A PIERI–CHEVALLEY FORMULA FOR $K(G/B)$ AND STANDARD MONOMIAL THEORY

P. LITTELMANN AND C.S. SESHADRI

1. Introduction

Let G be a semi-simple, simply connected algebraic group defined over an algebraically closed field k. Following the usual practice, we fix a Borel subgroup B of G, a maximal torus T contained in B and denote by W the Weyl group of G. Thus the Schubert varieties $X(\tau)$ in G/B are canonically indexed by W; in fact for $\tau \in W$, $X(\tau)$ is the closure of the B-orbit through τ, τ being identified canonically with a T-fixed point in G/B. Let Δ denote the weight lattice, identified canonically with the characters of T (and of B). Then for $\lambda \in \Delta$ we have a canonical line bundle $\mathcal{L}_\lambda$ on G/B. We denote by Δ^+ the set of dominant weights (i.e. $\mathcal{L}_\lambda$ has a non-zero section on G/B). More generally we can take a parabolic subgroup Q containing B. Let W_Q denote the Weyl group of Q. Then the Schubert varieties in G/Q are indexed by W/W_Q. We denote by Δ_Q the sublattice of Δ corresponding to characters of T which can be extended to Q, or equivalently the line bundle $\mathcal{L}_\lambda$ on G/B "goes down" to a line bundle on G/Q for the canonical morphism $G/B \longrightarrow G/Q$. We have similar definitions of Δ_Q^+ and Δ_Q^{++} and characterisations by means of the line bundles which they define on G/Q.

Recall that the Chow ring Ch(G/B) has a $\mathbb{Z}$-basis consisting of $[X(\tau)]$ – the class or object in Ch(G/B) defined by the cycle represented by the Schubert variety $X(\tau)$, $\tau \in W$. The class $[\mathcal{L}_\lambda]$ is defined by a codimension one cycle. Then we have the following expression for the product in the Chow ring:

$$[\mathcal{L}] \cdot [X(\tau)] = \sum a_i [X(\sigma_i)] \tag{1}$$

where $X(\sigma)$ is of codimension one in $X(\tau)$ and a_i is the intersection multiplicity along $X(\sigma_i)$. Further, a_i is given by a formula due to Chevalley ([Ch]). Since every weight is a linear combination of fundamental weights, to compute a_i, we could assume that λ is a fundamental weight ω, further if P is the maximal parabolic subgroup, $P \supset B$, associated to ω, by using the projection formula, one sees that one can indeed work in G/P i.e. $X(\tau) \subset G/P$ and $\mathcal{L}_\lambda$ is the ample generator of Pic G/P. When $G = SL(n)$, G/P is the Grassmannian and in the RHS of (1), σ_i runs over all the Schubert varieties of codimension one and $a_i = 1$ for all i. This is the classical formula of Pieri.

SUPPORTED BY TMR-GRANT ERB FMRX-CT97-0100.

Let $K(G/B)$ be the Grothendieck ring of the category of coherent sheaves on G/B. Then one knows (Grothendieck) that the classes $[\mathcal{O}_{X(\tau)}]$ in $K(G/B)$ of the structure sheaves $\mathcal{O}_{X(\tau)}$ of the Schubert varieties $X(\tau)$ form a $\mathbb{Z}$-basis of $K(G/B)$. Then it is natural to ask for a formula which generalizes (1), namely

$$[\mathcal{L}_\lambda]\cdot[\mathcal{O}_{X(\tau)}] = \sum_{\sigma\in W} a^\lambda_{\tau,\sigma}[X(\sigma)] \tag{2}$$

where the multiplication is in $K(G/B)$. More precisely, let $K_T(G/B)$ be the Grothendieck ring of T-equivariant sheaves on G/B. By [KK] one knows that again the classes $[\mathcal{O}_{X(\tau)}]_T$ of the structure sheaves of the Schubert varieties form a $\mathbb{Z}[\Delta]$ basis so that one asks for a formula

$$[\mathcal{L}_\lambda]_T\cdot[\mathcal{O}_{X(\tau)}]_T = \sum_{\sigma\in W} C^\lambda_{\tau,\sigma}[\mathcal{O}_{X(\sigma)}]_T \tag{3}$$

where $C^\lambda_{\tau,\sigma}$ are formal sums of characters of T with integral coefficients. Another basis of $K_T(G/B)$ is given by the ideal sheaves describing the boundary of a Schubert variety, i.e., let $\partial X(\tau)$ be the union of all $X(\sigma)\subset X(\tau)$ for $\sigma\neq\tau$, and let $\mathcal{I}_{X(\tau)}$ be defined by the exact sequence

$$0\longrightarrow \mathcal{I}_{X(\tau)}\longrightarrow \mathcal{O}_{X(\tau)}\longrightarrow \mathcal{O}_{\partial X(\tau)}\longrightarrow 0. \tag{4}$$

The classes $[\mathcal{I}_{X(\tau)}]$ form a basis of $K_T(X)$ and one may ask for a formula

$$[\mathcal{L}_\lambda]_T\cdot[\mathcal{I}_{X(\tau)}]_T = \sum_{\sigma\in W} D^\lambda_{\tau,\sigma}[\mathcal{I}_{X(\sigma)}]_T \tag{5}$$

or a formula for the coefficients in the mixed equation

$$[\mathcal{L}_\lambda]_T\cdot[\mathcal{I}_{X(\tau)}]_T = \sum_{\sigma\in W} M^\lambda_{\tau,\sigma}[\mathcal{O}_{X(\sigma)}]_T \tag{6}$$

Assume now that $\lambda\in\Delta^+$. Then one knows that in (1) all $a_i\geq 0$. An answer to (2) was given by Fulton and Lascoux [FL] for the case $G=SL(n)$; they provide a combinatorial formula to determine $a^\lambda_{\tau,\sigma}$. The general case was treated by Mathieu [M] who shows that $a^\lambda_{\tau,\sigma}\geq 0$ and more precisely that $C^\lambda_{\tau,\sigma}$ (as well as $D^\lambda_{\tau,\sigma}$) are effective i.e. formal sums of characters with positive integral coefficients. A striking solution to (3) has been given by Pittie and Ram [PR] as follows: Given an "L-S path η of shape λ on $X(\tau)$" let $\eta(1)$ be the *weight* of η and $e(\eta,\tau)$ (an element of W) be the *final direction* (or end element) of η with respect to τ (see §3 below). Then one has

$$[\mathcal{L}_\lambda]_T\cdot[\mathcal{O}_{X(\tau)}]_T = \sum_\pi [\mathcal{O}_{X(e(\pi,\tau))}]_T e^{-\pi(1)}$$

where π runs over all the L-S paths of shape λ on $X(\tau)$. Observe that

$$[\mathcal{L}_\lambda]_T\cdot[\mathcal{O}_{X(\tau)}]_T = [\mathcal{O}_{X(\tau)}\otimes\mathcal{L}_\lambda]_T.$$

Denoting $\mathcal{O}_{X(\tau)} \otimes \mathcal{L}_\lambda$ as $\mathcal{O}_{X(\tau)}(\lambda)$, the above result of Pittie and Ram takes the form

$$[\mathcal{O}_{X(\tau)}(\lambda)]_T = \sum_\pi [\mathcal{O}_{X(e(\pi,\tau))}]_T e^{-\pi(1)} \tag{7}$$

where π runs over all the L-S paths of shape λ on $X(\tau)$.

