October 21, 1998

# Frobenius splitting in characteristic zero and the quantum Frobenius map

Shrawan Kumar*
Department of Mathematics
University of North Carolina at Chapel Hill
Chapel Hill, NC 27599-3250
USA

Peter Littelmann**
Département de Mathématiques et IRMA
Université Louis Pasteur et Institut Universitaire de France
7, rue René Descartes
67084 Strasbourg
France

## Introduction

In the representation theory of a semisimple algebraic group $G$, the Schubert varieties $X(w)$, $w$ an element of the Weyl group $W$ of $G$, play a prominent rôle. An important breakthrough in the analysis of the geometry of these subvarieties of the flag variety $G/B$ was the introduction by Mehta-Ramanathan of the notion of a Frobenius split variety and compatibly split subvarieties (for varieties defined over a field of char. $p > 0$). They proved [MR] that $G/B$ (more generally any $G/P$ for a parabolic subgroup $P$) is a Frobenius split variety such that all of the Schubert subvarieties are compatibly split, in particular, one immediately obtains the Kodaira-Kempf vanishing theorem. Using this tool, it was shown for example that Schubert varieties are normal, Cohen-Macaulay and have rational singularities. Moreover, they are projectively normal, projectively Cohen-Macaulay, and are defined by quadratic relations in any embedding given by an ample homogeneous line bundle on $G/B$ (cf. [RR], [Ra1], [Ra2]). Of course, as is well-known, the normality of Schubert varieties is equivalent to the validity of the Demazure's character formula. Further, the Frobenius splitting was used by Mathieu to give a uniform proof that the category of finite dimensional $G$-representations (over char. $p > 0$) admitting a good filtration is stable under tensor product and more generally under the restriction to the semisimple part of a Levi subgroup (cf. [D], [Ma]).

Earlier, a different way to analyze the geometry of Schubert varieties was suggested by Seshadri and his school. They proposed to construct a standard monomial theory

* Supported by NSF grant no. DMS- 9622887.
** Supported by TMR-Grant ERB FMRX-CT97-0100.

for the homogeneous coordinate ring of an embedding $G/P \hookrightarrow \mathbb{P}(V(\omega))$ given by the orbit of a highest weight vector in a fundamental representation $V(\omega)$ (see for example [LS], [LLM] for comments on the development). In this approach, the extremal weight vectors play a prominent role. Using the quantum Frobenius map defined by Lusztig, the second author defined the "$\ell$-th root" of a product of extremal weight vectors in the quantum Demazure module $(V_v(\lambda)_w)^*$ at an $\ell$-th root of unity $v$. It turned out that this method presented the perfect tool to develop a standard monomial theory for arbitrary embeddings $X(w) \hookrightarrow \mathbb{P}(V(\lambda))$ of Schubert varieties, avoiding all case by case considerations. Many of the results proved by Frobenius splitting methods follow then also naturally by standard monomial theory, see [Li1], [Li2], [LLM].

The aim of this article it to systematically begin unifying these two approaches. For the Borel subgroup $B$ of $G$, let $\mathfrak{b}$ be its Lie algebra over the complex numbers $\mathbb{C}$ and $\mathfrak{b}^-$ be the opposite Borel subalgebra. We first establish a duality between the algebra $U_{\mathbb{Z}}(\mathfrak{b}^-)$ (resp. its quantum analogue $U_v(\mathfrak{b}^-)$), and the direct sum of the dual modules of all Weyl modules $\bigoplus_{\lambda\in P^+} V_{\mathbb{Z}}(\lambda)^*$ (resp. its quantum analogue $\bigoplus_{\lambda\in P^+} V_v(\lambda)^*$) (cf. Propositions 1 and 2), where $v$ as earlier is an $\ell$-th root of unity, $U_{\mathbb{Z}}(\mathfrak{b}^-)$ is the Kostant's $\mathbb{Z}$-lattice of the enveloping algebra $U(\mathfrak{b}^-)$, $U_v(\mathfrak{b}^-)$ is the Lusztig's $\tilde{\mathbb{Z}}$-lattice of the quantized algebra $U_q(\mathfrak{b}^-)$ and $\tilde{\mathbb{Z}}$ is the ring obtained from $\mathbb{Z}$ by adjoining all the roots of unities. Now Lusztig defined for $\ell$ an odd integer ( $\ell$ coprime to 3 if $G_2$ is a factor of $G$) a certain Frobenius homomorphism $\mathrm{Fr} : U_v(\mathfrak{b}^-) \to U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$ and also a certain splitting of it on the "$\mathfrak{n}^-$-part" (which we shall refer to as *Frobenius splitting* homomorphism) $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}(\mathfrak{n}^-) \to U_v(\mathfrak{n}^-)$, where $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-) := U_{\mathbb{Z}}(\mathfrak{b}^-) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$, $\mathfrak{n}^-$ is the nilradical of $\mathfrak{b}^-$, and $U_{\tilde{\mathbb{Z}}}(\mathfrak{n}^-)$, $U_v(\mathfrak{n}^-)$ have meaning similar to that of the corresponding $\mathfrak{b}^-$. We extend the definition of $\mathrm{Fr}'$ to $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$ (cf. Lemma 3). By using the duality mentioned above, we get maps $\mathrm{Fr}^* : \bigoplus_{\lambda\in P^+} V_{\tilde{\mathbb{Z}}}(\lambda)^* \to \bigoplus_{\lambda\in P^+} V_v(\lambda)^*$ respectively $\mathrm{Fr}^{'*} : \bigoplus_{\lambda\in P^+} V_v(\lambda)^* \to \bigoplus_{\lambda\in P^+} V_{\tilde{\mathbb{Z}}}(\lambda)^*$ (cf. Propositions 4 and 5).

Next we assume that $\ell$ is a prime, and show that, after a base change with an algebraically closed field of char. $p = \ell$, the map $\mathrm{Fr}^*$ has a natural interpretation as the $p$-th power map on the space of sections $H^0(G/B, \mathcal{L}_\lambda)$, and $\mathrm{Fr}^{'*}$ can naturally be interpreted as a splitting of this map. This is our principal result of the paper (cf. Theorem 1). So on the level of quantum groups, the map $\mathrm{Fr}^{'*}$ can be considered as a char. zero lift of the Frobenius splitting of $G/B$ in char. $p$, and it is exactly this map which has been used in [Li2] to define the "$\ell$-th root" of certain sections.

In a subsequent paper we will use Lusztig's Fr (resp. $\mathrm{Fr}^{'*}$) to define the Frobenius map (resp. Frobenius splitting map) at 'higher cohomology' level as well and show that $\mathrm{Fr}^{'*}$ after base-change provides a canonical (in the sense of Mathieu) splitting of $G/B$ compatibly splitting all of the Schubert subvarieties. In particular, this will provide a purely algebraic proof (via the quantum groups at roots of unity) of results of Mehta-Ramanathan mentioned above and also various results on the geometry of Schubert varieties mentioned in the first paragraph.

## 1. A pairing and its quantum analogue

For a complex semisimple Lie algebra $\mathfrak{g}$, associated to a Cartan matrix $C = (c_{i,j})_{1\le i,j\le n}$, fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the associated triangular decomposition and set $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$. We write $U$, $U^+$, $U^-$ for the enveloping algebras $U(\mathfrak{g})$, $U(\mathfrak{n}^+)$, $U(\mathfrak{n}^-)$ respectively.

Let $\omega_1, \dots, \omega_n$ be the set of fundamental weights in the weight lattice $P$ of $\mathfrak{g}$. Denote by $P^+$ the set of non-negative linear combinations $\mathbb{N}\omega_1 + \ldots + \mathbb{N}\omega_n$, $\mathbb{N}$ being the set of non-negative integers. We write $V(\lambda)$ for the complex irreducible representation of highest weight $\lambda \in P^+$. As an $\mathfrak{h}$-module, $V(\lambda)$ decomposes into the direct sum $\oplus_{\mu\in P} V(\lambda)_\mu$ of weight spaces. Here $V(\lambda)_\mu = \{v \in V(\lambda) \mid hv = \mu(h)v \ \forall h \in \mathfrak{h}\}$.

Let $\alpha_1, \dots, \alpha_n$ be the set of simple roots (corresponding to the Borel subalgebra $\mathfrak{b}$). For a simple root $\alpha_i$ choose $X_i \in \mathfrak{g}_{\alpha_i}$, $Y_i \in \mathfrak{g}_{-\alpha_i}$ and $H_i \in \mathfrak{h}$ such that $[H_i, X_j] = a_{i,j}X_j$, $[H_i, Y_j] = -a_{i,j}Y_j$, $[X_i, Y_j] = \delta_{i,j}H_i$. We denote by $U_\mathbb{Z}$ the Kostant form of $U$ over the ring of integers $\mathbb{Z}$, generated by the divided powers $X_i^{(k)} := X_i^k/k!$ and $Y_i^{(k)} := Y_i^k/k!$, $k \ge 0$. Let $U_\mathbb{Z}^+$, $U_\mathbb{Z}^-$ be the corresponding Kostant forms of $U^+$ and $U^-$, generated respectively by $\{X_i^{(k)}\}$ and $\{Y_i^{(k)}\}$, and let $U_\mathbb{Z}(\mathfrak{b}^-)$ be the Kostant form of $U(\mathfrak{b}^-)$ generated by $\{Y_i^{(k)}, \binom{H_i}{k}\}$, where $\binom{H_i}{k} = H_i(H_i - 1)\cdots(H_i - k + 1)/k!$. Then $U_\mathbb{Z}(\mathfrak{b}^-) \subset U_\mathbb{Z}$.

