We show that the monomials listed in the proposition form a subalgebra. Recall that \([\text{Lu}1, \text{Lemma } 34.1.2]\) \([x^j]_i = 0\) unless \(\ell_i\) divides \(j\), and \([x^j]_{\ell_i} = \binom{x}{j}\), where \(\binom{x}{j}\) is the ordinary binomial coefficient. Further, \(v^{d_i \ell_i xy} = v^{(r\ell_i)xy}\) for some \(r \in \mathbb{N}\), hence it is equal to \(\pm 1\). By specializing the relation in Lemma 1 e) at \(q = v\), we get:

\[
\begin{bmatrix}
K_i; 0 \\
x\ell_i
\end{bmatrix}
\begin{bmatrix}
K_i; 0 \\
y\ell_i
\end{bmatrix} = \sum_{j=0}^x \binom{x}{j} (x + y - j) (\pm 1) K_i^{-j\ell_i} \begin{bmatrix}
K_i; 0 \\
(x + y - j)\ell_i
\end{bmatrix},
\]

which proves that the monomials of this type span a subalgebra of \(U^0_v\). In \(U^0_A\) we have in addition the relation: \((0 < r < \ell_i, m \geq 0)\)

\[
\begin{bmatrix}
K_i; 0 \\
m\ell_i + r
\end{bmatrix} \prod_{s=1}^{r + m\ell_i} (q^{sd_i} - q^{-sd_i}) = \begin{bmatrix}
K_i; 0 \\
m\ell_i
\end{bmatrix} \prod_{s=1}^{r + m\ell_i} (K_i q^{d_i(-s+1)} - K_i^{-1} q^{d_i(s-1)}).
\]

If we specialize at \(q = v\), then \(v^{md_i \ell_i} = v^{-md_i \ell_i} = \pm 1\). Since this term occurs on both sides, we can cancel it and get:

\[
\begin{bmatrix}
K_i; 0 \\
m\ell_i + r
\end{bmatrix} \prod_{s=1}^r (v^{sd_i} - v^{-sd_i}) = \begin{bmatrix}
K_i; 0 \\
m\ell_i
\end{bmatrix} \prod_{s=1}^r (K_i v^{d_i(-s+1)} - K_i^{-1} v^{d_i(s-1)}).
\]

Note that \(\prod_{s=1}^r (v^{sd_i} - v^{-sd_i}) \neq 0\). Since \(K_i^{2\ell_i} = 1\), this implies that we can express \([K_i; 0]\) over \(C\) as a product of \([K_i; 0]_{m\ell_i + r}\) with a linear combination of 1, \(K_i, \ldots, K_i^{2\ell_i - 1}\). The linear independence of the monomials follows from the description of the basis for \(U^0_A\) above.

\[\square\]

**Proof of Proposition 2, continuation.** Let \(h \in U^0_v\) be such that \(hv_\lambda = 0\) for all \(\lambda \gg 0\). We write \(h\) (viewed as an element of \(U^0_v \otimes_\mathbb{Z} \mathbb{C}\)) as a linear combination

\[h = \sum_{m \in \mathbb{N}^n} \left( \sum_{i=1}^n \begin{bmatrix} K_i; 0 \\ \ell_i m_i \end{bmatrix} \right) \left( \sum_{\underline{e} \in \mathbb{N}^n} b_{m, \underline{e}} K_1^{e_1} \cdots K_n^{e_n} \right),\]

where \(m := (m_1, \ldots, m_n)\) and similarly \(\underline{e}\).

If \(v_\lambda \in V_\lambda(\lambda)\) is a highest weight vector of weight \(\lambda = \sum_{i=1}^n (a_i \ell_i + r_i) \omega_i\) with \(0 \leq r_i < \ell_i\), then, by [Lu1, Lemma 34.1.2],

\[
\begin{bmatrix}
K_i; 0 \\
\ell_i m_i
\end{bmatrix} v_\lambda = \begin{bmatrix} a_i \ell_i + r_i \\ \ell_i m_i \end{bmatrix} v_\lambda = \pm \begin{bmatrix} a_i \\ m_i \end{bmatrix} v_\lambda.
\]

From this it follows easily that \(hv_\lambda = 0\) for all \(\lambda \gg 0\) is equivalent to the condition

\[
\sum_{\underline{e}} b_{m, \underline{e}} K_1^{e_1} \cdots K_n^{e_n} v_\lambda = 0\]

for all \(m\) and all \(\lambda \gg 0\).
Suppose now we have such an element \( h = \sum_{\ell} b_\ell K_i^{e_1} \cdots K_n^{e_n} \neq 0 \) and \( hv_\lambda = 0 \) for all \( \lambda \gg 0 \). Since \( K_{2\ell_i} = 1 \), this is equivalent to saying that \( hv_\lambda = 0 \) for all \( \lambda = \sum_{i=1}^n a_i \omega_i \) such that \( 0 \leq a_i < 2\ell_i \).

The \( \mathbb{C} \)-subalgebra \( K \) of \( U_v^0 \otimes \mathbb{Z} \mathbb{C} \) generated by the \( K_i \) can be viewed as the group algebra of the group \( \prod_{i=1}^n \mathbb{Z}/2\ell_i \mathbb{Z} \). If \( \ell \) is odd, then \( \nu^{d_i} \) is a primitive \( \ell_i \)-th root of unity. The one-dimensional representations provided by the action of the \( K_i \)'s on the highest weight vectors in \( V_v(\lambda) \), \( \lambda = \sum_{i=1}^n a_i \omega_i \), \( 0 \leq a_i < 2\ell_i \), hence does not give a complete list of all irreducible representations of \( K \). The intersection of the kernels of these representations is the subalgebra generated by \( (K_{2\ell_i} - 1) \).