The aim of this paper is to show that (7) (and corresponding versions for (5) and (6)) is an easy consequence of Standard Monomial Theory (SMT) ([Li$_2$], [LLM]), and to present indications that an effective version of (7) should be roughly equivalent to SMT.

In fact, SMT implies a stronger (effective) result than (7) (see Theorem 5.3) because it provides an explicit filtration which, in the K-group, gives exactly the formula above:

$$\begin{cases} \text{There is a filtration } \mathcal{F} = \{\mathcal{F}^i\} \text{ of } \mathcal{O}_{X(\tau)}(\mathcal{L}_\lambda) \text{ by } B\text{-equivariant} \\ \mathcal{O}_{G/B}\text{-modules such that gr } \mathcal{O}_{X(\tau)}(\mathcal{L}_\lambda) = \bigoplus_\pi \mathcal{O}_{X(e(\pi,\tau))} \otimes \chi_{-\pi(1)}. \end{cases} \tag{8}$$

This means that $\mathcal{F}^i/\mathcal{F}^{i-1}$ as an $\mathcal{O}_{G/B}$ module is isomorphic to the structure sheaf of a Schubert variety of the form $\mathcal{O}_{X(\kappa)}$, where $\kappa = e(\pi, \tau)$, but the B-action is isomorphic to the canonical B-action on $\mathcal{O}_{X(\kappa)}$, twisted by the character $e^{-\pi(1)}$ (of B which comes from a character of T).

The result (8) goes back to considerations in ([G/P-IV]) and especially ([G/P-V)]) where filtrations closely related to (8) are constructed and play a crucial role in developing an SMT, for "classical type" i.e. on G/Q, where Q is a parabolic subgroup of classical type (see §6 below).

Now (8) generalizes suitably for Schubert varieties in G/Q, where Q is a parabolic subgroup as indicated in the beginning.

We shall briefly indicate the idea of proof of (8). The formal proof is given in the next sections. Let us suppose for simplicity that $\lambda \in \Delta_B^{++}$ so that $\mathcal{L}_\lambda$ is ample (as a consequence it is indeed very ample) on G/B. We denote $\mathcal{O}_{X(\tau)}(\lambda)$ by $\mathcal{O}_{X(\tau)}(1)$. An L-S path π of shape λ on $X(\tau)$ is a pair of sequences $\pi = (\underline{\tau}, \underline{a})$ of Weyl group elements and rational numbers, where $\underline{\tau}$ is of the form $\underline{\tau} = (\tau_1, \dots, \tau_r)$ such that $\tau \geq \tau_1$ and $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_r$ in the Bruhat order on W. We call $\tau_1 = i(\pi)$ the "initial" element of π and $\tau_r = e(\pi)$ the "end" element of π.

Then by SMT we have a basis $\{p_\pi\}$ of $H^0(\mathcal{O}_{X(\tau)}(1))$ indexed by L-S paths of shape λ such that p_π is a weight vector of weight $-\pi(1)$. We call a product

$$p_{\pi_1} \cdots p_{\pi_m} \in H^0(\mathcal{O}_{X(\tau)}(m))$$

a *standard monomial of length* m on $X(\tau)$ if $\tau \geq i(\pi_1)$ and $e(\pi_k) \geq i(\pi_{k+1})$ for $1 \leq k \leq m-1$. Then again by SMT the standard monomials of length m on $X(\tau)$ form a basis of $H^0(\mathcal{O}_{X(\tau)}(m))$. We set $R(\tau)_m = H^0(\mathcal{O}_{X(\tau)}(m))$ and $R(\tau)$ the graded ring

$$R(\tau) = \bigoplus_{m \geq 0} R(\tau)_m.$$

When $\tau = w_0$ is the longest word in the Weyl group, then $X(\tau) = G/B$ and in this case we write $R(\tau) = R$. By the properties of SMT, we see that we have the following direct sum decomposition of vector spaces

$$R(\tau)_m = \bigoplus_{\pi} p_\pi R(e(\pi))_{m-1}, \text{ i.e.}$$

$$R(\tau) = \bigoplus_{\pi} p_\pi R(e(\pi))(-1), \text{ or equivalently}$$

$$R(\tau)(1) = \bigoplus_{\pi} p_\pi R(e(\pi))$$

where π runs over L-S paths of shape λ on $X(\tau)$. Observe that $R(e(\pi))$ is a quotient ring of $R(\tau)$, in particular it is a $R(\tau)$ (or even R) module.

The important point is that there is a total order on L–S paths of shape λ on $X(\tau)$ such that the following filtration $F = \{F^k\}$ of $R(\tau)(1)$ by R submodules

$$F^k = \bigoplus_{\pi,\ \mathrm{ord}\ \pi \le k} p_\pi R(\tau)(1)$$

is B-stable, besides F^k/F^{k-1} is $B - R$ isomorphic to the module $R(e(\theta))$ twisted by the B–character $e^{-\theta(1)}$, θ being the L–S path of order k. We see that F defines then a filtration $\mathcal{F}$ of B–$\mathcal{O}_{G/B}$ submodules of $\mathcal{O}_{X(\tau)}(1)$ such that we get for the associated gradred sheaf $\mathrm{Gr}\,\mathcal{F}$:

$$\mathrm{Gr}\,\mathcal{F} \xrightarrow[B\text{–}\mathcal{O}_{X(\tau)} \text{ isomorphism}]{\sim} \bigoplus_{\pi} \mathcal{O}_{X(e(\pi))} \cdot e^{-\pi(1)}$$

which is (8).

We have seen that (8) is a consequence of SMT. On the other hand (8) is nearly equivalent to SMT (see § 6 below). Thus (8) is deeper than (7) which is equivalent to the character formulae of SMT (see Lemma 4.1); in fact the proof in [PR] uses only these character formulae ([Li$_1$]).