For $i = 1, \dots, n$, fix a highest weight vector $\bar{v}_{\omega_i} \in V(\omega_i)_{\omega_i}$ and let $V_\mathbb{Z}(\omega_i) := U_\mathbb{Z}\bar{v}_{\omega_i}$ be the corresponding $U_\mathbb{Z}$-stable $\mathbb{Z}$-lattice. For $\lambda = \sum a_i\omega_i \in P^+$ denote by $\bar{v}_\lambda$ the vector $\bar{v}_{\omega_1}^{\otimes a_1} \otimes \ldots \otimes \bar{v}_{\omega_n}^{\otimes a_n}$, and let $V_\mathbb{Z}(\lambda)$ be the lattice

$$U_\mathbb{Z}\bar{v}_\lambda := U_\mathbb{Z}(\bar{v}_{\omega_1}^{\otimes a_1} \otimes \ldots \otimes \bar{v}_{\omega_n}^{\otimes a_n}) \hookrightarrow V_\mathbb{Z}(\omega_1)^{\otimes a_1} \otimes \ldots \otimes V_\mathbb{Z}(\omega_n)^{\otimes a_n} .$$

The dual module $\mathrm{Hom}_\mathbb{Z}(V_\mathbb{Z}(\lambda), \mathbb{Z})$ is denoted by $V_\mathbb{Z}(\lambda)^*$. Setting $R_\mathbb{Z} := \bigoplus_{\lambda\in P^+} V_\mathbb{Z}(\lambda)^*$, we have a natural pairing

$$\Phi : U_\mathbb{Z}(\mathfrak{b}^-) \times R_\mathbb{Z} \to \mathbb{Z}, \quad \Phi(u, \xi) := \xi(u.\bar{v}_\lambda) \text{ for } u \in U_\mathbb{Z}(\mathfrak{b}^-),\ \xi \in V_\mathbb{Z}(\lambda)^*.$$

Denote by $U(\mathfrak{b}^-)^*$ the Hopf dual of $U(\mathfrak{b}^-)$ (under the standard Hopf algebra structure on $U(\mathfrak{b}^-)$), i.e., $U(\mathfrak{b}^-)^*$ is the subspace of linear forms $f \in \mathrm{Hom}(U(\mathfrak{b}^-), \mathbb{C})$ for which there exists a two-sided ideal $I \subset U(\mathfrak{b}^-)$ of finite codimension such that $f(I) = 0$. Recall that $U(\mathfrak{b}^-)^*$ is again a Hopf algebra. Let $U_\mathbb{Z}(\mathfrak{b}^-)^*$ be the $\mathbb{Z}$-submodule of forms $f \in U(\mathfrak{b}^-)^*$ such that $f(U_\mathbb{Z}(\mathfrak{b}^-)) \subset \mathbb{Z}$. Note that $U_\mathbb{Z}(\mathfrak{b}^-)^*$ is a $\mathbb{Z}$-subalgebra of $U(\mathfrak{b}^-)^*$. The following proposition can be viewed as a certain “half” of the Peter-Weyl type theorem. One may also consider it as an algebraic analogoue of the result of Bernstein, Gelfand and Gelfand that the ring $\mathbb{C}[G/U]$ of regular functions on $G/U$ is isomorphic to $\bigoplus_{\lambda\in P^+} V(\lambda)^*$, where $G$ is the corresponding simply connected complex algebraic group with Borel subgroup $B$ and unipotent radical $R_u(B) = U$ having $\mathfrak{n}^+$ as its Lie algebra.

**Proposition 1.**
*$\Phi$ is a non-degenerate pairing, identifying $R_{\mathbb{Z}}$ as a subalgebra of $U_{\mathbb{Z}}(\mathfrak{b}^-)^*$:*

$$\xi^\lambda \cdot \xi^\mu(u) := (\xi^\lambda \otimes \xi^\mu)|_{V_{\mathbb{Z}}(\lambda+\mu)}(u.\bar{v}_{\lambda+\mu})$$

*for $\xi^\lambda \in V_{\mathbb{Z}}(\lambda)^*$, $\xi^\mu \in V_{\mathbb{Z}}(\mu)^*$ and $u \in U_{\mathbb{Z}}(\mathfrak{b}^-)$.*

*Proof.* Denote by $Q$ the root lattice and set $Q^+ = \mathbb{N}\alpha_1 + \ldots + \mathbb{N}\alpha_n$. The Kostant form $U^0_{\mathbb{Z}}$ of the enveloping algebra $U(\mathfrak{h})$ has as basis the monomials $\binom{H_1}{k_1}\cdots\binom{H_n}{k_n}$, $k_i \in \mathbb{N}$. Fix a $\mathbb{Z}$-basis $\mathtt{B}$ of $U^-_{\mathbb{Z}}$ such that for any $\beta \in Q^+$ the elements in $\mathtt{B}_\beta := U^-_{\mathbb{Z},-\beta} \cap \mathtt{B}$ form a basis of the weight space $U^-_{\mathbb{Z},-\beta}$.

Suppose now that $u \in U_{\mathbb{Z}}(\mathfrak{b}^-)$ is such that $\Phi(u, R_{\mathbb{Z}}) = 0$. We can write $u = \sum_{\beta\in Q^+}\sum_{b\in\mathtt{B}_\beta} bh_b$, where $h_b \in U^0_{\mathbb{Z}}$. To prove $u = 0$, it is sufficient to show that for any $u \neq 0$, $u\bar{v}_\lambda \neq 0$ for some $\lambda \in P^+$. It is well known that if $\beta$ is fixed, then for $\lambda \gg 0$ (i.e. $a_i \gg 0$ for all $i = 1,\ldots,n$), the vectors $b\bar{v}_\lambda$, $b \in \mathtt{B}_\beta$, form a basis of the weight space $V_{\mathbb{Z}}(\lambda)_{\lambda-\beta}$. So if we choose $\lambda$ big enough, $u\bar{v}_\lambda = 0$ implies $\lambda(h_b) = 0$ for all $b$ and all $\lambda \gg 0$. But this is possible only if $h_b = 0$.

Consider $f = \sum \xi^\lambda$. Among the $\xi^\lambda \neq 0$ fix $\xi^{\lambda_o}$ such that $\lambda_o$ is maximal in the lexicographic ordering, i.e., if $\lambda_o = \sum a_i\omega_i$ and $\lambda = \sum b_i\omega_i$ is such that $\xi^\lambda \neq 0$, then there exists a $j \leq n$ such that $a_i = b_i$ for $i < j$ and $a_j > b_j$. Set $H_{\lambda_o} := \prod_{i=1}^n \binom{H_i}{a_i}$. Note that $H_{\lambda_o}\bar{v}_{\lambda_o} = \bar{v}_{\lambda_o}$ and $H_{\lambda_o}\bar{v}_\lambda = 0$ for all $\lambda$ such that $\xi^\lambda \neq 0$, $\lambda \neq \lambda_o$. Since $V_{\mathbb{Z}}(\lambda_o) = U^-_{\mathbb{Z}}\bar{v}_{\lambda_o}$, we can find $u \in U^-_{\mathbb{Z}}$ such that $\xi^{\lambda_o}(u\bar{v}_{\lambda_o}) \neq 0$. It follows that $\Phi(uH_{\lambda_o}, f) = \sum \xi^\lambda(uH_{\lambda_o}\bar{v}_\lambda) = \xi^{\lambda_o}(u\bar{v}_{\lambda_o}) \neq 0$. This proves that $\Phi$ is non-degenerate.

To see that $R_{\mathbb{Z}}$ forms a subalgebra of $U_{\mathbb{Z}}(\mathfrak{b}^-)$, note that the co-product $\Delta$ induces a natural $U_{\mathbb{Z}}(\mathfrak{b}^-)$-module structure on $V_{\mathbb{Z}}(\lambda)\otimes V_{\mathbb{Z}}(\mu)$. By the definition of the product, we have $\xi^\lambda \cdot \xi^\mu(u) = \xi^\lambda \otimes \xi^\mu(\Delta(u))$. For $\Delta(u) = \sum u_1 \otimes u_2$ we have $\xi^\lambda \otimes \xi^\mu(\Delta(u)) = \sum \xi^\lambda(u_1\bar{v}_\lambda)\cdot\xi^\mu(u_2\bar{v}_\mu)$.

Now the map $u\bar{v}_{\lambda+\mu} \mapsto u(\bar{v}_\lambda \otimes \bar{v}_\mu)$, $u \in U_{\mathbb{Z}}(\mathfrak{b}^-)$, induces an isomorphism between $V_{\mathbb{Z}}(\lambda+\mu)$ and the $U_{\mathbb{Z}}$-submodule $U_{\mathbb{Z}}(\mathfrak{b}^-)(\bar{v}_\lambda\otimes\bar{v}_\mu)$ of $V_{\mathbb{Z}}(\lambda)\otimes V_{\mathbb{Z}}(\mu)$. The restriction map induces hence a map $\mathrm{res} : V_{\mathbb{Z}}(\lambda)^* \otimes V_{\mathbb{Z}}(\mu)^* \to V_{\mathbb{Z}}(\lambda+\mu)^*$. It follows that $\xi^\lambda \cdot \xi^\mu(u) = \mathrm{res}(\xi^\lambda \otimes \xi^\mu)(u\bar{v}_{\lambda+\mu})$. ◇

**Remark 1.** By using the Peter-Weyl theorem, we get an isomorphism $\mathbb{C}[G/U] \simeq \bigoplus_{\lambda\in P^+} V(\lambda)^*$. Let $B^-$ be the opposite Borel subgroup ($\mathrm{Lie}\, B^- = \mathfrak{b}^-$). Since $B^-$ is open and dense in $G/U$, we have an inclusion $\mathbb{C}[G/U] \hookrightarrow \mathbb{C}[B^-]$. But by [J, Part I, §§7.10 and 7.18], we have $\mathbb{C}[B^-] \simeq U(\mathfrak{b}^-)^*$, and hence we have $\bigoplus_{\lambda\in P^+} V(\lambda)^* \hookrightarrow U(\mathfrak{b}^-)^*$. So this gives an alternative geometric derivation of the above proposition over $\mathbb{C}$.

We have a similar construction for the quantum group $U_q := U_q(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$. Let $d_1,\ldots,d_n \geq 1$ be minimal integers such that $(d_ic_{i,j})$ is a symmetric matrix, and let $\tilde{\mathbb{Z}}$ be the ring obtained from $\mathbb{Z}$ by adjoining all roots of unities.

We fix a positive integer $\ell$. If $\ell$ is odd, then let $\phi$ be the $\ell$-cyclotomic polynomial, and if $\ell$ is even, then let $\phi$ be the $2\ell$-cyclotomic polynomial. Let $A = \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials and fix a homomorphism $A/(\phi) \hookrightarrow \tilde{\mathbb{Z}}$, where $(\phi)$ is the ideal in $A$ generated by $\phi$. Denote by $v$ the image of $q$ in $\tilde{\mathbb{Z}}$.