If \( \ell \) is even, then \( \nu^{d_i} \) is a primitive \( 2\ell_i \)-th root of unity. The one-dimensional representations provided by the action of the \( K_i \) on the highest weight vectors in \( V_v(\lambda) \), \( \lambda = \sum_{i=1}^n a_i \omega_i \), \( 0 \leq a_i < 2\ell_i \), hence give a complete list of all irreducible representations, so \( h = 0 \).

The description of the multiplication can be proved as in the classical case. ○

2. The Frobenius maps

We recall in this section the definition of the quantum Frobenius maps \( Fr \) and \( Fr' \) defined by Lusztig on \( U_v^- \) respectively \( U^-_Z := U_v^- \otimes \mathbb{Z} \hat{\mathbb{Z}} \). Fix a positive integer \( \ell \). To simplify the arguments we assume that \( \ell \) is odd, and if \( g \) has simple factors of type \( G_2 \), then we assume \( \ell \) to be coprime to 3 in addition. Note that these conditions imply \( \ell_i = \ell \) for all \( i \); we will make some remarks at the end of this section concerning the cases \( \ell = 2, 3 \).

G. Lusztig has constructed two algebra homomorphisms (Theorem 35.1.7 and 35.1.8 in [Lu1]):

\[
Fr : U_v^- \rightarrow U^-_Z \quad \text{and} \quad Fr' : U^-_Z \rightarrow U^-_v,
\]

which are defined on the generators by

\[
Fr(F_i^{(k)}) := \begin{cases} 
0 & \text{if } \ell \nmid k \\
y_i^{(k/\ell)} & \text{if } \ell \mid k
\end{cases}
\]

respectively \( Fr'(Y_i^{(k)}) := F_i^{(\ell k)} \).

The composition \( Fr \circ Fr' \) is obviously the identity map on \( U^-_Z \). One can of course similarly define \( Fr : U_v^+ \rightarrow U^+_Z \) and \( Fr' : U^+_Z \rightarrow U^+_v \). The map \( Fr \) can be extended to an algebra homomorphism \( Fr : U_v \rightarrow U^-_Z \) (see [Lu1, Theorem 35.1.9] or [Lu3, Theorem 8.10, §8.11 and Corollary 8.14]):

**Proposition 3.** The map defined by \( u \mapsto Fr(u) \) for \( u \in U_v^- \) or \( u \in U_v^+ \) and

\[
K_i \mapsto 1, \quad \begin{bmatrix} K_i; 0 \\ m \end{bmatrix} \mapsto \begin{cases} 
0 & \text{if } \ell \nmid m \\
\left( \frac{H_i}{m/\ell} \right) & \text{if } \ell \mid m
\end{cases} \quad i = 1, \ldots, n,
\]

extends the Frobenius maps for \( U_v^- \) and \( U_v^+ \) to a surjective \( \hat{\mathbb{Z}} \)-algebra homomorphism \( Fr : U_v \rightarrow U^-_Z \). Moreover, \( Fr \) is a Hopf algebra homomorphism.

The map \( Fr' \) can *not* be extended to a homomorphism defined on \( U^-_Z \). Though, we can extend it to a homomorphism defined on \( U^-_Z(b^-) \), the price for the extension being that
the range $U_v(b^-)$ is to be replaced by $U_v(b^-)/J$. Here $J$ is the ideal defined in the last section (Proposition 2).

**Lemma 3.** The map defined by $u \mapsto Fr'(u)$ for $u \in U^-_\mathbb{Z}$ and $(H_{m}) \mapsto \left[ K_{i}; 0 \right]_{m}$ extends $Fr'$ to a $\mathbb{Z}$-algebra homomorphism (again denoted by) $Fr' : U^-_\mathbb{Z}(b^-) \to U_v(b^-)/J$.

We refer to $Fr'$ as the Frobenius splitting homomorphism.

**Proof.** Recall that $\nu^\ell = 1$, $\left[ x^\ell \right]_j = 0$ unless $\ell$ divides $j$ and $\left[ x^\ell \right]_{j\ell} = \left( \frac{x}{j} \right)$. Further, $K_i^\ell = 1$ in $U_v(b^-)/J$. Hence Lemma 1e) implies that

$$Fr'(\left( H_i \atop x \right))Fr'(\left( H_i \atop y \right)) = \left[ K_i; 0 \right]_{x^\ell} \left[ K_i; 0 \right]_{y^\ell} = \sum_{j=0}^x \left( \begin{array}{c} x \\ j \end{array} \right) \left( x + y - j \right) \left( \begin{array}{c} K_i; 0 \\ \ell(x+y-j) \end{array} \right)$$

$$= Fr'(\sum_{j=0}^x \left( \begin{array}{c} x \\ j \end{array} \right) \left( x + y - j \right) \left( \begin{array}{c} H_i \\ x + y - j \end{array} \right))$$

$$= Fr'(\left( H_i \atop x \right))Fr'(\left( H_i \atop y \right)).$$

Now, for $y \geq 0$, we have:

$$Fr'(\left( H_i + y \atop x \right)) = Fr'(\sum_{s=0}^x \left( \begin{array}{c} x \\ s \end{array} \right) \left( H_i \atop x-s \right)) = \sum_{s=0}^x \left[ y^\ell \atop s^\ell \right]_{i} \left[ K_i; 0 \atop \ell(x-s) \right] = \left[ K_i; y^\ell \atop \ell x \right],$$