In the same way we obtain combinatorial formulae and corresponding effective filtrations giving an answer to the questions (5) and (6), or to the question to express in $K_T(G/B \times G/B)$ the class of a Schubert variety diagonally embedded in $G/B \times G/B$, see section 4 and section 5.

2. Some coherent sheaves and $\mathbb{N}$-graded modules

We keep the notations as in §1. If Q is a parabolic subgroup of G, we set $X = G/Q$ so that for $\lambda \in \Delta_Q$, $\mathcal{L}_\lambda$ denotes the corresponding line bundle on X. For $\lambda \in \Delta_Q^+$, the space of global sections $H^0(X, \mathcal{L}_\lambda)$ is the dual space of what is called the Weyl module $V(\lambda)$. For $\rho \in \Delta_Q^{++}$ consider the $\mathbb{N}$–graded algebra

$$R := \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{L}_{n\rho}).$$

The structure sheaf $\mathcal{O}_X$ can be easily recovered from R as the sheaf of functions which are locally quotients of sections of the same degree. In the same way one can associate to a $\mathbb{N}$-graded R–module a sheaf of $\mathcal{O}_X$–modules, for example, the module R itself corresponds to $\mathcal{O}_X$. Other examples are obtained in the following way: Fix a coset $\tau \in W/W_Q$ and for $\lambda \in \Delta_Q^+$ set

$$M_\tau^n(\lambda) := H^0(X(\tau), \mathcal{L}_{n\rho+\lambda}).$$

The $\mathbb{N}$-graded R-modules

$$M_\tau = \bigoplus_{n\in\mathbb{N}} M_\tau^n(0) \text{ and } M_\tau(\lambda) = \bigoplus_{n\in\mathbb{N}} M_\tau^n(\lambda)$$

correspond to the $\mathcal{O}_X$-sheaves $\mathcal{O}_{X(\tau)}$ and $\mathcal{O}_{X(\tau)}(\lambda) = \mathcal{L}_\lambda \otimes_{\mathcal{O}_X} \mathcal{O}_X(\tau)$ respectively. Similarly, consider the "boundary" $\partial X(\tau) = \bigcup_{\sigma<\tau} X(\sigma)$ of the Schubert variety $X(\tau)$, and set

$$I_\tau^n(\lambda) := \operatorname{Ker}\big(\operatorname{res} : H^0(X(\tau), \mathcal{L}_{n\rho+\lambda}) \to H^0(\partial X(\tau), \mathcal{L}_{n\rho+\lambda})\big).$$

The $\mathbb{N}$-graded R-modules

$$I_\tau = \bigoplus_{n\in\mathbb{N}} I_\tau^n(0) \text{ and } I_\tau(\lambda) = \bigoplus_{n\in\mathbb{N}} I_\tau^n(\lambda)$$

correspond to the $\mathcal{O}_X$-sheaves $\mathcal{I}_{X(\tau)}$ and $\mathcal{I}_{X(\tau)}(\lambda) = \mathcal{L}_\lambda \otimes_{\mathcal{O}_X} \mathcal{I}_{X(\tau)}$ respectively, where $\mathcal{I}_{X(\tau)}$ is defined by the exact sequence:

$$0 \longrightarrow \mathcal{I}_{X(\tau)} \longrightarrow \mathcal{O}_{X(\tau)} \longrightarrow \mathcal{O}_{\partial X(\tau)} \longrightarrow 0$$

We have a duality between the sections on Schubert varieties and the "Demazure modules": For a dominant weight λ and a $\tau \in W/W_\lambda$ fix a non-zero weight vector $v_\tau \in V(\lambda)$ of weight $\tau(\lambda)$ and denote $V_\tau(\lambda)$ the k-span of the orbit $B.v_\tau$. This B-submodule $V_\tau(\lambda)$ of $V(\lambda)$ is called a *Demazure submodule.* The embeding $G/Q \hookrightarrow \mathbb{P}(V(\lambda)$ induces a natural map between the dual space $V(\lambda)^*$ of $V(\lambda)$ and the global sections $H^0(G/Q, \mathcal{L}_\lambda)$. Similarly, the image of the Schubert variety $X(\tau)$ is contained in $\mathbb{P}(V_\tau(\lambda))$ and one gets an induced map between the dual space $V_\tau(\lambda)^*$ and $H^0(X(\tau), \mathcal{L}_\lambda)$. The fact that these maps are isomorphisms can be proved by Frobenius splitting or standard monomial theory (see [RR], [Li$_1$]).

3. Standard monomial theory

We keep the same notation as in section 2. We now recall quickly the notion of L–S paths and the associated basis $\{p_\pi\}$ of $H^0(Y, \mathcal{L}_\lambda)$, where $Y \subset G/Q$ is a union of Schubert varieties and $\lambda \in \Delta_Q^+$. The straightening law stated in Theorem 3.13 is new in this generality but implicitly already contained in [Li$_1$] and [LLM].

Set $\Delta_\mathbb{R} := \Delta \otimes_\mathbb{Z} \mathbb{R}$, the real span of the weight lattice. Let "$\leq$" be the Bruhat order on W/W_λ, recall that $W_Q \subset W_\lambda$. We identify a pair $\pi = (\underline{\tau}, \underline{a})$ of sequences:

- $\underline{\tau} : \tau_0 > \tau_1 > \ldots > \tau_r$, a sequence of linearly ordered cosets in W/W_λ;
- $\underline{a} : 0 < a_1 < \ldots < a_r < 1$, a sequence of rational numbers;

with the piecewise linear path $\pi : [0,1] \to \Delta_{\mathbb{R}}$ defined by (we set $a_0 := 0$):

$$\pi(t) := \sum_{i=0}^{j-1}(a_{i+1} - a_i)\tau_i(\lambda) + (t - a_j)\tau_j(\lambda) \text{ for } a_j \leq t \leq a_{j+1}.$$

Let $l(\cdot)$ be the length function on W/W_λ, and denote by $\beta^\vee$ the coroot of a positive root β. Let $\tau > \sigma$ be two elements of W/W_λ and let $0 < a < 1$ be a rational number. By an *a-chain* for the pair (τ, σ) we mean a sequence of cosets $\kappa_0, \ldots, \kappa_s$ in W/W_λ such that $\tau = \kappa_0$, $\sigma = \kappa_s$, $l(\kappa_i) = l(\kappa_{i-1}) - 1$, and the positive roots $\beta_1, \ldots, \beta_s$ such that

$$\tau = \kappa_0 > \kappa_1 = s_{\beta_1}\tau > \kappa_2 = s_{\beta_2}s_{\beta_1}\tau > \ldots > \kappa_s = s_{\beta_s}\cdot\ldots\cdot s_{\beta_1}\tau = \sigma,$$

satisfy the integrality property: $a(\kappa_i(\lambda), \beta_i^\vee) \in \mathbb{Z}$ for all $i = 1, \ldots, s$.