We denote the generators of the quantum group $U_q$ over $\mathbb{C}(q)$ by $E_i, F_i, K_i^{\pm}$, and let $U_A$ be the Lusztig form of $U_q$ over $A$ generated by the divided powers $E_i^{(m)} := E_i^m/[m]_i!$ and $F_i^{(m)} := F_i^m/[m]_i!$ and the $K_i^{\pm 1}$ [Lu3, §1]. Recall that the Gaussian numbers $[m]_i$ are defined by $[m]_i := (q^{d_i m} - q^{-d_i m})/(q^{d_i} - q^{-d_i})$, and $[m]_i! := [1]_i \dots [m]_i$.

We denote by $U_A^+, U_A^-, U_A^0$ the subalgebras of $U_A$ generated by the $E_i^{(m)}$, the $F_i^{(m)}$, and the $\{K_i^{\pm}, \left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]\}$ respectively for $1 \le i \le n$ and $m \in \mathbb{N}$. Recall that the latter is defined by

$$\begin{bmatrix} K_i; c \\ m \end{bmatrix} := \prod_{s=1}^{m} \frac{K_i q^{(c-s+1)d_i} - K_i^{-1} q^{(-c+s-1)d_i}}{q^{sd_i} - q^{-sd_i}}, \text{ for } c \in \mathbb{Z} \text{ and } m \in \mathbb{N},$$

and $U_A^0$ has as a basis the monomials of the form $\prod_{i=1}^{n} (\left[\begin{smallmatrix} K_i;0 \\ m_i \end{smallmatrix}\right] K_i^{e_i})$, where the $m_i$ are non-negative integers and $e_1, \dots, e_n \in \{0, 1\}$ [Lu3, Theorem 6.7 (c)]. Let $U_A(\mathfrak{b}^-)$ be the subalgebra generated by the $F_i^{(m)}$, $K_i^{\pm}$ and the $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]$. The following statements can be found in [Lu2] or [CP], or can be easily deduced from [Lu3, §6.4].

**Lemma 1.**

*a)* $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right] \left[\begin{smallmatrix} K_i;-m \\ t \end{smallmatrix}\right] = \left[\begin{smallmatrix} m+t \\ m \end{smallmatrix}\right]_i \left[\begin{smallmatrix} K_i;0 \\ m+t \end{smallmatrix}\right]$.

*b)* $\left[\begin{smallmatrix} K_i;c \\ t' \end{smallmatrix}\right] = \sum_{j=0}^{t} \left[\begin{smallmatrix} t \\ j \end{smallmatrix}\right]_i q^{d_i(tt'-cj)} K_i^{-j} \left[\begin{smallmatrix} K_i;c-t \\ t'-j \end{smallmatrix}\right]$ *, for any* $t \le t'$.

*c)* $\left[\begin{smallmatrix} K_i;c \\ t \end{smallmatrix}\right] = \sum_{s=0}^{t} q^{d_i c(t-s)} \left[\begin{smallmatrix} c \\ s \end{smallmatrix}\right]_i K_i^{-s} \left[\begin{smallmatrix} K_i;0 \\ t-s \end{smallmatrix}\right]$ *, for* $c > 0$.

*d)* $\left[\begin{smallmatrix} K_i;-c \\ t \end{smallmatrix}\right] = \sum_{s=0}^{t} (-1)^s q^{d_i c(t-s)} \left[\begin{smallmatrix} c+s-1 \\ s \end{smallmatrix}\right]_i K_i^{s} \left[\begin{smallmatrix} K_i;0 \\ t-s \end{smallmatrix}\right]$ *, for* $c > 0$.

*e)* $\left[\begin{smallmatrix} K_i;0 \\ x \end{smallmatrix}\right] \left[\begin{smallmatrix} K_i;0 \\ y \end{smallmatrix}\right] = \sum_{j=0}^{x} \left[\begin{smallmatrix} x \\ j \end{smallmatrix}\right]_i \left[\begin{smallmatrix} x+y-j \\ x \end{smallmatrix}\right]_i q^{d_i xy} K_i^{-j} \left[\begin{smallmatrix} K_i;0 \\ x+y-j \end{smallmatrix}\right]$,

where $\left[\begin{smallmatrix} t \\ j \end{smallmatrix}\right]_i := \prod_{s=1}^{j} \frac{q^{(t-s+1)d_i} - q^{(-t+s-1)d_i}}{q^{sd_i} - q^{-sd_i}}$, for $t \in \mathbb{Z}$ and $j \in \mathbb{N}$.

We denote by $U_v$, $U_v^+$, $U_v^0$, $U_v^-$ and $U_v(\mathfrak{b}^-)$ the algebras over $\tilde{\mathbb{Z}}$ obtained from the corresponding forms defined over $A$ by base change $A \to A/(\phi) \hookrightarrow \tilde{\mathbb{Z}}$. For $\lambda \in P^+$ let $V_q(\lambda)$ be the irreducible representation of $U_q$ over $\mathbb{C}(q)$ with highest weight $\lambda$. As in the classical case, we fix for $i = 1, \dots, n$ a highest weight vector $v_{\omega_i} \in V_q(\omega_i)_{\omega_i}$, and let $V_A(\omega_i) = U_A v_{\omega_i}$ be the corresponding $A$-lattice. For $\lambda = \sum a_i \omega_i \in P^+$ denote by $v_\lambda$ the vector $v_{\omega_1}^{\otimes a_1} \otimes \dots \otimes v_{\omega_n}^{\otimes a_n}$, and let $V_A(\lambda)$ be the lattice

$$U_A v_\lambda := U_A(v_{\omega_1}^{\otimes a_1} \otimes \dots \otimes v_{\omega_n}^{\otimes a_n}) \hookrightarrow V_A(\omega_1)^{\otimes a_1} \otimes \dots \otimes V_A(\omega_n)^{\otimes a_n}.$$

Then $U_A v_\lambda$ is indeed $A$-free. We denote by $V_v(\lambda)$ the corresponding representation $V_A(\lambda) \otimes_A \tilde{\mathbb{Z}}$ of $U_v$. Recall that $K_i$ acts on a weight vector $v_\mu \in V_v(\lambda)_\mu$ by multiplication with $v^{d_i \mu(H_i)}$. As in the classical case, let $R_v$ denote the direct sum $\bigoplus_{\lambda \in P^+} V_v(\lambda)^*$, where $V_v(\lambda)^* := \mathrm{Hom}_{\tilde{\mathbb{Z}}}(V_v(\lambda), \tilde{\mathbb{Z}})$.

Let $\ell_i \in \mathbb{N}$ be minimal such that $d_i\ell_i \equiv 0 \bmod \ell$ (recall: $d_i \in \{1,2,3\}$). Then $v^{d_i}$ is a primitive $\ell_i$-th root of unity if $\ell$ is odd and a primitive $2\ell_i$-th root of unity if $\ell$ is even. Note that in either case $K_i^{2\ell_i} = 1$ in $U_v$ (cf. [Lu2, Lemma 4.4 (a)]), as can be easily seen from the following relation in $U_A^0$:

$$\begin{bmatrix} K_i;0 \\ \ell_i \end{bmatrix} \prod_{j=1}^{\ell_i} (q^{d_i j} - q^{-d_i j}) = \prod_{j=1}^{\ell_i} (K_i q^{d_i(-j+1)} - K_i^{-1} q^{d_i(j-1)}).$$

If $\ell$ is odd, then $K_i^{\ell_i} v_\mu = v_\mu$ for all $\mu \in P$ and all weight vectors $v_\mu$, in particular, $K_i^{\ell_i}$ are in the center of $U_v(\mathfrak{b}^-)$. Denote by $J'$ the ideal of $U_v(\mathfrak{b}^-) \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ generated by $(K_i^{\ell_i} - 1)$, $i = 1, \ldots, n$, and let $J$ be the ideal $J' \cap U_v(\mathfrak{b}^-)$ in $U_v(\mathfrak{b}^-)$. Observe that $U_v(\mathfrak{b}^-)$ embeds inside $U_v(\mathfrak{b}^-) \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ since $U_v(\mathfrak{b}^-)$ is $\tilde{\mathbb{Z}}$-free.

Define the pairing $\Phi_v : U_v(\mathfrak{b}^-) \times R_v \to \tilde{\mathbb{Z}}$ by $(u, \xi^\lambda) \mapsto \xi^\lambda(uv_\lambda)$ for $u \in U_v(\mathfrak{b}^-)$ and $\xi^\lambda \in V_v(\lambda)^*$.

**Proposition 2.**

*a) If $\ell$ is odd, then the pairing $\Phi_v$ has radical precisely equal to $(J, 0)$. The induced pairing $\Phi_v' : U_v(\mathfrak{b}^-)/J \times R_v \to \tilde{\mathbb{Z}}$ is hence non-degenerate.*

*b) If $\ell$ is even, then the pairing $\Phi_v$ is non-degenerate.*

*c) The induced map $\psi_v : R_v \to U_v(\mathfrak{b}^-)^*$ is injective, and the image is a subalgebra of $U_v(\mathfrak{b}^-)^*$, where the multiplication of $\xi^\lambda \in V_v(\lambda)^*$ and $\xi^\mu \in V_v(\mu)^*$ is given by:*

$$\big(\xi^\lambda \cdot \xi^\mu\big)(u) := (\xi^\lambda \otimes \xi^\mu)|_{V_v(\lambda+\mu)}(u.v_{\lambda+\mu}) \quad \text{for } u \in U_v(\mathfrak{b}^-).$$

*Proof.* Let $v_\lambda \in V_v(\lambda)$ be the fixed highest weight vector. Recall that $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right] v_\lambda = \left[\begin{smallmatrix} \lambda(H_i) \\ m \end{smallmatrix}\right]_i v_\lambda$, in particular, $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right] v_\lambda = 0$ if $m > \lambda(H_i)$. The same argument as above in the classical case shows that the map $\psi_v : R_v \to U_v(\mathfrak{b}^-)^*$ is injective.

Suppose now $u \in U_v(\mathfrak{b}^-)$ is such that $\Phi_v(u, R_v) = 0$. We can find linearly independent $u_1, \ldots, u_t \in U_v^-$ and some $h_1, \ldots, h_t \in U_v^0$ such that $u = \sum_{i=1}^t u_i h_i$. To say that $u$ is in the radical of the pairing is equivalent to saying that $uv_\lambda = 0$ for all highest weight vectors $v_\lambda \in V_v(\lambda)$, $\lambda \in P^+$. Since the $u_i$ are linearly independent, the vectors $u_i v_\lambda$ are linearly independent for $\lambda \gg 0$. So $uv_\lambda = 0$ for all $\lambda \gg 0$ is equivalent to $h_i v_\lambda = 0$ for all $i = 1, \ldots, t$ and all $\lambda \gg 0$. In the next lemma we determine a basis of $U_v^0 \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ to find those $h \in U_v^0 \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ satisfying this property.