because the other terms in the expression (Lemma 1e) for $\left[ K_i; y^\ell \atop \ell x \right]$ vanish. Similarly, for $y > 0$, we get:

$$Fr'(\left( H_i - y \atop x \right)) = Fr'(\sum_{s=0}^x (-1)^s \left( \begin{array}{c} y + s - 1 \\ s \end{array} \right) \left( H_i \atop x-s \right))$$

$$= \sum_{s=0}^x (-1)^s \left( y + s - 1 \right) \left[ K_i; 0 \atop \ell(x-s) \right]$$

$$= \sum_{s=0}^x (-1)^s \left[ y^\ell + sl - 1 \atop s^\ell \right]_{i} \left[ K_i; 0 \atop \ell(x-s) \right].$$

To prove the last equality, note that $(-1)^s = (-1)^s$, and (see [Lu1, Lemma 34.1.2])

$$\left[ y^\ell + sl - 1 \atop s^\ell \right]_{i} = (y^s + r - 1) \left[ \ell - 1 \atop r \right]_{i} = (-1)^s. Suppose s' = s + r with 0 < r < \ell. Note that

$$\left[ y^\ell + sl - 1 \atop s^\ell \right]_{i} = \left( y^s + r \right) \left[ r - 1 \atop r \right] = 0,$$

so Lemma 1 gives (for $y > 0$):

$$Fr'(\left( H_i - y \atop x \right)) = \sum_{s'=0}^{x} (-1)^{s'} \nu^{d_i y (x-s')} \left[ y^\ell + s' - 1 \atop s' \right]_{i} \left[ K_i; 0 \atop x\ell - s' \right] = \left[ K_i; -\ell y \atop x\ell \right].$$

From this we conclude that (cf. [Lu3, §6.5])

$$Fr'(\left( H_i \atop x \right))Fr'(\left( y_{j}^{(y)} \atop x\ell \right)) = \left[ K_i; 0 \atop x\ell \right] F_j(y^\ell) = F_j(y^\ell) \left[ K_i; -y^\ell c_{i,j} \atop x\ell \right] = Fr'(\left( y_{j}^{(y)} \atop x \right)) Fr'(\left( H_i - y_{c_{i,j}} \atop x \right)).$$
which shows that the map respects the defining relations between the generators of $U_{Z}(b^{-})$.

\[ \text{Remark 2.} \] The assumption that $\ell$ is coprime to 3 if $g$ admits simple factors of type $G_2$ is not necessary for Proposition 3 and Lemma 3. Actually, the construction makes sense for arbitrary $\ell$, but we have to redefine the maps; for details see [Lu1, Chapter 35]. In the following we mainly concentrate in the remarks on $\ell = 2, 3$, but, with the appropriate adaptions (similarly to those in [L2]), the constructions hold also in the general case.

As before, let $\ell_i$ be minimal such that $d_i \ell_i \equiv 0 \mod \ell$, and denote by $C^\#$ the matrix $(c_{i,j} \ell_j / \ell_i)$. This is the Cartan matrix of the root system having the roots $\alpha_i^\# := \ell_i \alpha_i$ as simple roots and $H_i^\# := H_i / \ell_i$ as co-roots. Its weight lattice is the subset $P^\# := \{ \lambda \in P \mid \lambda(H_i) \in \ell_i \mathbb{Z} \forall i \}$ of $P$. Note if $\mu \in P^\# \subset P$ and $v_\mu$ is a weight vector in a $U_v^0$-representation, then:

\[
\begin{bmatrix}
K_i \ k \ell_i \\
m \ell_i
\end{bmatrix}_i v_\mu = \begin{bmatrix} \mu(H_i) + k \ell_i \\ m \ell_i \end{bmatrix}_i v_\mu = \begin{bmatrix} \ell_i \mu(H_i^\#) + k \ell_i \\ m \ell_i \end{bmatrix}_i v_\mu = \begin{bmatrix} \mu(H_i^\#) + k \\ m \end{bmatrix}_i v_\mu.
\]

Denote by $g^\#$ the corresponding Lie algebra and let $U^\#$ be its enveloping algebra. We use the notation $X_i^\#, Y_i^\#$ and $H_i^\#$ for the generators. If $g$ is simply laced or $\ell$ is a prime $> 3$, then $C^\# = C$. But if $\ell = 3$, then $C^\#$ is obtained from $C$ by transposing the $2 \times 2$ submatrices corresponding to simple factors of type $G_2$. If $\ell = 2$, then the same has to be applied for simple factors of type $F_4, B_n$ and $C_n$. The Frobenius homomorphisms $\text{Fr} : U_v^- \to U_{Z}^\#(b^\#,-)$ respectively $\text{Fr}' : U_{Z}^\#(b^\#,-) \to U_v^-$ are defined by

\[
\text{Fr}(F_i^{(k)}) := \begin{cases}
0 & \text{if } \ell_i \mid k; \\
Y_i^{(k/\ell_i)} & \text{if } \ell_i \nmid k;
\end{cases}
\text{ respectively } \text{Fr}'(Y_i^{(k)}) := F_i^{(\ell,k)}.
\]

If $\ell = 3$, then we extend the Frobenius map to a homomorphism $\text{Fr} : U_v^- (b^-) \to U_{Z}^\#(b^\#,-)$ by setting $\text{Fr}(K_i) = 1$, $\text{Fr}(K_i^{0,m}) = [H_i^\#]_{m/\ell_i}$ if $\ell_i$ divides $m$ and $\text{Fr}(K_i^{0,m}) = 0$ otherwise. Similarly, one can extend $\text{Fr}'$ to a homomorphism $U_{Z}^\#(b^\#,-) \to U_v^- (b^-)/J$ by setting $\text{Fr}(K_i^{0,m}) = [K_i^{0,m}]$. The details of the proof are left to the reader.