Definition 3.1. A pair $\pi = (\underline{\tau}, \underline{a})$ is called an *L-S path of shape* λ if for all $i = 1, \ldots, r$ there exists an a_i-chain for the pair (τ_{i-1}, τ_i).

Let $\pi = (\underline{\tau}, \underline{a})$ be an L-S path of shape λ. Then π is called an *L-S path on a Schubert variety* $X(\tau) \subset G/Q$ if $\tau \geq \tau_0 \mod W_\lambda$.

More generally, if $Y = \bigcup_{j=1}^r X(\sigma_j)$ is a union of Schubert varieties, then π is called an *L–S path on* Y if $\sigma_j \geq \tau_0 \mod W_\lambda$ for at least one j. The endpoint $\pi(1)$ is called the *weight* of the L-S path.

The set of all L–S paths of shape λ is denoted $B(\lambda)$ and the set of L–S paths of shape λ on $X(\tau)$ respectively on the union Y is denoted $B_\tau(\lambda)$ respectively $B_Y(\lambda)$.

Example 3.2. Suppose that $\lambda = \omega$ is a minuscule fundamental weight, i.e., $(\omega, \beta^\vee) = 0$ or 1 for a positive root β. Then $B(\omega) = \{(\sigma) \mid \sigma \in W/W_\omega\}$. Recall that the weight spaces in $V(\omega)_\mu$ are at most one-dimensional, and $V(\omega)_\mu \neq 0$ if and only if $\mu = \sigma(\omega)$ for some $\sigma \in W/W_\omega$. Note that $\operatorname{Char} V(\omega) = \sum_{\pi \in B(\omega)} e^{\pi(1)}$.

Example 3.3. The set $\{(\sigma) \mid \sigma \in W/W_\lambda\}$ is a subset of $B(\lambda)$ for all dominant weights $\lambda \in X^+$.

Example 3.4. Suppose that $\lambda = \omega$ is a fundamental weight of classical type, i.e., $(\omega, \beta^\vee) = 0, 1$ or 2 for a positive root β. An equivalent definition is to demand that the multiplicities a_i in (1) are all smaller or equal to two for all $\tau \in W/W_\omega$. Note that the fundamental weights of the classical groups have this property. The L-S paths of shape ω are either of the form $\pi = (\sigma)$, $\sigma \in W/W_\lambda$ or $\pi = (\sigma, \tau; \frac{1}{2})$, where in the latter case one has $\sigma > \tau$ and there exists a sequence of codimension one Schubert subvarieties

$$X(\sigma) = X(\kappa_0) \supset X(\kappa_1) \supset \ldots \supset X(\kappa_s) = X(\tau)$$

such that $X(\kappa_i)$ is of "multiplicity two" in $X(\kappa_{i-1})$ (see section 6 below). To have a uniform notation we write just $\pi = (\sigma, \tau)$ for the path $(\sigma, \tau; \frac{1}{2})$ and $\pi = (\sigma, \sigma)$ for the path (σ). These pairs of Weyl group cosets corresponding to L–S paths of shape ω are called *admissible pairs*. Then

$$B(\omega) = \{\pi \mid \pi = (\sigma, \tau) \text{ an admissible pair }\}.$$

and for a Schubert variety $X(\kappa) \subset G/P$ we have

$$B(\omega)_\kappa = \{\pi \mid \pi = (\sigma, \tau) \text{ an admissible pair }, \kappa \geq \sigma\}.$$

For example for the adjoint representation of the group Spin_{2m} one has as admissible pairs those pairs $(\sigma, \tau) \in W/W_\omega \times W/W_\omega$ such that:

$$B(\omega) = \left\{(\sigma, \tau) \mid \begin{array}{l} \text{either } \sigma = \tau; \\ \text{or } \tau(\omega) = \alpha \text{ is a simple root and } s_\alpha \tau = \sigma. \end{array} \right\}$$

The L-S paths provide a combinatorial tool to calculate characters:

Theorem 3.5. [Li$_1$][Li$_2$] *The character of the Weyl module $V(\lambda)$ is equal to* $\mathrm{Char}\, B(\lambda) := \sum_{\pi \in B(\lambda)} e^{\pi(1)}$*, and the character of the Demazure module $V_\tau(\lambda)$ is equal to the sum* $\mathrm{Char}\, B_\tau(\lambda) = \sum_{\pi \in B_\tau(\lambda)} e^{\pi(1)}$.

The theorem suggests that the set of L-S paths can be used as an indexing system for a basis of $V_\tau(\lambda)$ respectively its dual space, $H^0(X(\tau), \mathcal{L}_\lambda)$. In [Li$_2$], global sections $p_\pi \in H^0(G/Q, \mathcal{L}_\lambda)$, $\pi \in B(\lambda)$, have been constructed with the following properties:

Theorem 3.6. [Li$_2$] *The set $\{p_\pi \mid \pi \in B(\lambda)\}$ forms a basis of the space of global sections $H^0(G/Q, \mathcal{L}_\lambda)$. The basis vectors p_π are T-eigenvectors of weight $-\pi(1)$ respectively. Further, let Y be a union of Schubert varieties. The restriction $p_\pi|_Y$ is non-zero if and only if $\pi \in B_Y(\lambda)$, and the restrictions $\{p_\pi|_Y \mid \pi \in B_Y(\lambda)\}$ form a basis of T-eigenvectors of $H^0(Y, \mathcal{L}_\lambda)$.*

We write in the following $p_\pi \in H^0(Y, \mathcal{L}_\lambda)$ also for the restriction of p_π to Y. The basis $\{p_\pi \mid \pi \in B_Y(\lambda)\}$ is called the *path basis* of $H^0(Y, \mathcal{L}_\lambda)$. We introduce partial lexicographic orders on $B(\lambda)$ which are induced from the Bruhat order on W/W_λ: Let

$$\pi = (\tau_0, \ldots, \tau_r; a_1, \ldots, a_r), \eta = (\kappa_0, \ldots, \kappa_s; b_1, \ldots, b_s) \in B(\lambda)$$

then we say $\pi \geq \eta$ if $\tau_0 > \kappa_0$, or $\tau_0 = \kappa_0$ and $a_1 > b_1$, or $\tau_0 = \kappa_0$, $a_1 = b_1$ and $\tau_1 > \kappa_1$, etc. Similarly, let $\geq^r$ be the induced reverse partial lexicographic order, i.e., we say $\pi \geq^r \eta$ if $\tau_r > \kappa_s$ or $\tau_r = \kappa_s$ and $1 - a_r > 1 - b_s$, or $\tau_r = \kappa_s$ and $1 - a_r = 1 - b_s$ and $\tau_{r-1} > \kappa_{s-1}$, etc.