**Lemma 2.**
*The complex algebra $U_v^0 \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ has as basis the monomials $\prod_{i=1}^n (\left[\begin{smallmatrix} K_i;0 \\ \ell_i m_i \end{smallmatrix}\right] K_i^{e_i})$, where $m_i \in \mathbb{N}$ and $0 \le e_i < 2\ell_i$.*

*Proof.* If $\ell_i = 1$ (which can of course only happen if $\ell = 2, d_i = 2$ or $\ell = 3, d_i = 3$ or $\ell = 1$), then the corresponding $i$-part of the monomial is exactly of the form $\left[\begin{smallmatrix} K_i;0 \\ m_i \end{smallmatrix}\right]$ or

$\left[\begin{smallmatrix}K_i;0\\ m_i\end{smallmatrix}\right]K_i$, as in the basis of $U_A^0$ mentioned before. So, without loss of generality, we may assume $\ell_i > 1$.

We show that the monomials listed in the proposition form a subalgebra. Recall that [Lu1, Lemma 34.1.2] $\left[\begin{smallmatrix}x\ell_i\\ j\end{smallmatrix}\right]_i = 0$ unless $\ell_i$ divides $j$, and $\left[\begin{smallmatrix}x\ell_i\\ j\ell_i\end{smallmatrix}\right]_i = \binom{x}{j}$, where $\binom{x}{j}$ is the ordinary binomial coefficient. Further, $v^{d_i\ell_i^2xy} = v^{(r\ell)\ell_ixy}$ for some $r \in \mathbb{N}$, hence it is equal to $\pm 1$. By specializing the relation in Lemma 1 *e)* at $q = v$, we get:

$$\begin{bmatrix}K_i;0\\ x\ell_i\end{bmatrix}\begin{bmatrix}K_i;0\\ y\ell_i\end{bmatrix} = \sum_{j=0}^{x}\binom{x}{j}\binom{x+y-j}{x}(\pm 1)K_i^{-j\ell_i}\begin{bmatrix}K_i;0\\ (x+y-j)\ell_i\end{bmatrix},$$

which proves that the monomials of this type span a subalgebra of $U_v^0$. In $U_A^0$ we have in addition the relation: $(0 < r < \ell_i,\ m \geq 0)$

$$\begin{bmatrix}K_i;0\\ m\ell_i + r\end{bmatrix}\prod_{s=1+m\ell_i}^{r+m\ell_i}(q^{sd_i} - q^{-sd_i}) = \begin{bmatrix}K_i;0\\ m\ell_i\end{bmatrix}\prod_{s=1+m\ell_i}^{r+m\ell_i}(K_iq^{d_i(-s+1)} - K_i^{-1}q^{d_i(s-1)}).$$

If we specialize at $q = v$, then $v^{md_i\ell_i} = v^{-md_i\ell_i} = \pm 1$. Since this term occurs on both sides, we can cancel it and get:

$$\begin{bmatrix}K_i;0\\ m\ell_i + r\end{bmatrix}\prod_{s=1}^{r}(v^{sd_i} - v^{-sd_i}) = \begin{bmatrix}K_i;0\\ m\ell_i\end{bmatrix}\prod_{s=1}^{r}(K_iv^{d_i(-s+1)} - K_i^{-1}v^{d_i(s-1)}).$$

Note that $\prod_{s=1}^{r}(v^{sd_i} - v^{-sd_i}) \neq 0$. Since $K^{2\ell_i} = 1$, this implies that we can express $\left[\begin{smallmatrix}K_i;0\\ m\ell_i+r\end{smallmatrix}\right]$ over $\mathbb{C}$ as a product of $\left[\begin{smallmatrix}K_i;0\\ m\ell_i\end{smallmatrix}\right]$ with a linear combination of $1, K_i, \ldots, K_i^{2\ell_i-1}$. The linear independence of the monomials follows from the description of the basis for $U_A^0$ above. ◇

*Proof of Proposition 2, continuation.* Let $h \in U_v^0$ be such that $hv_\lambda = 0$ for all $\lambda \gg 0$. We write $h$ (viewed as an element of $U_v^0 \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$) as a linear combination

$$h = \sum_{\underline{m}\in\mathbb{N}^n}\left(\prod_{i=1}^{n}\begin{bmatrix}K_i;0\\ \ell_im_i\end{bmatrix}\right)\Bigg(\sum_{\substack{\underline{e}\in\mathbb{N}^n\\ 0\leq e_i<2\ell_i}} b_{\underline{m},\underline{e}}K_1^{e_1}\cdots K_n^{e_n}\Bigg),$$

where $\underline{m} := (m_1, \ldots, m_n)$ and similarly $\underline{e}$.

If $v_\lambda \in V_v(\lambda)$ is a heighest weight vector of weight $\lambda = \sum_{i=1}^{n}(a_i\ell_i + r_i)\omega_i$ with $0 \leq r_i < \ell_i$, then, by [Lu1, Lemma 34.1.2],

$$\begin{bmatrix}K_i;0\\ \ell_im_i\end{bmatrix}v_\lambda = \begin{bmatrix}a_i\ell_i + r_i\\ \ell_im_i\end{bmatrix}_i v_\lambda = \pm\binom{a_i}{m_i}v_\lambda.$$

From this it follows easily that $hv_\lambda = 0$ for all $\lambda \gg 0$ is equivalent to the condition $\sum_{\underline{e}} b_{\underline{m},\underline{e}}K_1^{e_1}\cdots K_n^{e_n}v_\lambda = 0$ for all $\underline{m}$ and all $\lambda \gg 0$.

Suppose now we have such an element $h = \sum_{\underline{e}} b_{\underline{e}} K_1^{e_1} \cdots K_n^{e_n} \neq 0$ and $hv_\lambda = 0$ for all $\lambda \gg 0$. Since $K_i^{2\ell_i} = 1$, this is equivalent to saying that $hv_\lambda = 0$ for all $\lambda = \sum_{i=1}^n a_i\omega_i$ such that $0 \leq a_i < 2\ell_i$.

The $\mathbb{C}$-subalgebra $K$ of $U_v^0 \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ generated by the $K_i$ can be viewed as the group algebra of the group $\prod_{i=1}^n \mathbb{Z}/2\ell_i\mathbb{Z}$. If $\ell$ is odd, then $v^{d_i}$ is a primitive $\ell_i$-th root of unity. The one-dimensional representations provided by the action of the $K_i$'s on the highest weight vectors in $V_v(\lambda)$, $\lambda = \sum_{i=1}^n a_i\omega_i$, $0 \leq a_i < 2\ell_i$, hence does not give a complete list of all irreducible representations of $K$. The intersection of the kernels of these representations is the subalgebra generated by $(K_i^{\ell_i} - 1)$.

If $\ell$ is even, then $v^{d_i}$ is a primitive $2\ell_i$-th root of unity. The one-dimensional representations provided by the action of the $K_i$ on the highest weight vectors in $V_v(\lambda)$, $\lambda = \sum_{i=1}^n a_i\omega_i$, $0 \leq a_i < 2\ell_i$, hence give a complete list of all irreducible representations, so $h = 0$.

The description of the multiplication can be proved as in the classical case. $\diamond$

## 2. The Frobenius maps

We recall in this section the definition of the quantum Frobenius maps Fr and Fr$'$ defined by Lusztig on $U_v^-$ respectively $U_{\tilde{\mathbb{Z}}}^- := U_{\mathbb{Z}}^- \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$. Fix a positive integer $\ell$. To simplify the arguments we assume that $\ell$ is odd, and if $\mathfrak{g}$ has simple factors of type $\mathtt{G}_2$, then we assume $\ell$ to be coprime to 3 in addition. Note that these conditions imply $\ell_i = \ell$ for all $i$; we will make some remarks at the end of this section concerning the cases $\ell = 2, 3$.

G. Lusztig has constructed two algebra homomorphisms (Theorem 35.1.7 and 35.1.8 in [Lu1]):

$\mathrm{Fr} : U_v^- \to U_{\tilde{\mathbb{Z}}}^-$ and $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^- \to U_v^-$, which are defined on the generators by

$$\mathrm{Fr}(F_i^{(k)}) := \begin{cases} 0 & \text{if } \ell \not| k \\ \mathrm{Y}_i^{(k/\ell)} & \text{if } \ell | k \end{cases}; \quad \text{respectively} \quad \mathrm{Fr}'(\mathrm{Y}_i^{(k)}) := F_i^{(\ell k)}.$$

The composition $\mathrm{Fr} \circ \mathrm{Fr}'$ is obviously the identity map on $U_{\tilde{\mathbb{Z}}}^-$. One can of course similarly define $\mathrm{Fr} : U_v^+ \to U_{\tilde{\mathbb{Z}}}^+$ and $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^+ \to U_v^+$. The map Fr can be extended to an algebra homomorphism $\mathrm{Fr} : U_v \to U_{\tilde{\mathbb{Z}}}$ (see [Lu1, Theorem 35.1.9] or [Lu3, Theorem 8.10, §8.11 and Corollary 8.14]):

**Proposition 3.** *The map defined by* $u \mapsto \mathrm{Fr}(u)$ *for* $u \in U_v^-$ *or* $u \in U_v^+$ *and*

$$K_i \mapsto 1, \quad \begin{bmatrix} K_i; 0 \\ m \end{bmatrix} \mapsto \begin{cases} 0 & \text{if } \ell \not| m \\ \binom{H_i}{m/\ell} & \text{if } \ell | m \end{cases}; \quad i = 1, \ldots, n,$$

*extends the Frobenius maps for* $U_v^-$ *and* $U_v^+$ *to a surjective* $\tilde{\mathbb{Z}}$*-algebra homomorphism* $\mathrm{Fr} : U_v \to U_{\tilde{\mathbb{Z}}}$. *Moreover,* Fr *is a Hopf algebra homomorphism.*

The map Fr$'$ can *not* be extended to a homomorphism defined on $U_{\tilde{\mathbb{Z}}}$. Though, we can extend it to a homomorphism defined on $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$, the price for the extension being that the range $U_v(\mathfrak{b}^-)$ is to be replaced by $U_v(\mathfrak{b}^-)/J$. Here $J$ is the ideal defined in the last section (Proposition 2).