The definitions of $\text{Fr} : U_v^- \to U_{Z}^\#$ and $\text{Fr}' : U_{Z}^\# \to U_v^-$ given above make sense for arbitrary positive integer $\ell$. To avoid problems with the definition of the extensions for $\ell = 2$, we assume that $\ell = 2d$, where $d$ is the smallest common multiple of $d_1, \ldots, d_n$. Since $\ell_i = \ell / d_i = 2(d/d_i)$, we know that all the $\ell_i$ are even. Denote by $(U_v^0)_{ev}$ the subalgebra of $U_v^0$ generated by $[K_i^{0,m}]$ and $K_i^m$, $m$ even, and $K_i [K_i^{0,m}]$ for $m$ odd, and let $(U_v^-)_{ev}$ be the subalgebra of $U_v^-$ spanned by the monomials of weight $-2\beta, \beta \in Q^+$. Let $(U_v^-)_{ev}$ be the subalgebra of $U_v^- (b^-)$ generated by $(U_v^-)_{ev}$ and $(U_v^0)_{ev}$. Note that $\text{Fr}'(Y_i^{(k)}) = F_i^{(\ell,k)} \in (U_v^-)_{ev}$ because the $\ell_i$ are even.

Using Lemma 1, it is easy to verify that $(U_v^-)_{ev}$ is spanned by the elements of the form $u \prod_{i=1}^n ([K_i^{0,m}] K_i^{e_i})$, where $u \in (U_v^-)_{ev}$, $m_i \in \mathbb{N}$ and $e_i \in \{0, 1\}$ with $m_i + e_i$ even.
The elements $(K^e_i - 1)$ are in the center of the even subalgebra. As in the odd case, let $J'$ be the ideal of $U_v(b^-)_{ev} \otimes \mathbb{Z} \mathbb{C}$ generated by the elements $(K^e_i - 1)$, $i = 1, \ldots, n$, and let $J$ be the (two sided) ideal $J' \cap U_v(b^-)_{ev}$.

Denote by $V_v(\lambda)_{ev}$ the direct sum $\bigoplus_{\mu} V_v(\lambda)_{ev}$ of all weight spaces corresponding to the weights of the form $\mu = \lambda - 2\beta$, $\beta \in \mathbb{Z}^+$. and set $R_{v, ev} := \bigoplus_{\lambda \in \mathbb{Z}^+} (V_v(\lambda)_{ev})^*$. Proposition 2 can then be reformulated as: The pairing $\Phi_{J, u, \xi}$ to an algebra homomorphism $\psi_{v, ev}$.

In particular, the induced map $\psi_v : U_{v, ev} \to (U_v(b^-)_{ev})^*$ is injective, and the image is a subalgebra of $(U_v(b^-)_{ev})^*$. The Frobenius maps can also be extended correspondingly: the map defined by $u \mapsto Fr(u)$ for $u \in U_{v, ev}$, $K^2_{i} \to 1$, $K_i[K_i;0] \to 0$ for $m$ odd, $\gamma_{m} \to 0$ if $m$ is even and $\ell_i \gamma_{m}$ and $\gamma_{m/\ell_i}$ if $m$ is even and $\ell_i m$, extends Fr to an algebra homomorphism $Fr : U_v(b^-)_{ev} \to U_{\mathbb{Z}}(b^-)_{ev}$. Similarly, the map defined by $u \mapsto Fr'(u)$ for $u \in U_{\mathbb{Z}}(b^-)_{ev}$ and $(h^m_{i}) \to (K_i[K_i;0] \to 0)$. The proofs are very similar to the proofs above.

3. The dual maps $Fr^*$ and $Fr'$

We assume again that $\ell$ is an odd integer and moreover coprime to $3$ if $g$ has simple factors of type $G_2$. We make some remarks concerning the general case at the end of the section. The ideal $J$ (see Proposition 2) is in the kernel of $Fr$, so we get an induced map $Fr^*$ between the Hopf dual $U_{\mathbb{Z}}(b^-)^*$ of $U_v(b^-)$ and the Hopf dual $U_v(b^-)^*$ of $U_v(b^-)$.