To define complete flags it is convenient to have a total order on $B(\lambda)$. Fix a total order "$\succeq$" on W/W_λ refining the Bruhat order. As above, we denote by $\succeq$ (respectively $\succeq^r$) the induced lexicographic order (respectively reverse lexicographic order) on the set of L-S paths. Let $\pi_1, \pi_2, \ldots, \pi_N$ be an enumeration of the L-S paths in $B_\tau(\lambda)$ such that $\pi_1 \succ \ldots \succ \pi_N$, and denote

$$H^0(X(\tau), \mathcal{L}_\lambda)_j = \langle p_{\pi_1}, p_{\pi_2}, p_{\pi_3}, \ldots, p_{\pi_j} \rangle_k$$

the subspace of $H^0(X(\tau), \mathcal{L}_\lambda)$ spanned by the $\{p_{\pi_i} \mid i \leq j\}$. Let F be the corresponding complete flag:

$$F : 0 \subset H^0(X(\tau), \mathcal{L}_\lambda)_1 \subset \ldots \subset H^0(X(\tau), \mathcal{L}_\lambda)_N = H^0(X(\tau), \mathcal{L}_\lambda).$$

Theorem 3.7. [Li$_2$] *The complete flag F is B-stable.*

For the purpose of this paper it is only necessary to consider the product of two path basis elements $p_\pi p_\eta$, where $\pi \in B_\tau(\lambda)$, $\eta \in B_\tau(\rho)$ and $\rho \in \Delta_Q^{++}$, $\lambda \in \Delta_Q^+$. For this reason we give below just the simplified version of the definition of a standard monomial. Recall that $W_Q = W_\rho$ and $W_Q \subseteq W_\lambda$.

Let $\pi = (\tau_0, \dots, \tau_r; \dots) \in B(\lambda)$ and $\eta = (\sigma_0, \dots; \dots) \in B(\rho)$, and let $p_\pi \in H^0(G/Q, \mathcal{L}_\lambda)$ and $p_\eta \in H^0(G/Q, \mathcal{L}_\rho)$ be the corresponding sections.

If $\lambda \in \Delta_Q^{++}$, then the pair $(\pi, \eta) \in B(\lambda) \times B(\rho)$ and the monomial $p_\pi p_\eta \in H^0(G/B, \mathcal{L}_{\lambda+\rho})$ are called *standard* if

$$e(\pi) \geq i(\eta), \text{ i.e., } \tau_r \geq \sigma_0.$$

The pair and the monomial are called *standard on* $X(\tau)$ if in addition $\tau \geq \tau_1$. In the general case the definition is slightly more involved because to compare cosets for different subgroups one has to consider liftings of elements.

Definition 3.8. The pair $(\pi, \eta) \in B(\lambda) \times B(\rho)$ and the section $p_\pi p_\eta \in H^0(G/Q, \mathcal{L}_{\lambda+\rho})$ are called *standard* if there exists a $\kappa \in W/W_Q$ such that $\kappa \equiv \tau_r \mod W_\lambda$ and $\kappa \geq \sigma_0$. The pair and the monomial are called *standard on the Schubert variety* $X(\tau)$ if one can choose elements $\kappa_0, \dots, \kappa_r \in W/W_Q$ such that $\kappa_i \equiv \tau_i \mod W/W_\lambda$ and $\tau \geq \kappa_0 > \dots > \kappa_r \geq \sigma_0$.

Deodhar's Lemma [D] asserts that for a L–S path $\pi = (\underline{\tau}, \underline{a})$ on $X(\tau)$ there exist unique maximal $\kappa_0, \dots, \kappa_r \in W/W_Q$ such that

$$\tau \geq \kappa_0 > \dots > \kappa_r \quad \text{and} \quad \kappa_j \equiv \tau_j \mod W_\lambda \quad \text{for } j = 0, \dots, r$$

i.e., any other sequence $\kappa_0', \dots, \kappa_r'$ with these properties is such that $\kappa_0 \geq \kappa_0', \dots, \kappa_r \geq \kappa_r'$.

It follows that $p_\pi p_\eta$ is standard if and only if $\kappa_r \geq \sigma_0$. Note that these maximal κ_i depend not only on the path π but also on τ.

Definition 3.9. The element $\kappa_r \in W/W_Q$ is henceforth called the *final direction of* π *with respect to* τ and is denoted $e(\pi, \tau)$. If π is not an L–S path on $X(\tau)$, then we say that $e(\pi, \tau)$ is not defined.

If π is not an L-S path on $X(\tau)$, then we say that $e(\pi, \tau)$ is not defined. So the set of L-S paths such that $e(\pi, \tau) = \sigma$ for a given element $\sigma \in W/W_Q$ is the subset of the set of L-S paths on $X(\tau)$ such that the final direction with respect to τ is σ.

Similarly, if the pair $(\pi, \eta) \in B(\lambda) \times B(\rho)$ is standard, then Deodhar's Lemma [D] asserts that for $\pi = (\underline{\tau}, \underline{a}), \eta = (\underline{\sigma}, \underline{b})$ there exist unique minimal $\kappa_0, \dots, \kappa_r \in W/W_Q$ such that

$$\kappa_0 > \dots > \kappa_r \geq \sigma_0 \quad \text{and} \quad \kappa_j \equiv \tau_j \mod W_\lambda \quad \text{for } j = 0, \dots, r$$

It follows that $p_\pi p_\eta$ is standard on $X(\tau)$ if and only if $\tau \geq \kappa_0$. Note that these κ_i depend not only on the path π but also on $\sigma = \sigma_0$.

Definition 3.10. The element $\kappa_0 \in W/W_Q$ is henceforth called the *first direction of* π *with respect to* η *respectively* σ and is denoted $i(\pi, \eta)$ respectively $i(\pi, \sigma)$. If $p_\pi p_{(\sigma)}$ is not standard on $X(\tau)$, then we say that $i(\pi, \tau)$ is not defined.