**Lemma 3.** *The map defined by* $u \mapsto \mathrm{Fr}'(u)$ *for* $u \in U_{\tilde{\mathbb{Z}}}^-$ *and* $\binom{H_i}{m} \mapsto \left[\begin{smallmatrix} K_i;0 \\ \ell m \end{smallmatrix}\right]$ *extends* $\mathrm{Fr}'$ *to a* $\tilde{\mathbb{Z}}$*-algebra homomorphism (again denoted by)* $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-) \to U_v(\mathfrak{b}^-)/J$.

*We refer to* $\mathrm{Fr}'$ *as the Frobenius splitting homomorphism.*

*Proof.* Recall that $v^\ell = 1$, $\left[\begin{smallmatrix} x\ell \\ j \end{smallmatrix}\right] = 0$ unless $\ell$ divides $j$ and $\left[\begin{smallmatrix} x\ell \\ j\ell \end{smallmatrix}\right] = \binom{x}{j}$. Further, $K_i^\ell = 1$ in $U_v(\mathfrak{b}^-)/J$. Hence Lemma 1*e)* implies that

$$\begin{aligned}
\mathrm{Fr}'(\binom{H_i}{x})\mathrm{Fr}'(\binom{H_i}{y}) = \begin{bmatrix} K_i;0 \\ x\ell \end{bmatrix}\begin{bmatrix} K_i;0 \\ y\ell \end{bmatrix} &= \sum_{j=0}^{x} \binom{x}{j}\binom{x+y-j}{x}\begin{bmatrix} K_i;0 \\ \ell(x+y-j) \end{bmatrix} \\
&= \mathrm{Fr}'(\sum_{j=0}^{x} \binom{x}{j}\binom{x+y-j}{x}\binom{H_i}{x+y-j}) \\
&= \mathrm{Fr}'(\binom{H_i}{x}\binom{H_i}{y}).
\end{aligned}$$

Now, for $y \geq 0$, we have:

$$\mathrm{Fr}'(\binom{H_i+y}{x}) = \mathrm{Fr}'(\sum_{s=0}^{x}\binom{y}{s}\binom{H_i}{x-s}) = \sum_{s=0}^{x}\begin{bmatrix} y\ell \\ s\ell \end{bmatrix}_i \begin{bmatrix} K_i;0 \\ \ell(x-s) \end{bmatrix} = \begin{bmatrix} K_i;\ell y \\ \ell x \end{bmatrix},$$

because the other terms in the expression (Lemma 1 *e*) for $\left[\begin{smallmatrix} K_i;\ell y \\ \ell x \end{smallmatrix}\right]$ vanish. Similarly, for $y > 0$, we get:

$$\begin{aligned}
\mathrm{Fr}'(\binom{H_i-y}{x}) &= \mathrm{Fr}'(\sum_{s=0}^{x}(-1)^s\binom{y+s-1}{s}\binom{H_i}{x-s}) \\
&= \sum_{s=0}^{x}(-1)^s\binom{y+s-1}{s}\begin{bmatrix} K_i;0 \\ \ell(x-s) \end{bmatrix} \\
&= \sum_{s=0}^{x}(-1)^{s\ell}\begin{bmatrix} y\ell+s\ell-1 \\ s\ell \end{bmatrix}_i\begin{bmatrix} K_i;0 \\ \ell(x-s) \end{bmatrix}.
\end{aligned}$$

To prove the last equality, note that $(-1)^{s\ell} = (-1)^s$, and (see [Lu1, Lemma 34.1.2]) $\left[\begin{smallmatrix} y\ell+s\ell-1 \\ s\ell \end{smallmatrix}\right]_i = \binom{y+s-1}{s}\left[\begin{smallmatrix} \ell-1 \\ 0 \end{smallmatrix}\right]_i = \binom{y+s-1}{s}$. Suppose $s' = s\ell + r$ with $0 < r < \ell$. Note that $\left[\begin{smallmatrix} y\ell+s'-1 \\ s' \end{smallmatrix}\right]_i = \pm\binom{y+s}{s}\left[\begin{smallmatrix} r-1 \\ r \end{smallmatrix}\right] = 0$, so Lemma 1 gives (for $y > 0$):

$$\mathrm{Fr}'(\binom{H_i-y}{x}) = \sum_{s'=0}^{x\ell}(-1)^{s'}v^{d_i y\ell(x\ell-s')}\begin{bmatrix} y\ell+s'-1 \\ s' \end{bmatrix}_i\begin{bmatrix} K_i;0 \\ x\ell-s' \end{bmatrix} = \begin{bmatrix} K_i;-\ell y \\ x\ell \end{bmatrix}.$$

From this we conclude that (cf. [Lu3, §6.5])

$$\mathrm{Fr}'(\binom{H_i}{x})\mathrm{Fr}'(Y_j^{(y)}) = \begin{bmatrix} K_i;0 \\ x\ell \end{bmatrix}F_j^{(y\ell)} = F_j^{(y\ell)}\begin{bmatrix} K_i;-y\ell c_{i,j} \\ x\ell \end{bmatrix} = \mathrm{Fr}'(Y_j^{(y)})\mathrm{Fr}'(\binom{H_i-yc_{i,j}}{x}),$$

which shows that the map respects the defining relations between the generators of $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$. ◇

**Remark 2.** The assumption that $\ell$ is coprime to 3 if $\mathfrak{g}$ admits simple factors of type $\mathtt{G}_2$ is not necessary for Proposition 3 and Lemma 3. Actually, the construction makes sense for arbitrary $\ell$, but we have to redefine the maps; for details see [Lu1, Chapter 35]. In the following we mainly concentrate in the remarks on $\ell = 2, 3$, but, with the appropriate adaptions (similarly to those in [L2]), the constructions hold also in the general case.

As before, let $\ell_i$ be minimal such that $d_i\ell_i \equiv 0 \bmod \ell$, and denote by $C^\#$ the matrix $(c_{i,j}\ell_j/\ell_i)$. This is the Cartan matrix of the root system having the roots $\alpha_i^\# := \ell_i\alpha_i$ as simple roots and $H_i^\# := H_i/\ell_i$ as co-roots. Its weight lattice is the subset $P^\# := \{\lambda \in P \mid \lambda(H_i) \in \ell_i\mathbb{Z}\ \forall i\}$ of $P$. Note if $\mu \in P^\# \subset P$ and $v_\mu$ is a weight vector in a $U_v^0$-representation, then:

$$\begin{bmatrix} K_i; k\ell_i \\ m\ell_i \end{bmatrix} v_\mu = \begin{bmatrix} \mu(H_i) + k\ell_i \\ m\ell_i \end{bmatrix}_i v_\mu = \begin{bmatrix} \ell_i\mu(H_i^\#) + k\ell_i \\ m\ell_i \end{bmatrix}_i v_\mu = \binom{\mu(H_i^\#) + k}{m} v_\mu \, .$$

Denote by $\mathfrak{g}^\#$ the corresponding Lie algebra and let $U^\#$ be its enveloping algebra. We use the notation $X_i^\#, Y_i^\#$ and $H_i^\#$ for the generators. If $\mathfrak{g}$ is simply laced or $\ell$ is a prime $> 3$, then $C^\# = C$. But if $\ell = 3$, then $C^\#$ is obtained from $C$ by transposing the $2 \times 2$ submatrices corresponding to simple factors of type $\mathtt{G}_2$. If $\ell = 2$, then the same has to be applied for simple factors of type $\mathtt{F}_4$, $\mathtt{B}_n$ and $\mathtt{C}_n$. The Frobenius homomorphisms $\mathrm{Fr} : U_v^- \to U_{\tilde{\mathbb{Z}}}^{\#^-}$ respectively $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^{\#^-} \to U_v^-$ are defined by

$$\mathrm{Fr}(F_i^{(k)}) := \begin{cases} 0 & \text{if } \ell_i \not| k; \\ Y_i^{\#(k/\ell_i)} & \text{if } \ell_i | k; \end{cases} \qquad \text{respectively} \quad \mathrm{Fr}'(Y_i^{\#(k)}) := F_i^{(\ell_i k)}.$$

If $\ell = 3$, then we extend the Frobenius map to a homomorphism $\mathrm{Fr} : U_v(\mathfrak{b}^-) \to U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^{\#^-})$ by setting $\mathrm{Fr}(K_i) = 1$, $\mathrm{Fr}(\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]) = \left[\begin{smallmatrix} H_i^\# \\ m/\ell_i \end{smallmatrix}\right]$ if $\ell_i$ divides $m$ and $\mathrm{Fr}(\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]) = 0$ otherwise. Similarly, one can extend $\mathrm{Fr}'$ to a homomorphism $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^{\#^-}) \to U_v(\mathfrak{b}^-)/J$ by setting $\mathrm{Fr}(\left[\begin{smallmatrix} H_i^\# \\ m \end{smallmatrix}\right]) = \left[\begin{smallmatrix} K_i;0 \\ m\ell_i \end{smallmatrix}\right]$. The details of the proof are left to the reader.

The definitions of $\mathrm{Fr} : U_v^- \to U_{\tilde{\mathbb{Z}}}^{\#^-}$ and $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^{\#^-} \to U_v^-$ given above make sense for arbitrary positive integer $\ell$. To avoid problems with the definition of the extensions for $\ell = 2$, we assume that $\ell = 2d$, where $d$ is the smallest common multiple of $d_1, \ldots, d_n$. Since $\ell_i = \ell/d_i = 2(d/d_i)$, we know that all the $\ell_i$ are even. Denote by $(U_v^0)_{ev}$ the subalgebra of $U_v^0$ generated by $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]$ and $K_i^m$, $m$ even, and $K_i\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]$ for $m$ odd, and let $(U_v^-)_{ev}$ be the subalgebra of $U_v^-$ spanned by the monomials of weight $-2\beta$, $\beta \in Q^+$. Let $U_v(\mathfrak{b}^-)_{ev}$ be the subalgebra of $U_v(\mathfrak{b}^-)$ generated by $(U_v^-)_{ev}$ and $(U_v^0)_{ev}$. Note that $\mathrm{Fr}'(Y_i^{\#(k)}) = F_i^{(\ell_i k)} \in U_v(\mathfrak{b}^-)_{ev}$ because the $\ell_i$ are even.