The $U_v \otimes \mathbb{Z} \mathbb{C}$-module $V_v(\lambda) \otimes \mathbb{C}$, $\lambda \in \mathbb{P}^+$, is in general not a simple module. Denote by $L_v(\lambda)$ its simple quotient. By Lustzig [Lu2, Proposition 7.2], $E_{i}$, $F_{i}$ and $K_{i} - 1$ operate trivially on $L_v(\ell \lambda)$ for $\lambda \in \mathbb{P}^+$. Further, as in [Lu2, Proposition 7.5], the Frobenius splittings $Fr : U_{\mathbb{Z}} \to U_{\mathbb{Z}}$ and $Fr' : U_{\mathbb{Z}} \to U_{\mathbb{Z}}$ can be glued together to a surjective homomorphism (in fact an isomorphism) $F : U \simeq U_{\mathbb{Z}} \otimes \mathbb{Z} \mathbb{C} \to U_{v} \otimes \mathbb{Z} \mathbb{C}/(E_{i}, F_{i}, K_{i} - 1)$, and $L_v(\ell \lambda)$ becomes via $F$ a simple $U$-module $V(\lambda)$ of highest weight $\lambda$. We can also view $L_v(\ell \lambda)$ the other way around: We start with the irreducible $U$-module $V(\lambda)$ and make it into an $U_{v} \otimes \mathbb{Z} \mathbb{C}$-module $V(\lambda)^{Fr}$ via the Frobenius homomorphism $Fr : U_v \otimes \mathbb{Z} \mathbb{C} \to U$ (cf. Proposition 3). Then, Fr being surjective, $V(\lambda)^{Fr}$ is an irreducible $U_v \otimes \mathbb{Z} \mathbb{C}$-module. It is easy to see that $V(\lambda)^{Fr}$ is, in fact, isomorphic with $L_v(\ell \lambda)$. For each fundamental weight $\omega_i$ ($1 \leq i \leq n$), choose an isomorphism $\varphi_i : V(\omega_i)^{Fr} \simeq L_v(\ell \omega_i)$ such that $\tilde{v}_{\omega_i} \in V(\omega_i)$ corresponds to $v_{\ell \omega_i} \in L_v(\ell \omega_i)$ (cf. §1 for the notation $\tilde{v}_{\omega_i}$ and $v_{\omega_i}$). Since Fr is a Hopf algebra homomorphism, the isomorphisms $\varphi_i$ give rise to an $U_v \otimes \mathbb{Z} \mathbb{C}$-module isomorphism $\varphi : V(\lambda)^{Fr} \simeq L_v(\ell \lambda)$ (for all $\lambda \in \mathbb{P}^+$) so that $\tilde{v}_\lambda \in V(\lambda)$ corresponds to $v_{\ell \lambda} \in L_v(\ell \lambda)$. In the sequel, we fix such an isomorphism $\varphi_\lambda$ for each $\lambda \in \mathbb{P}^+$. For any $\lambda \in \mathbb{P}^+$ let $L_v(\lambda)_{\mathbb{Z}}$ be the $U_v$-submodule of $L_v(\lambda)$ generated by $v_{\lambda}$.

We thus get a ‘natural’ $U_v$-module isomorphism $L_v(\ell \lambda)_{\mathbb{Z}} \to V_{\mathbb{Z}}(\lambda)^{Fr}$ and hence the
Remark 3. Recall that we cannot extend $\text{Fr}^\ast$ to an algebra homomorphism on the full enveloping algebra, so $V_v(\ell \lambda)$ is not naturally endowed with a structure as an $U^-\hat{\otimes}Z$-module. The inclusion $V^-\hat{\otimes}Z \hookrightarrow V_v(\ell \lambda)$ hence does not give rise to an $U^-\hat{\otimes}Z$-equivariant map $V_v(\ell \lambda)^\ast \to V^-\hat{\otimes}Z(\ell \lambda)^\ast$. But, using the Frobenius maps $\text{Fr}^\prime : U^-\hat{\otimes}Z \to U^-\hat{\otimes}Z$ and $\text{Fr}^\prime : U^+\hat{\otimes}Z \to U^+\hat{\otimes}Z$ 

Define the map $\text{Fr}^\prime : R^-\hat{\otimes}Z \to R_v$, as the direct sum of the composite maps $V^-\hat{\otimes}Z(\ell \lambda)^\ast \to (L_v(\ell \lambda)\hat{\otimes}Z)^\ast \to V_v(\ell \lambda)^\ast$, where the last map is the dual of the quotient map $V_v(\ell \lambda) \to L_v(\ell \lambda)$, and $R^-\hat{\otimes}Z := R^-\otimes Z \hat{\otimes}Z$.

**Proposition 4.** The map $\text{Fr}^\prime$ is nothing but the restriction of $\text{Fr}^\prime$ to $R^-\hat{\otimes}Z$ under the identification of $R^-\hat{\otimes}Z$ (resp. $R_v$) as a subalgebra of $U^-\hat{\otimes}Z(\ell \lambda)^\ast$ (resp. $U_v(\ell \lambda)^\ast$) induced by the pairing $\Phi$ (resp. $\Phi_v$), cf. Propositions 1,2.

Equivalently, for any $X \in U_v(\ell \lambda)$ and $\xi \in R^-\hat{\otimes}Z$ we have

$$(1) \quad \Phi(\text{Fr}^\prime \lambda, \xi) = \Phi_v(X, \text{Fr}^\prime \lambda).$$

**Proof.** Equivalence of the two assertions is easy and the identity (1) follows readily from the definition of $\text{Fr}^\prime$.

From now on, we will denote (by abuse of notation) $\text{Fr}^\prime$ by $\text{Fr}^\ast$ itself.

Similarly the Hopf algebra homomorphism $\text{Fr}^\prime$ gives rise to the dual map $\text{Fr}^\prime \ast : (U_v(\ell \lambda)/J)^\ast \to U^-\hat{\otimes}Z(\ell \lambda)^\ast$. As above, one proves that the dual map $\text{Fr}^\prime \ast$ induces in fact a map $R_v \to R^-\hat{\otimes}Z$.

To describe this map more explicitly, let $\lambda \in P^+$ be a dominant weight. For the Weyl module $V_v(\ell \lambda)$ for $U_v$ denote by $V_v(\ell \lambda)^\perp$ the direct sum $\bigoplus_{\mu \in \ell P} V_v(\ell \lambda)_\mu$ of all weight spaces corresponding to the weights in $\ell P$. If $\mu = \ell \mu_1$ is a weight in $\ell P$, then so is the weight $\mu \pm n \ell \alpha_i = \ell (\mu_1 \pm n \alpha_i)$. It follows that $V_v(\ell \lambda)^\perp$ is stable under the action of all the $E_i^{(n\mu)}$ and $F_i^{(n\mu)}$.