Note again: If $\lambda \in \Delta_Q^{++}$,i.e., if $\mathcal{L}_\lambda$ is ample on G/Q, then the *first direction* and *final direction* are just the first and last element occuring in the parameterization of the path π, so if $\tau \geq \tau_0 \geq \tau_r \geq \sigma$, then:

$$i(\pi, \sigma) = \tau_0, \quad e(\pi, \tau) = \tau_r.$$

Geometrically they can be characterized as follows:

- $X(i(\pi, \sigma))$ is the smallest Schubert variety such that the restriction of p_π does not identically vanish.
- $X^-(e(\pi, \tau))$ is the smallest opposite Schubert variety such that the restriction of p_π does not identically vanish.

Here the opposite Schubert $X^-(\sigma)$ is defined as follows: for $\sigma \in W$ let e_σ be the coset $\sigma B/B \in G/B$, let $B^- \subset w_0 B w_0$ be the opposite Borel subgroup, then $X^-(\sigma) = \overline{B^- e_\sigma}$ is the closure of the corresponding B^--orbit.

Denote $B(\lambda, \rho)$ the set of standard pairs $(\pi, \eta) \in B(\lambda) \times B(\rho)$ and let $B_\tau(\lambda, \rho)$ be the subset of pairs standard on $X(\tau)$.

The notion generalizes to a union of Schubert varieties Y by demanding that a pair is standard on Y if it is standard on at least one irreducible component. Let $B_Y(\lambda, \rho) \subset B(\lambda, \rho)$ be the subset of pairs standard on Y

Theorem 3.11. [Li$_2$] *The character of the Weyl module $V(\lambda+\rho)$ is equal to* $\operatorname{Char} B(\lambda, \rho) = \sum_{(\pi,\eta)\in B(\lambda,\rho)} e^{\pi(1)+\eta(1)}$*, and the character of the Demazure module $V_\tau(\lambda+\rho)$ is equal to the sum* $\operatorname{Char} B_\tau(\lambda, \rho) = \sum_{(\pi,\eta)\in B_\tau(\lambda,\rho)} e^{\pi(1)+\eta(1)}$.

Using the standard monomials, we get the corresponding version for a basis indexed by the standard pairs:

Theorem 3.12. [Li$_2$] *The monomials $\{p_\pi p_\eta \mid \pi, \mu \in B(\lambda, \rho)\}$ form a basis of T-eigenvectors of $H^0(G/Q, \mathcal{L}_{\lambda+\rho})$, the weight of the basis vector $p_\pi p_\eta$ is $-\pi(1) - \eta(1)$. Further, the monomials $\{p_\pi p_\eta \mid \pi, \mu \in B_Y(\lambda, \rho)\}$ form a basis of $H^0(Y, \mathcal{L}_{\lambda+\rho})$.*

It remains to consider monomials which are not standard, the theorem above implies that a product which is not standard can be expressed as a linear combination of standard monomials. We extend the partial orders $\geq$ and $\geq^r$ on L-S paths (reverse) lexicographically to pairs of L-S paths.

Theorem 3.13. *Let $\pi \in B_\tau(\lambda)$ and $\pi' \in B_\tau(\rho)$ be such that the product $p_\pi p_{\pi'} \in H^0(X(\tau), \mathcal{L}_{\lambda+\rho})$ is not standard on $X(\tau)$. Then there exist standard monomials $p_\eta p_{\eta'}$, standard on $X(\tau)$, such that $(\eta, \eta') \geq (\pi, \pi') \geq^r (\eta, \eta')$ and*

$$p_\pi p_{\pi'} = \sum a_{\eta,\eta'} p_\eta p_{\eta'}.$$

Proof. The proof of Theorem 3.12 (in [Li$_2$], Theorem 4) is given by constructing a basis

$$\{v_{\eta,\eta'} \mid (\eta, \eta') \in B_\tau(\lambda, \rho)\} \subset V_\tau(\lambda + \rho) \hookrightarrow V(\lambda) \otimes V(\rho)$$

which has the following properties: $p_\eta p_{\eta'}(v_{\eta,\eta'}) \neq 0$, and if $p_\delta p_{\delta'}$ is standard on $X(\tau)$, then $p_\delta p_{\delta'}(v_{\eta,\eta'}) \neq 0$ only if $(\eta, \eta') \geq (\delta, \delta')$. It follows directly from the construction that the assumption on the standardness of $p_\delta p_{\delta'}$ can be omitted. So this basis has in fact the properties: $p_\eta p_{\eta'}(v_{\eta,\eta'}) \neq 0$, and

$$\delta \in B_\tau(\lambda), \delta' \in B_\tau(\rho) \Rightarrow \big(p_\delta p_{\delta'}(v_{\eta,\eta'}) \neq 0 \text{ only if } (\eta,\eta') \geq (\delta,\delta')\big).$$

In [LLM] one has presented a different algorithm to construct a basis of $V(2\lambda) \hookrightarrow V(\lambda) \otimes V(\lambda)$. Using the same arguments as in the proof of Proposition 7.3 [LLM] (which were applied there only to the case $\lambda = \rho$), one sees that this algorithm provides a basis

$$\{u_{\eta,\eta'} \mid (\eta,\eta') \in B_\tau(\lambda,\rho)\} \subset V_\tau(\lambda+\rho) \hookrightarrow V(\lambda) \otimes V(\rho)$$

such that $p_\eta p_{\eta'}(u_{\eta,\eta'}) \neq 0$, and

$$\delta \in B_\tau(\lambda), \delta' \in B_\tau(\rho) \Rightarrow \big(p_\delta p_{\delta'}(u_{\eta,\eta'}) \neq 0 \text{ only if } (\delta,\delta') \geq^r (\eta,\eta')\big).$$

Let now $\pi \in B_\tau(\lambda), \pi' \in B_\tau(\rho)$ be such that $p_\pi p_{\pi'}$ is not standard on $X(\tau)$. By Theorem 3.12, there exist standard monomials $p_\eta p_{\eta'}$, standard on $X(\tau)$, such that

$$p_\pi p_{\pi'} = \sum a_{\eta,\eta'} p_\eta p_{\eta'} \text{ in } H^0(X(\tau), \mathcal{L}_{\lambda+\rho}).$$

Among those pairs (η,η') such that $a_{\eta,\eta'} \neq 0$ let (δ,δ') be a minimal element with respect to $\geq$. The minimality implies:

$$p_\pi p_{\pi'}(v_{\delta,\delta'}) = \sum a_{\eta,\eta'} p_\eta p_{\eta'}(v_{\delta,\delta'}) = a_{\delta,\delta'} p_\delta p_{\delta'}(v_{\delta,\delta'}) \neq 0$$

and hence $(\delta,\delta') \geq (\pi,\pi')$. This shows of course $(\eta,\eta') \geq (\pi,\pi')$ for all pairs (η,η') such that $a_{\eta,\eta'} \neq 0$.