Using Lemma 1, it is easy to verify that $U_v(\mathfrak{b}^-)_{ev}$ is spanned by the elements of the form $u\prod_{i=1}^n(\left[\begin{smallmatrix} K_i;0 \\ m_i \end{smallmatrix}\right]K_i^{e_i})$, where $u \in (U_v^-)_{ev}$, $m_i \in \mathbb{N}$ and $e_i \in \{0, 1\}$ with $m_i + e_i$ even.

The elements $(K_i^{\ell_i} - 1)$ are in the center of the even subalgebra. As in the odd case, let $J'$ be the ideal of $U_v(\mathfrak{b}^-)_{ev} \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ generated by the elements $(K_i^{\ell_i} - 1)$, $i = 1, \ldots, n$, and let $J$ be the (two sided) ideal $J' \cap U_v(\mathfrak{b}^-)_{ev}$.

Denote by $V_v(\lambda)_{ev}$ the direct sum $\bigoplus_\mu V_v(\lambda)_\mu$ of all weight spaces corresponding to the weights of the form $\mu = \lambda - 2\beta$, $\beta \in Q^+$, and set $R_{v,ev} := \bigoplus_{\lambda \in 2P^+} (V_v(\lambda)_{ev})^*$. Proposition 2 can then be reformulated as: The pairing $\Phi_v : U_v(\mathfrak{b}^-)_{ev} \times R_{v,ev} \to \tilde{\mathbb{Z}}$ defined by $(u, \xi^\lambda) \mapsto \xi^\lambda(uv_\lambda)$ for $u \in U_v(\mathfrak{b}^-)_{ev}$ and $\xi^\lambda \in (V_v(\lambda)_{ev})^*$, has as radical precisely $(J, 0)$, and hence the induced pairing $U_v(\mathfrak{b}^-)_{ev}/J \times R_{v,ev} \to \tilde{\mathbb{Z}}$ is non-degenerate. In particular, the induced map $\psi_v : R_{v,ev} \to (U_v(\mathfrak{b}^-)_{ev})^*$ is injective, and the image is a subalgebra of $(U_v(\mathfrak{b}^-)_{ev})^*$. The Frobenius maps can also be extended correspondingly: the map defined by $u \mapsto \mathrm{Fr}(u)$ for $u \in U_{v,ev}^-$, $K_i^2 \mapsto 1$, $K_i \left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right] \mapsto 0$ for $m$ odd, $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right] \mapsto 0$ if $m$ is even and $\ell_i \not| m$, and $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right] \mapsto \binom{H_i^\#}{m/\ell_i}$ if $m$ is even and $\ell_i | m$, extends Fr to an algebra homomorphism $\mathrm{Fr} : U_v(\mathfrak{b}^-)_{ev} \to U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^{\#-})$. Similarly, the map defined by $u \mapsto \mathrm{Fr}'(u)$ for $u \in U_{\tilde{\mathbb{Z}}}^{\#-}$ and $\binom{H_i^\#}{m} \mapsto \left[\begin{smallmatrix} K_i;0 \\ \ell_i m \end{smallmatrix}\right]$ extends $\mathrm{Fr}'$ to an algebra homomorphism $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^{\#-}) \to U_v(\mathfrak{b}^-)_{ev}/J$. The proofs are very similar to the proofs above, and hence the details are left to the reader.

## 3. The dual maps $\mathrm{Fr}^*$ and $\mathrm{Fr}'^*$

We assume again that $\ell$ is an odd integer and moreover coprime to 3 if $\mathfrak{g}$ has simple factors of type $\mathtt{G}_2$. We make some remarks concerning the general case at the end of the section. The ideal $J$ (see Proposition 2) is in the kernel of Fr, so we get an induced map $\mathrm{Fr}^*$ between the Hopf dual $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)^*$ of $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$ and the Hopf dual $U_v(\mathfrak{b}^-)^*$ of $U_v(\mathfrak{b}^-)$.

The $U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$-module $V_v(\lambda) \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$, $\lambda \in P^+$, is in general not a simple module. Denote by $L_v(\lambda)$ its simple quotient. By Lustzig [Lu2, Proposition 7.2], $E_i, F_i$ and $K_i - 1$ operate trivially on $L_v(\ell\lambda)$ for $\lambda \in P^+$. Further, as in [Lu2, Proposition 7.5], the Frobenius splittings $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^- \to U_v^-$ and $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^+ \to U_v^+$ can be glued together to a surjective homomorphism (in fact an isomorphism) $F : U \simeq U_{\tilde{\mathbb{Z}}} \otimes_{\tilde{\mathbb{Z}}} \mathbb{C} \to U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}/\langle E_i, F_i, K_i - 1\rangle$, and $L_v(\ell\lambda)$ becomes via $F$ a simple $U$-module $V(\lambda)$ of highest weight $\lambda$. We can also view $L_v(\ell\lambda)$ the other way around: We start with the irreducible $U$-module $V(\lambda)$ and make it into an $U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$-module $V(\lambda)^{\mathrm{Fr}}$ via the Frobenius homomorphism $\mathrm{Fr} : U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C} \to U$ (cf. Proposition 3). Then, Fr being surjective, $V(\lambda)^{\mathrm{Fr}}$ is an irreducible $U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$-module. It is easy to see that $V(\lambda)^{\mathrm{Fr}}$ is, in fact, isomorphic with $L_v(\ell\lambda)$. For each fundamental weight $\omega_i$ $(1 \le i \le n)$, choose an isomorphism $\varphi_i : V(\omega_i)^{\mathrm{Fr}} \simeq L_v(\ell\omega_i)$ such that $\bar{v}_{\omega_i} \in V(\omega_i)$ corresponds to $v_{\ell\omega_i} \in L_v(\ell\omega_i)$ (cf. §1 for the notation $\bar{v}_{\omega_i}$ and $v_{\omega_i}$). Since Fr is a Hopf algebra homomorphism, the isomorphisms $\varphi_i$ give rise to an $U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$-module isomorphism $\varphi_\lambda : V(\lambda)^{\mathrm{Fr}} \simeq L_v(\ell\lambda)$ (for all $\lambda \in P^+$) so that $\bar{v}_\lambda \in V(\lambda)$ corresponds to $v_{\ell\lambda} \in L_v(\ell\lambda)$. In the sequel, we fix such an isomorphism $\varphi_\lambda$ for each $\lambda \in P^+$. For any $\lambda \in P^+$ let $L_v(\lambda)_{\tilde{\mathbb{Z}}}$ be the $U_v$-submodule of $L_v(\lambda)$ generated by $v_\lambda$.

We thus get a 'natural' $U_v$-module isomorphism $L_v(\ell\lambda)_{\tilde{\mathbb{Z}}} \to V_{\tilde{\mathbb{Z}}}(\lambda)^{\mathrm{Fr}}$ and hence the dual map $\left(V_{\tilde{\mathbb{Z}}}(\lambda)^{\mathrm{Fr}}\right)^* \to (L_v(\ell\lambda)_{\tilde{\mathbb{Z}}})^*$, where $V_{\tilde{\mathbb{Z}}}(\lambda) := V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$.

Define the map $\mathrm{Fr}^\vee : R_{\tilde{\mathbb{Z}}} \to R_v$, as the direct sum of the composite maps $V_{\tilde{\mathbb{Z}}}(\lambda)^* \to (L_v(\ell\lambda)_{\tilde{\mathbb{Z}}})^* \to V_v(\ell\lambda)^*$, where the last map is the dual of the quotient map $V_v(\ell\lambda) \to L_v(\ell\lambda)_{\tilde{\mathbb{Z}}}$, and $R_{\tilde{\mathbb{Z}}} := R_{\mathbb{Z}} \otimes {}_{\mathbb{Z}}\tilde{\mathbb{Z}}$.

**Proposition 4.** *The map* $\mathrm{Fr}^\vee$ *is nothing but the restriction of* $\mathrm{Fr}^*$ *to* $R_{\tilde{\mathbb{Z}}}$ *under the identification of* $R_{\tilde{\mathbb{Z}}}$ *(resp.* $R_v$*) as a subalgebra of* $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)^*$ *(resp.* $U_v(\mathfrak{b}^-)^*$*) induced by the pairing* $\Phi$ *(resp.* $\Phi_v$*), cf. Propositions 1,2.*

*Equivalently, for any* $X \in U_v(\mathfrak{b}^-)$ *and* $\xi \in R_{\tilde{\mathbb{Z}}}$ *we have*

$$(1) \quad \Phi(\mathrm{Fr}X, \xi) = \Phi_v(X, \mathrm{Fr}^\vee \xi).$$

*Proof.* Equivalence of the two assertions is easy and the identity (1) follows readily from the definition of $\mathrm{Fr}^\vee$. $\diamond$

From now on, we will denote (by abuse of notation) $\mathrm{Fr}^\vee$ by $\mathrm{Fr}^*$ itself.

Similarly the Hopf algebra homomorphism $\mathrm{Fr}'$ gives rise to the dual map $\mathrm{Fr}'^* : (U_v(\mathfrak{b}^-)/J)^* \to U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)^*$. As above, one proves that the dual map $\mathrm{Fr}'^*$ induces in fact a map $R_v \to R_{\tilde{\mathbb{Z}}}$.

To describe this map more explicitly, let $\lambda \in P^+$ be a dominant weight. For the Weyl module $V_v(\ell\lambda)$ for $U_v$ denote by $V_v(\ell\lambda)^{\frac{1}{\ell}}$ the direct sum $\bigoplus_{\mu \in \ell P} V_v(\ell\lambda)_\mu$ of all weight spaces corresponding to the weights in $\ell P$. If $\mu = \ell\mu_1$ is a weight in $\ell P$, then so is the weight $\mu \pm n\ell\alpha_i = \ell(\mu_1 \pm n\alpha_i)$. It follows that $V_v(\ell\lambda)^{\frac{1}{\ell}}$ is stable under the action of all the $F_i^{(n\ell)}$ and $E_i^{(n\ell)}$.