We make $V_v(\ell \lambda)^\perp$ into an $U^-\hat{\otimes}Z$-module respectively $U^+\hat{\otimes}Z$-module via the homomorphism $\text{Fr}^\prime$ (i.e. by letting $X_i^{(m)}$ act as $E_i^{(m)}$ and $Y_i^{(m)}$ act as $F_i^{(m)}$). A simple calculation (see for example [Li2] for details) shows that if we let $(H_i^{(m)})$ act as $[K_i^{(m)}]$, then this defines a $U^-\hat{\otimes}Z$-module structure on $V_v(\ell \lambda)^\perp$, and the submodule generated by the highest weight vector $v_{\ell \lambda}$ is isomorphic to $V^-\hat{\otimes}Z(\ell \lambda)$. Again we choose an isomorphism so that $v_{\ell \lambda}$ corresponds to $\bar{v}_{\ell \lambda}$.

Similar to Proposition 4, we obtain:

**Proposition 5.** The restriction of the dual map $\text{Fr}^\prime \ast$ to $V_v(\ell \lambda)^\ast$ is the dual map of the inclusion $V^-\hat{\otimes}Z(\ell \lambda) \hookrightarrow V_v(\ell \lambda)$, and $\text{Fr}^\prime \ast |_{V_v(\mu)^\ast} = 0$ for $\mu \not\in \ell P^+$. 

Remark 3. Recall that we cannot extend $\text{Fr}^\prime$ to an algebra homomorphism on the full enveloping algebra, so $V_v(\ell \lambda)$ is not naturally endowed with a structure as an $U^-\hat{\otimes}Z$-module. The inclusion $V^-\hat{\otimes}Z \hookrightarrow V_v(\ell \lambda)$ hence does not give rise to an $U^-\hat{\otimes}Z$-equivariant map $V_v(\ell \lambda)^\ast \to V^-\hat{\otimes}Z(\ell \lambda)^\ast$. But, using the Frobenius maps $\text{Fr}^\prime : U^-\hat{\otimes}Z \to U^-\hat{\otimes}Z$ and $\text{Fr}^\prime : U^+\hat{\otimes}Z \to U^+\hat{\otimes}Z$ —
$U_v^+$, we can make $V_v(\ell \lambda)$ into an $U_{\hat{Z}}^-$-module, and, by the definition of the inclusion $V_{\hat{Z}}(\lambda) \hookrightarrow V_v(\ell \lambda)^{\mathbb{Z}} \hookrightarrow V_v(\ell \lambda)$, the map $V_{\hat{Z}}(\lambda) \hookrightarrow V_v(\ell \lambda)$ is equivariant with respect to the action of $U_{\hat{Z}}^+$ and $U_{\hat{Z}}^-$, and hence so is the dual map.

**Remark 4.** The composition

$$U_{\hat{Z}}(b^-) \xrightarrow{\text{Fr}} U_v(b^-)/J \xrightarrow{\text{Fr}} U_{\hat{Z}}(b^-)$$

is the identity map, and hence so is $R_{\hat{Z}} \xrightarrow{\text{Fr}} R_{\hat{Z}} \xrightarrow{\text{Fr}} R_{\hat{Z}}$.

**Remark 5.** If $\ell = 2d$, then $\text{Fr}^*$ induces a map $R_{\hat{Z}} \xrightarrow{\text{Fr}} R_{v, ev}$, which is the direct sum of the duals of the quotient maps $V_v(\ell \lambda) \rightarrow L_v(\lambda)_{\hat{Z}} = V_{\hat{Z}}(\lambda^\#)$, for $\lambda^\# \in P^\#$, and the restriction of the dual map $\text{Fr}^*$ to $V_v(\lambda)^*$ is the dual map of the inclusion $V_{\hat{Z}}(\lambda^\#) \hookrightarrow V_v(\lambda)$, and $\text{Fr}^*|_{V_v(\mu)} = 0$ for $\mu^\# \not\in P^\#$. To see this, let $\lambda^\# \in P^\# \subset P$ be a dominant weight, we write just $\lambda$ for the weight if we view it as a $U_v$-weight. We make $V_{\hat{Z}}(\lambda^\#)$ as above into an $U_v^-$ and $U_v^+$-module by using the Frobenius map $\text{Fr}$, and we let $[K_m]$ act on $V_{\hat{Z}}(\lambda^\#)$ as $(H_m^\mu)_{\mathbb{Z}}$ if $m$ is divisible by $\ell_i$, and as 0 if $\ell_i \nmid m$.

Then as above, the three actions glue together to give a $U_v$-module structure on $V_{\hat{Z}}(\lambda^\#)$ such that $u \in J$ acts trivially. Thus $V_{\hat{Z}}(\lambda^\#)$ becomes in this way a highest weight module for $U_v$ of heighest weight $\lambda$. Now $V(\lambda^\#)$ is a simple module for $U(\mathfrak{g}^\#)$ and hence for $U_v \otimes_{\hat{Z}} \mathbb{C}$. So, as above, we can view $\text{Fr}^*$ as the dual map of the quotient map $V_v(\lambda) \rightarrow L_v(\lambda) = V_{\hat{Z}}(\lambda^\#)$.

Actually, with the appropriate adaptions (for details see [Lu1, Chapter 35]) one can reformulate the results for arbitrary $\ell$.