Similarly, among those pairs (η,η') such that $a_{\eta,\eta'} \neq 0$ let (δ,δ') be a maximal element with respect to $\geq^r$. The maximality implies:

$$p_\pi p_{\pi'}(u_{\delta,\delta'}) = \sum a_{\eta,\eta'} p_\eta p_{\eta'}(u_{\delta,\delta'}) = a_{\delta,\delta'} p_\delta p_{\delta'}(u_{\delta,\delta'}) \neq 0$$

and hence $(\pi,\pi') \geq^r (\delta,\delta')$. The latter proves by the maximality that $(\pi,\pi') \geq^r (\eta,\eta')$ for all pairs (η,η') such that $a_{\eta,\eta'} \neq 0$. □

4. The combinatorial approach

We keep the notation of the preceeding sections. For every dominant weight $\lambda \in \Delta_Q^+$ we have a $\mathbb{Z}[\Delta]$-linear map $\chi_\lambda : K_T(X) \longrightarrow \mathbb{Z}[\Delta]$ defined on the class of a T-equivariant coherent sheaf $\mathcal{F}$ as follows:

$$\chi_\lambda([\mathcal{F}]) = \sum_{i\geq 0} (-1)^i \operatorname{Char} H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}_\lambda).$$

This collection of maps is "generically injective" in the following sense:

Lemma 4.1. *Let $\rho \in \Delta_Q^{++}$. If two elements of $K_T(X)$ do not coincide, say $\sum a_{\mathcal{F}}[\mathcal{F}] \neq \sum a_{\mathcal{F}'}[\mathcal{F}']$, then there exists an $n \in \mathbb{N}$ such that*

$$\chi_{n\rho}(\sum a_{\mathcal{F}}[\mathcal{F}]) \neq \chi_{n\rho}(\sum a_{\mathcal{F}'}[\mathcal{F}']).$$

Proof. It suffices to show that if $\sum a_{\mathcal{F}}[\mathcal{F}] \neq 0$ in $K_T(X)$, then there exists an $n \in \mathbb{N}$ such that $\chi_{n\rho}(\sum a_{\mathcal{F}}[\mathcal{F}]) \neq 0$. So let

$$\sum a_{\mathcal{F}}[\mathcal{F}] = \sum_{\kappa \in W/W_Q} a_\kappa[\mathcal{O}_{X(\kappa)}] \neq 0, \quad a_\kappa \in \mathbb{Z}[\Delta]. \tag{9}$$

We fix an ordering on the set of weights as follows: write $\nu \succeq_{\mathbb{Q}} \nu'$ if $\nu - \nu'$ is a sum of positive roots with non-negative rational coefficients. We denote with the same symbol the induced ordering on W/W_Q, i.e., we write $\sigma \succeq_{\mathbb{Q}} \delta$ if $\sigma(\rho) \succeq_{\mathbb{Q}} \delta(\rho)$.

Fix $\tau \in W/W_Q$ minimal with respect to $\succeq_{\mathbb{Q}}$ such that $a_\tau \neq 0$. The map $\chi_{n\rho}$ is $\mathbb{Z}[\Delta]$-linear, so we can "normalize" the coefficient a_τ as follows: by multiplying the element with some e^μ, $\mu \in \Delta$, if necessary, we can assume that $a_\tau = c_0 e^0 + \sum c_\nu e^\nu$ with $c_\nu \in \mathbb{Z}$, the coefficient c_0 for the zero weight is non-zero, and the other coefficients are non-zero only if $\nu \not\succeq_{\mathbb{Q}} 0$, i.e., the weight is not greater or equal to the zero weight. We claim for $n \gg 0$:

$$\chi_{n\rho}(\sum a_{\mathcal{F}}[\mathcal{F}]) = c_0 e^{-\tau(n\rho)} + \textit{other terms} \tag{10}$$

corresponding to weights $\neq -\tau(n\rho)$. In particular: $\chi_{n\rho}(\sum a_{\mathcal{F}}[\mathcal{F}]) \neq 0$. To prove the claim, recall that for $\kappa \in W/W_Q$ one has:

$$\operatorname{Char} H^0(X(\kappa), \mathcal{L}_{n\rho}) = e^{-\kappa(n\rho)} + \sum_{\nu \in \Delta, \nu \succ_{\mathbb{Q}} \kappa(n\rho)} c_\nu e^{-\nu}. \tag{11}$$

Since the non-zero weights occuring in a_τ are all $\not\succeq 0$, it follows by (11):

$$\chi_{n\rho}(a_\tau[\mathcal{O}_{X(\tau)}]) = a_\tau \operatorname{Char} H^0(X(\tau), \mathcal{L}_{n\rho}) = c_0 e^{-\tau(n\rho)} + \textit{other terms}$$

The weight $-\tau(n\rho)$ occurs in $a_\tau \operatorname{Char} H^0(\mathcal{O}_{X(\tau)}(n\rho))$ with coefficient c_0. Note that this holds for all $n \gg 0$.

To control the contribution coming from the other summands in (9) recall that τ was chosen to be minimal with respect to $\succeq_{\mathbb{Q}}$ among those elements such that $a_\kappa \neq 0$. Now $\tau \not\succeq_{\mathbb{Q}} \kappa$ implies $\tau(\rho) \not\succeq_{\mathbb{Q}} \kappa(\rho)$ and hence $\tau(n\rho) \not\succeq_{\mathbb{Q}} \kappa(n\rho)$ for all $n \geq 1$. This implies that

$$\kappa(n\rho) = \tau(n\rho) + \sum_{\alpha \in I_1} r_\alpha \alpha - \sum_{\alpha \in I_2} t_\alpha \alpha, \tag{12}$$

where I_1, I_2 are disjoint sets of simple roots, $I_1 \neq \emptyset$, and the r_α, t_α are positive rational numbers. Further, for any given $m \in \mathbb{N}$ one can choose an $n_0 \in \mathbb{N}$ such that $r_\alpha \geq m$ for all $n \geq n_0$ and all $\alpha \in I_1$.