We make $V_v(\ell\lambda)^{\frac{1}{\ell}}$ into an $U_{\tilde{\mathbb{Z}}}^-$-module respectively $U_{\tilde{\mathbb{Z}}}^+$-module via the homomorphism $\mathrm{Fr}'$ (i.e. by letting $X_i^{(m)}$ act as $E_i^{(\ell n)}$ and $Y_i^{(m)}$ act as $F_i^{(\ell n)}$). A simple calculation (see for example [Li2] for details) shows that if we let $\binom{H_i}{m}$ act as $\left[\begin{smallmatrix} K_i;0 \\ m\ell \end{smallmatrix}\right]$, then this defines a $U_{\tilde{\mathbb{Z}}}$-module structure on $V_v(\ell\lambda)^{\frac{1}{\ell}}$, and the submodule generated by the highest weight vector $v_{\ell\lambda}$ is isomorphic to $V_{\tilde{\mathbb{Z}}}(\lambda)$. Again we choose an isomorphism so that $v_{\ell\lambda}$ corresponds to $\bar{v}_\lambda$.

Similar to Proposition 4, we obtain:

**Proposition 5.** *The restriction of the dual map* $\mathrm{Fr}'^*$ *to* $V_v(\ell\lambda)^*$ *is the dual map of the inclusion* $V_{\tilde{\mathbb{Z}}}(\lambda) \hookrightarrow V_v(\ell\lambda)$*, and* $\mathrm{Fr}'^*|_{V_v(\mu)^*} = 0$ *for* $\mu \notin \ell P^+$.

**Remark 3.** Recall that we can not extend $\mathrm{Fr}'$ to an algebra homomorphism on the full enveloping algebra, so $V_v(\ell\lambda)$ is not naturally endowed with a structure as an $U_{\tilde{\mathbb{Z}}}$-module. The inclusion $V_{\tilde{\mathbb{Z}}}(\lambda) \hookrightarrow V_v(\ell\lambda)$ hence does not give rise to an $U_{\tilde{\mathbb{Z}}}$-equivariant map $V_v(\ell\lambda)^* \to V_{\tilde{\mathbb{Z}}}(\lambda)^*$. But, using the Frobenius maps $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^- \to U_v^-$ and $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^+ \to$

$U_v^+$, we can make $V_v(\ell\lambda)$ into an $U_{\tilde{\mathbb{Z}}}^-$- respectively $U_{\tilde{\mathbb{Z}}}^+$-module, and, by the definition of the inclusion $V_{\tilde{\mathbb{Z}}}(\lambda) \hookrightarrow V_v(\ell\lambda)^{\frac{1}{\ell}} \hookrightarrow V_v(\ell\lambda)$, the map $V_{\tilde{\mathbb{Z}}}(\lambda) \hookrightarrow V_v(\ell\lambda)$ is equivariant with respect to the action of $U_{\tilde{\mathbb{Z}}}^+$ and $U_{\tilde{\mathbb{Z}}}^-$, and hence so is the dual map.

**Remark 4.** The composition

$$U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-) \xrightarrow{\mathrm{Fr}'} U_v(\mathfrak{b}^-)/J \xrightarrow{\mathrm{Fr}} U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$$

is the identity map, and hence so is $R_{\tilde{\mathbb{Z}}} \xrightarrow{\mathrm{Fr}^*} R_v \xrightarrow{\mathrm{Fr}'^*} R_{\tilde{\mathbb{Z}}}$.

**Remark 5.** If $\ell = 2d$, then $\mathrm{Fr}^*$ induces a map $R_{\tilde{\mathbb{Z}}} \to R_{v,ev}$, which is the direct sum of the duals of the quotient maps $V_v(\lambda) \to L_v(\lambda)_{\tilde{\mathbb{Z}}} = V_{\tilde{\mathbb{Z}}}(\lambda^\#)$, for $\lambda^\# \in P^\#$, and the restriction of the dual map $\mathrm{Fr}'^*$ to $V_v(\lambda)^*$ is the dual map of the inclusion $V_{\tilde{\mathbb{Z}}}(\lambda^\#) \hookrightarrow V_v(\lambda)$, and $\mathrm{Fr}'^*|_{V_v(\mu)^*} = 0$ for $\mu^\# \notin P^\#$. To see this, let $\lambda^\# \in P^\# \subset P$ be a dominant weight, we write just $\lambda$ for the weight if we view it as a $U_v$-weight. We make $V_{\tilde{\mathbb{Z}}}(\lambda^\#)$ as above into an $U_v^-$- and $U_v^+$-module by using the Frobenius map Fr, and we let $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]$ act on $V_{\tilde{\mathbb{Z}}}(\lambda^\#)$ as $\binom{H_i^\#}{m/\ell_i}$ if $m$ is divisible by $\ell_i$, and as 0 if $\ell_i \not| m$.

Then as above, the three actions glue together to give a $U_v$-module structure on $V_{\tilde{\mathbb{Z}}}(\lambda^\#)$ such that $u \in J$ acts trivially. Thus $V_{\tilde{\mathbb{Z}}}(\lambda^\#)$ becomes in this way a highest weight module for $U_v$ of heighest weight $\lambda$. Now $V(\lambda^\#)$ is a simple module for $U(\mathfrak{g}^\#)$ and hence for $U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$. So, as above, we can view $\mathrm{Fr}^*$ as the dual map of the quotient map $V_v(\lambda) \to L_v(\lambda) = V_{\tilde{\mathbb{Z}}}(\lambda^\#)$.

Actually, with the appropriate adaptions (for details see [Lu1, Chapter 35]) one can reformulate the results for arbitrary $\ell$.

## 4. Base change

In the following we assume that $\ell = p$ is in fact an odd prime, and further $p > 3$ if $\mathfrak{g}$ has factors of type $\mathtt{G}_2$. Let $k$ be an algebraically closed field of char. $p$. We consider $k$ as a $\tilde{\mathbb{Z}}$-module by extending the canonical map $\mathbb{Z} \to k$ to a ring homomorphism $\tilde{\mathbb{Z}} \to k$ (where the first map is given by the projection $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ and the inclusion $\mathbb{Z}/p\mathbb{Z} \subset k$, and the extension $\tilde{\mathbb{Z}} \to k$ follows from the 'Going-up theorem' [M, Theorem 9.3]). We denote by $U_k, U_{v,k}$ the corresponding algebras obtained by the base change.

Note that the image of $v$ in $k$ is 1. Let $J_p$ be the ideal in $U_{v,k}$ generated by the central elements $(K_i - 1)$, $i = 1, \ldots, n$; then the quotient $U_{v,k}/J_p$ is naturally isomorphic to $U_k$. Further, let $\mu \in P^+$ be a dominant weight and $V_{v,k}(\mu)$ be the corresponding Weyl module for $U_{v,k}$. Since all the $(K_i - 1)$ operate trivially on $V_{v,k}(\mu)$, this becomes in a natural way the Weyl module $V_k(\mu)$ for $U_k$. Let $L_k(\mu)$ be the $U_k$-module $L_v(\mu)_{\tilde{\mathbb{Z}}} \otimes_{\tilde{\mathbb{Z}}} k$.

We are now left with only one algebra, namely $U_k$. The module $L_k(p\lambda)$ is, as a $U_k$-module the same as the $U_k$-module $V_k(\lambda)^{\mathrm{Fr}}$, where as in §3, $V_k(\lambda)^{\mathrm{Fr}}$ is the same $k$-vector space as $V_k(\lambda)$, but its $U_k$-module structure has been twisted via the Frobenius map $\mathrm{Fr}_k : U_k \to U_k$ given by $F_i^{(m)} \mapsto F_i^{(m/p)}; E_i^{(m)} \mapsto E_i^{(m/p)}$, if $m$ is divisible by $p$ and 0 otherwise.

We are going to twist the $k$-vector space structure of $V_k(\lambda)^{\mathrm{Fr}}$. Let $\phi : k \mapsto k$ be the ring homomorphism given by the inverse of the $p$-th power map, i.e., $z \mapsto z^{1/p}$, and denote by $V_k(\lambda)^{(1)}$ the $k$-vector space (and $U_k$-module) having as underlying abelian group the same as $V_k(\lambda)$, but where the scalar multiplication has been twisted by $\phi$: $a * v := \phi(a)v$, and where $U_k$ acts as on $V_k(\lambda)^{\mathrm{Fr}}$. (Note that the operation of $U_k$ is linear also with respect to the twisted scalar multiplication.) The $U_k$-module $V_k(\lambda)^{(1)}$ can be seen explicitly as a quotient of $V_k(p\lambda)$ as follows: The map $V_k(\lambda)^{(1)} \to S^pV_k(\lambda)$, defined by $v \mapsto v^p$ (which is linear because of the twisted scalar multiplication), induces an isomorphism onto the image of the canonical map $V_k(p\lambda) \to S^pV_k(\lambda)$ which sends the highest weight vector $v_{p\lambda} \in V_k(p\lambda)$ to the highest weight vector $v_\lambda^p \in S^pV_k(\lambda)$.

Let $G$ be the semisimple simply connected algebraic group over $k$ corresponding to the Lie algebra $\mathfrak{g}$ and let $B \subset G$ be the Borel subgroup corresponding to the Lie algebra $\mathfrak{b}$. Let $\mathcal{L}_\lambda$ be the line bundle on $X := G/B$ corresponding to a weight $-\lambda$. Recall that for $\lambda$ dominant, we have $H^0(X, \mathcal{L}_\lambda)$, as $G$-module, isomorphic to $V_k(\lambda)^*$. It follows from the considerations above that the dual map $\mathrm{Fr}^* : H^0(X, \mathcal{L}_\lambda)^{(1)} \to H^0(X, \mathcal{L}_{p\lambda})$ is just the $p$-th power map sending a section $s \in H^0(X, \mathcal{L}_\lambda)$ to $s^p \in H^0(X, \mathcal{L}_{p\lambda})$ (again, recall that this map is linear with respect to the twisted scalar multiplication).

The inclusion $V_k(\lambda) \hookrightarrow V_k(p\lambda)$ respectively $\mathrm{Fr}'^* : H^0(X, \mathcal{L}_{p\lambda}) \to H^0(X, \mathcal{L}_\lambda)$, the associated dual map, does not have such an equivariant interpretation, but Remark 4 implies that $\mathrm{Fr}'^*$ is a section to $\mathrm{Fr}^*$. Observe that $\mathrm{Fr}'^*$ restricted to $H^0(X, \mathcal{L}_\lambda)$ is zero if $\lambda \notin pP^+$.