**4. Base change**

In the following we assume that $\ell = p$ is in fact an odd prime, and further $p > 3$ if $\mathfrak{g}$ has factors of type $\mathfrak{g}_2$. Let $k$ be an algebraically closed field of char. $p$. We consider $k$ as a $\hat{Z}$-module by extending the canonical map $\mathbb{Z} \rightarrow k$ to a ring homomorphism $\hat{Z} \rightarrow k$ (where the first map is given by the projection $\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ and the inclusion $\mathbb{Z}/p\mathbb{Z} \subset k$, and the extension $\hat{Z} \rightarrow k$ follows from the ‘Going-up theorem’ [M, Theorem 9.3]). We denote by $U_k, U_{v,k}$ the corresponding algebras obtained by the base change.

Note that the image of $v$ in $k$ is 1. Let $J_p$ be the ideal in $U_{v,k}$ generated by the central elements $(K_i - 1)$, $i = 1, \ldots, n$; then the quotient $U_{v,k}/J_p$ is naturally isomorphic to $U_k$. Further, let $\mu \in P^+$ be a dominant weight and $V_{v,k}(\mu)$ be the corresponding Weyl module for $U_{v,k}$. Since all the $(K_i - 1)$ operate trivially on $V_{v,k}(\mu)$, this becomes in a natural way the Weyl module $V_k(\mu)$ for $U_k$. Let $L_k(\mu)$ be the $U_k$-module $L_v(\mu)_{\hat{Z}} \otimes_{\hat{Z}} k$.

We are now left with only one algebra, namely $U_k$. The module $L_k(p\lambda)$ is, as a $U_k$-module the same as the $U_k$-module $V_k(\lambda)^{\text{Fr}}$, where as in §3, $V_k(\lambda)^{\text{Fr}}$ is the same $k$-vector space as $V_k(\lambda)$, but its $U_k$-module structure has been twisted via the Frobenius
map $\text{Fr}_k : U_k \to U_k$ given by $F_i^{(m)} \mapsto F_i^{(m/p)}; E_i^{(m)} \mapsto E_i^{(m/p)}$, if $m$ is divisible by $p$ and 0 otherwise.

We are going to twist the $k$-vector space structure of $V_k(\lambda)^{Fr}$. Let $\phi : k \to k$ be the ring homomorphism given by the inverse of the $p$-th power map, i.e., $z \mapsto z^{1/p}$, and denote by $V_k(\lambda)^{(1)}$ the $k$-vector space (and $U_k$-module) having as underlying abelian group the same as $V_k(\lambda)$, but where the scalar multiplication has been twisted by $\phi$: $a \cdot v := \phi(a)v$, and where $U_k$ acts as on $V_k(\lambda)^{Fr}$. (Note that the operation of $U_k$ is linear also with respect to the twisted scalar multiplication.) The $U_k$-module $V_k(\lambda)^{(1)}$ can be seen explicitly as a quotient of $V_k(p\lambda)$ as follows: The map $V_k(\lambda)^{(1)} \to S^p V_k(\lambda)$, defined by $v \mapsto v^p$ (which is linear because of the twisted scalar multiplication), induces an isomorphism onto the image of the canonical map $V_k(p\lambda) \to S^p V_k(\lambda)$ which sends the highest weight vector $v_{p\lambda} \in V_k(p\lambda)$ to the highest weight vector $v_\lambda^p \in S^p V_k(\lambda)$.

Let $G$ be the semisimple simply connected algebraic group over $k$ corresponding to the Lie algebra $\mathfrak{g}$ and let $B \subset G$ be the Borel subgroup corresponding to the Lie algebra $\mathfrak{b}$. Let $L_\lambda$ be the line bundle on $X := G/B$ corresponding to a weight $-\lambda$. Recall that for $\lambda$ dominant, we have $H^0(X, L_\lambda)$, as $G$-module, isomorphic to $V_k(\lambda)^*$. It follows from the considerations above that the dual map $\text{Fr}^* : H^0(X, L_\lambda)^{(1)} \to H^0(X, L_{p\lambda})$ is just the $p$-th power map sending a section $s \in H^0(X, L_\lambda)$ to $s^p \in H^0(X, L_{p\lambda})$ (again, recall that this map is linear with respect to the twisted scalar multiplication).

The inclusion $V_k(\lambda) \hookrightarrow V_k(p\lambda)$ respectively $\text{Fr}^* : H^0(X, L_{p\lambda}) \to H^0(X, L_\lambda)$, the associated dual map, does not have such an equivariant interpretation, but Remark 4 implies that $\text{Fr}^*$ is a section to $\text{Fr}^*$. Observe that $\text{Fr}^*$ restricted to $H^0(X, L_\lambda)$ is zero if $\lambda \notin pP^+$.

**Theorem 1.** The dual map $\text{Fr}^* : H^0(X, L_\lambda) \to H^0(X, L_{p\lambda})$ is the map $s \mapsto s^p$ sending a section to its $p$-th power, and $\text{Fr}^* : H^0(X, L_{p\lambda}) \to H^0(X, L_\lambda)$ provides a splitting of this map. For any $s \in H^0(X, L_{j\lambda})$ and $f \in H^0(X, L_{m\lambda})$, the Frobenius map satisfies the following properties:

(a) $\text{Fr}^*(s^pf) = s\text{Fr}^*(f)$, and

(b) $\text{Fr}^*(X_i^{(pq)}f) = X_i^{(q)}\text{Fr}^*(f)$, for all $1 \leq i \leq n$, and $q \in \mathbb{N}$.