**Theorem 1.** *The dual map* $\mathrm{Fr}^* : H^0(X, \mathcal{L}_\lambda) \to H^0(X, \mathcal{L}_{p\lambda})$ *is the map* $s \mapsto s^p$ *sending a section to its p-th power, and* $\mathrm{Fr}'^* : H^0(X, \mathcal{L}_{p\lambda}) \to H^0(X, \mathcal{L}_\lambda)$ *provides a splitting of this map. For any* $s \in H^0(X, \mathcal{L}_{j\lambda})$ *and* $f \in H^0(X, \mathcal{L}_{m\lambda})$*, the Frobenius map satisfies the following properties:*

$$(a)\ \ \mathrm{Fr}'^*(s^pf) = s\mathrm{Fr}'^*(f),\ \text{and}$$

$$(b)\ \ \mathrm{Fr}'^*(X_i^{(pq)}f) = X_i^{(q)}\mathrm{Fr}'^*(f),\ \text{for all}\ 1 \le i \le n,\ \text{and}\ q \in \mathbb{N}.$$

**Remark 6.** For notational convenience assume that $\lambda \notin pP^+$. The property (a) implies that $\mathrm{Fr}'^*$ induces a graded Frobenius endomorphism of the graded algebra $S := \bigoplus_{m\ge0} H^0(X, \mathcal{L}_{m\lambda})$, more specifically, $\mathrm{Fr}'^*$ maps the homogeneous elements of degree not divisible by $p$ to zero and if $f$ is of degree $qp$ then $\mathrm{Fr}'^*(f)$ is of degree $q$, the map is additive: $\mathrm{Fr}'^*(s_1 + s_2) = \mathrm{Fr}'^*(s_1) + \mathrm{Fr}'^*(s_2)$, and $\mathrm{Fr}'^*(s_1^ps_2) = s_1\mathrm{Fr}'^*(s_2)$. The second property implies that $\mathrm{Fr}'^*$ is the *canonical* splitting, see [Ma]. In particular, $\mathrm{Fr}'^*$ maps $B$-modules to $B$ modules.

*Proof.* It remains to prove that the two properties (a) and (b) hold. For notational convenience assume that $\lambda \notin pP^+$. If $m$ is not divisible by $p$, then in both the equalities all the terms on the right and left are zero, so the properties hold in this case trivially. Suppose now that $m$ is divisible by $p$, say $m = pq$. Again, both the properties hold trivially if $f$ is a weight vector of a weight not divisible by $p$, so in the following we may assume that $f$ is a weight vector corresponding to a weight divisible by $p$. The element $f$ is hence an element of $(V_k(pq\lambda)^{\frac{1}{p}})^*$. Recall that the embedding $\iota : V_k(q\lambda) \hookrightarrow V_k(pq\lambda)^{\frac{1}{p}}$ satisfies $\iota(X_i^{(s)}v) = X_i^{(ps)}\iota(v)$, so the corresponding property holds also for the dual map $\mathrm{Fr}'^* : H^0(X, \mathcal{L}_{pq\lambda}) \to H^0(X, \mathcal{L}_{q\lambda})$. This implies the second property in the theorem above.

Abbreviate the module $L_v(\lambda)_{\tilde{\mathbb{Z}}}$ by $L_v(\lambda)$. To prove the first property, consider the following diagram of Weyl modules (for $U_{\tilde{\mathbb{Z}}}$ respectively $U_v$) defined over $\tilde{\mathbb{Z}}$: There are two inclusions of $U_{\tilde{\mathbb{Z}}}$-modules using the Frobenius map: $V_{\tilde{\mathbb{Z}}}((q+j)\lambda) \hookrightarrow V_v(p(q+j)\lambda)^{\frac{1}{p}}$, and the other inclusion is $V_{\tilde{\mathbb{Z}}}(j\lambda) \otimes V_{\tilde{\mathbb{Z}}}(q\lambda) \hookrightarrow L_v(pj\lambda) \otimes V_{\tilde{\mathbb{Z}}}(pq\lambda)^{\frac{1}{p}}$, using the fact that $L_v(pj\lambda) = V_{\tilde{\mathbb{Z}}}(j\lambda)^{\mathrm{Fr}}$ as $U_v$-module. Note that $E_i^{(mp)}$ acts on $V_{\tilde{\mathbb{Z}}}(j\lambda)$ as $X_i^{(m)}$, so $\mathrm{Fr}'(X_i^{(m)})$ acts on $L_v(pj\lambda) = V_{\tilde{\mathbb{Z}}}(j\lambda)$ as $X_i^{(m)}$. Then we have two inclusions of $U^-_{\tilde{\mathbb{Z}}}$- respectively $U^+_{\tilde{\mathbb{Z}}}$-modules which act on the $U_v$-modules via $\mathrm{Fr}'$: The inclusions are $V_v(p(q+j)\lambda)^{\frac{1}{p}} \hookrightarrow V_v(p(q+j)\lambda)$ and $L_v(pj\lambda) \otimes V_{\tilde{\mathbb{Z}}}(pq\lambda)^{\frac{1}{p}} \hookrightarrow L_v(pj\lambda) \otimes V_{\tilde{\mathbb{Z}}}(pq\lambda)$. In addition we have two maps between Weyl modules: $V_{\tilde{\mathbb{Z}}}((j+q)\lambda) \to V_{\tilde{\mathbb{Z}}}(j\lambda) \otimes V_{\tilde{\mathbb{Z}}}(q\lambda)$ and $V_v(p(j+q)\lambda) \to V_v(pj\lambda) \otimes V_v(pq\lambda)$.

$$
\begin{array}{ccccc}
 & V_v(p(q+j)\lambda)^{\frac{1}{p}} \hookrightarrow & V_v(p(q+j)\lambda) & \to V_v(pj\lambda) \otimes V_v(pq\lambda) \\
 \nearrow & & & \\
V_{\tilde{\mathbb{Z}}}((j+q)\lambda) & & & \downarrow \\
 \searrow & & & \\
 & V_{\tilde{\mathbb{Z}}}(j\lambda) \otimes V_{\tilde{\mathbb{Z}}}(q\lambda) \hookrightarrow & L_v(pj\lambda) \otimes V_v(pq\lambda)^{\frac{1}{p}} & \hookrightarrow L_v(pj\lambda) \otimes V_v(pq\lambda)
\end{array}
$$

The vertical map is the identity on the second factor and the projection on the first. All these maps are equivariant with respect to the $U^{\pm}_{\tilde{\mathbb{Z}}}$-actions on these spaces, they all map the highest weight vector (resp. the tensor product of highest weight vectors) to a highest weight vector (resp. the tensor product of highest weight vectors). It follows that the diagram is commutative and provides two different ways to construct a map $V_{\tilde{\mathbb{Z}}}((j+q)\lambda) \to L_v(pj\lambda) \otimes V_v(pq\lambda)$.

Over the field $k$, the dual of the bottom row is the map $H^0(X, \mathcal{L}_{j\lambda}) \otimes H^0(X, \mathcal{L}_{pq\lambda}) \to H^0(X, \mathcal{L}_{(q+j)\lambda})$ defined by $s \otimes f \mapsto s\mathrm{Fr}'^*(f)$ for sections $s \in H^0(X, \mathcal{L}_{j\lambda})$ and $f \in H^0(X, \mathcal{L}_{pq\lambda})$.

The dual of the top row provides a decomposition of this map in the following way: $s \otimes f \in H^0(X, \mathcal{L}_{j\lambda}) \otimes H^0(X, \mathcal{L}_{pq\lambda})$ is first mapped to $s^p \otimes f \in H^0(X, \mathcal{L}_{pj\lambda}) \otimes H^0(X, \mathcal{L}_{pq\lambda})$, then to the product $s^p f \in H^0(X, \mathcal{L}_{p(j+q)\lambda})$, and then to $\mathrm{Fr}'^*(s^p f) \in H^0(X, \mathcal{L}_{(j+q)\lambda})$. Since the two maps are the same, it follows $\mathrm{Fr}'^*(s^p f) = s\mathrm{Fr}'^*(f)$. This proves (b). $\diamond$

**References.**

**[CP]** V. Chari, A. Pressley, *A guide to quantum groups*, Cambridge University Press, (1994).

**[D]** S. Donkin, *Rational representations of algebraic groups*, LNM **1140**, Springer-Verlag (1985).

**[J]** J.C. Jantzen, *Representations of algebraic groups*, Academic Press, Orlando (1987).

**[LS]** V. Lakshmibai, and C.S. Seshadri, *Standard monomial theory*, in: Proc. Hyderabad Conference on Algebraic Groups, (S. Ramanan et. al., eds.), Manoj Prakashan, Madras (1991), pp. 279-323.

**[LLM]** V. Lakshmibai, P. Littelmann and P. Magyar, *Standard monomial theory and applications*, in: Representation Theory and Geometry, (B. Broer et. al., eds.), Kluwer Academic Publishers (1998), pp. 319–364.

**[Li1]** P. Littelmann, *The path model, the quantum Frobenius map and standard monomial theory*, in: Algebraic Groups and Their Representations (R. Carter and J. Saxl, eds.), Kluwer Academic Publishers (1998), pp. 175–212.

**[Li2]** P. Littelmann, *Contracting modules and standard monomial theory*, JAMS **11**, (1998), pp. 551–567.

**[Lu1]** G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics **110**, Birkhäuser-Verlag, Boston (1993).

**[Lu2]** G. Lusztig, *Modular representations and quantum groups*, in: Classical Groups and Related Topics, Beijing, (A. J. Hahn, D. G. James and Zhe Xian Wan, eds.), Contemporary Mathematics **82**, AMS, Providence, RI (1987), pp. 59–77.

**[Lu3]** G. Lusztig, *Quantum groups at roots of 1*, Geom. Dedicata **35**, (1990), pp. 89–113.

**[Ma]** O. Mathieu, *Filtrations of G-modules*, Ann. Scient. ENS **23**, (1990), pp. 625–644.

**[M]** H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press (1989).

**[MR]** V. B. Mehta, A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. Math. **122** (1985), pp. 27–40.

**[RR]** S. Ramanan, A. Ramanathan, *Projective normality of flag varieties and Schubert varieties*, Invent. math. **79** (1985), pp. 225–246.

**[Ra1]** A. Ramanathan, *Schubert varieties are arithmetically Cohen-Macaulay*, Invent. math. **80** (1985), pp. 283–294.

**[Ra2]** A. Ramanathan, *Equations defining Schubert varieties and Frobenius splitting of diagonals*, Publ. Math. Inst. Hautes Etud. Sci. **65**, (1987), pp. 61–90.