**Remark 6.** For notational convenience assume that $\lambda \notin pP^+$. The property (a) implies that $\text{Fr}^*$ induces a graded Frobenius endomorphism of the graded algebra $S := \bigoplus_{m \geq 0} H^0(X, L_{m\lambda})$, more specifically, $\text{Fr}^*$ maps the homogeneous elements of degree not divisible by $p$ to zero and if $f$ is of degree $qp$ then $\text{Fr}^*(f)$ is of degree $q$, the map is additive: $\text{Fr}^*(s_1 + s_2) = \text{Fr}^*(s_1) + \text{Fr}^*(s_2)$, and $\text{Fr}^*(s_1^ps_2) = s_1\text{Fr}^*(s_2)$. The second property implies that $\text{Fr}^*$ is the canonical splitting, see [Ma]. In particular, $\text{Fr}^*$ maps $B$-modules to $B$ modules.
The vertical map is the identity on the second factor and the projection on the first.

and

$V$ addition we have two maps between Weyl modules:

$V$, that and the other inclusion is $V$ satisfies $\iota(X_1^{(s)}v) = X_1^{(ps)}\iota(v)$, so the corresponding property holds also for the dual map $Fr^*: H^0(X, \mathcal{L}_{pq\lambda}) \to H^0(X, \mathcal{L}_{q\lambda})$. This implies the second property in the theorem above.

Abbreviate the module $L_v(\lambda)_{\overline{Z}}$ by $L_v(\lambda)$. To prove the first property, consider the following diagram of Weyl modules (for $U_{\overline{Z}}$ respectively $U_v$) defined over $\overline{Z}$: There are two inclusions of $U_{\overline{Z}}$-modules using the Frobenius map: $V_{\overline{Z}}((q + j)\lambda) \hookrightarrow V_v(p(q + j)\lambda)^{\overline{Z}}$, and the other inclusion is $V_{\overline{Z}}(j\lambda) \otimes V_{\overline{Z}}(q\lambda) \hookrightarrow L_v(pj\lambda) \otimes V_{\overline{Z}}(pq\lambda)$, using the fact that $L_v(pj\lambda) = V_{\overline{Z}}((j\lambda)\overline{F_r})$ as $U_v$-module. Note that $E_{i_1(m_1)}^{(mp)}$ acts on $V_{\overline{Z}}(j\lambda)$ as $X_{i_1(m_1)}^{(m)}$, so $Fr^*(X_{i_1(m_1)}^{(m)})$ acts on $L_v(pj\lambda) = V_{\overline{Z}}(j\lambda)$ as $X_{i_1(m_1)}^{(m)}$. Then we have two inclusions of $U_{\overline{Z}}^-$-respectively $U_v$-modules which act on the $U_v$-modules via $Fr^*$: The inclusions are $V_v(p(q + j)\lambda)^{\overline{Z}} \hookrightarrow V_v(p(q + j)\lambda)$ and $L_v(pj\lambda) \otimes V_{\overline{Z}}(pq\lambda)$.

In addition we have two maps between Weyl modules: $V_{\overline{Z}}((j + q)\lambda) \to V_{\overline{Z}}(j\lambda) \otimes V_{\overline{Z}}(q\lambda)$ and $V_v(p(j + q)\lambda) \to V_v(pj\lambda) \otimes V_v(pq\lambda)$.

\[
\begin{array}{c}
V_{\overline{Z}}((j + q)\lambda) \\ \downarrow \\
V_{\overline{Z}}(j\lambda) \otimes V_{\overline{Z}}(q\lambda) \hookrightarrow L_v(pj\lambda) \otimes V_v(pq\lambda) \\
V_v(p(q + j)\lambda)^{\overline{Z}} \hookrightarrow V_v(p(q + j)\lambda) \rightarrow V_v(pj\lambda) \otimes V_v(pq\lambda)
\end{array}
\]

The vertical map is the identity on the second factor and the projection on the first. All these maps are equivariant with respect to the $U_{\overline{Z}}^\pm$-actions on these spaces, they all act on the highest weight vector (resp. the tensor product of highest weight vectors) to a highest weight vector (resp. the tensor product of highest weight vectors). It follows that the diagram is commutative and provides two different ways to construct a map $V_{\overline{Z}}((j + q)\lambda) \to L_v(pj\lambda) \otimes V_v(pq\lambda)$.

Over the field $k$, the dual of the bottom row is the map $H^0(X, \mathcal{L}_{j\lambda}) \otimes H^0(X, \mathcal{L}_{pq\lambda}) \to H^0(X, \mathcal{L}_{(q+j)\lambda})$ defined by $s \otimes f \mapsto sFr^*(f)$ for sections $s \in H^0(X, \mathcal{L}_{j\lambda})$ and $f \in H^0(X, \mathcal{L}_{pq\lambda})$.

The dual of the top row provides a decomposition of this map in the following way: $s \otimes f \in H^0(X, \mathcal{L}_{j\lambda}) \otimes H^0(X, \mathcal{L}_{pq\lambda})$ is first mapped to $s^p \otimes f \in H^0(X, \mathcal{L}_{pj\lambda}) \otimes H^0(X, \mathcal{L}_{pq\lambda})$, then to the product $s^pf \in H^0(X, \mathcal{L}_{p(j+q)\lambda})$, and then to $Fr^*(s^pf) \in H^0(X, \mathcal{L}_{(j+q)\lambda})$. Since the two maps are the same, it follows $Fr^*(s^pf) = sFr^*(f)$. This proves (b). \hfill \diamond
References.