Since $a_\kappa = \sum b_\mu e^\mu$ is fixed, for n big enough one has for all weights such that $b_\mu \neq 0$:

$$\kappa(n\rho) + \mu = \tau(n\rho) + \sum_{\alpha \in I_1} r'_\alpha \alpha - \sum_{\alpha \in I'_2} t'_\alpha \alpha,$$

where I_1 is as above, I_1, I'_2 are disjoint, $r'_\alpha > 0$ and $t'_\alpha \in \mathbb{Q}$. In particular, $\kappa(n\rho) + \mu \neq \tau(n\rho)$. Since $\nu \succeq \kappa(n\rho)$ for the other weights such that $e^{-\nu}$ occurs with nonzero coefficients in (11), the same reasoning applies, i.e., the coefficient of $e^{-\tau(n\rho)}$ in $a_\kappa \operatorname{Char} H^0(X(\kappa), \mathcal{L}_{n\rho})$ is zero.

By choosing $n \gg 0$ big enough for all κ such that $a_\kappa \neq 0$, this proves (10) and hence the lemma. $\square$

Remark 4.2. The lemma can also be proved using the non-degenerate form on $K_T(X)$ defined in [KK].

Remark 4.3. If $\lambda \in \Delta_Q^{++}$, then we can take $\lambda = \rho$. In this case one does not need the full strength of standard monomial theory for the proofs of the corollaries below, one needs only the combinatorial charcter formulae. In fact, by the normality of Schubert varieties (see for example [S_2]) one knows that the character $\mathrm{Char}\, H^0(X(\kappa), \mathcal{L}_{n\lambda})$ is given by the Demazure character formula and the higher cohomology groups vanish for $n \gg 0$, so by [Li_1] the character is combinatorially given by the path character formula.

The Pieri-Chevalley formula (7) is now a simple consequence of the lemma above. The character formula in Theorem 3.11, the definition of standard pairs in 3.9 and 3.10 imply together: For all $\rho \in \Delta_Q^{++}$

$$\begin{aligned}\mathrm{Char}\, H^0(X(\tau), \mathcal{L}_{\lambda+\rho}) &= \sum_{(\pi,\eta)\in B_\tau(\lambda,\rho)} e^{\pi(1)+\eta(1)} \\ &= \sum_{\pi\in B_\tau(\lambda)} e^{\pi(1)}\, \mathrm{Char}\, H^0(X(e(\pi,\tau), \mathcal{L}_\lambda).\end{aligned}$$

It follows that χ_ρ applied to the left and right hand side of (7) gives the same character for all $\rho \in \Delta_Q^{++}$, so in fact the two elements of $K_T(X)$ are the same. Recall the Definitions 3.9 and 3.10 of $i(\pi,\sigma)$ and $e(\pi,\tau)$:

Corollary 4.4. *Let $B_\tau(\lambda)$ be the set of L-S paths $\pi = (\underline{\tau}, \underline{a})$ of shape λ on $X(\tau)$, so $\tau_0 \leq \tau \mod W_\lambda$. Then*

$$\begin{aligned}[\mathcal{O}_{X(\tau)}(\lambda)]_T &= \sum_{\pi\in B_\tau(\lambda)} e^{-\pi(1)}[\mathcal{O}_{X(e(\pi,\tau))}]_T \\ &= \sum_{\substack{\sigma\in W/W_Q\\ \sigma\leq\tau}} \Big(\sum_{\substack{\pi\in B_\tau(\lambda)\\ e(\pi,\tau)=\sigma}} e^{-\pi(1)}\Big)[\mathcal{O}_{X(\sigma)}]_T\end{aligned}$$

For $\lambda \in \Delta_Q^{++}$ the result can be reformulated as:

Corollary 4.5. *The coefficient $C^\lambda_{\tau,\sigma}$ of $[\mathcal{O}_{X(\sigma)}]_T$ in $[\mathcal{O}_{X(\tau)}(\lambda)]_T$ is $\sum e^{-\pi(1)}$, where the sum runs over all L-S paths of shape λ ending in σ and starting with an element smaller or equal to τ.*

The character formula in Theorem 3.11 implies that the character of the kernel of the restriction map $H^0(X(\tau), \mathcal{L}_{\lambda+\rho}) \to H^0(\partial X(\tau), \mathcal{L}_{\lambda+\rho})$ is equal to $\sum e^{-\pi(1)}$, where the sum runs over all pairs $(\pi,\eta) \in B_\tau(\lambda,\rho)$ such that $i(\pi,\eta) = \tau$. Now the same arguments as above prove the combinatorial solution to (5):

Corollary 4.6. *Let $B_\tau(\lambda)^0 = \{\pi = (\underline{\tau}, \underline{a}) \mid \exists\, \kappa \in W/W_Q, i(\pi,\kappa) = \tau\}$, then*

$$\begin{aligned}[\mathcal{I}_{X(\tau)}(\lambda)]_T &= \sum_{\pi \in B_\tau(\lambda)^0} e^{-\pi(1)}[\mathcal{I}_{X(e(\pi,\tau))}]_T \\ &= \sum_{\substack{\sigma \in W/W_Q \\ \sigma \le \tau}} \Big(\sum_{\substack{\pi \in B_\tau(\lambda) \\ i(\pi,\sigma)=\tau}} e^{-\pi(1)} \Big)[\mathcal{I}_{X(\sigma)}]_T\end{aligned}$$

Again, for $\lambda \in \Delta_Q^{++}$ the result can be reformulated as:

Corollary 4.7. *The coefficient $D^\lambda_{\tau,\sigma}$ of $[\mathcal{I}_{X(\sigma)}]_T$ in $[\mathcal{I}_{X(\tau)}(\lambda)]_T$ is $\sum e^{-\pi(1)}$, where the sum runs over all L-S paths of shape λ starting in τ and ending with an element greater or equal to σ.*

The "symmetry" between the coefficients showing up in the corollaries above, i.e., the switching from $i(\pi,\sigma)$ to $e(\pi,\tau)$ can also be seen combinatorially. Let π^* be the path defined by $\pi^*(t) = \pi(1-t) - \pi(1)$. This is again an L-S path, but of shape $\lambda^* = w_0(-\lambda)$, and $(\pi^*)^* = \pi$. For an element δ in W/W_λ let δ^* be the unique element in W/W_{λ^*} such that $-\delta(\lambda) = \delta^*(\lambda^*)$. Then for $\pi = (\underline{\tau}, \underline{a})$ we have $\pi^* = (\underline{\tau}^*, \underline{a}^*)$ where

$$\underline{\tau}^* = (\tau_r^*, \dots, \tau_0^*) \text{ and } \underline{a}^* = (1-a_r, \dots, 1-a_1).$$

Denote $Q^* \supset B$ the parabolic subgroup "dual" to Q. We define an order reversing map

$$W/W_Q \to W/W_{Q^*}, \quad \sigma \mapsto \sigma^*,$$

as follows: Denote by σ also its unique minimal representative in W, then $\sigma^* \in W/W_{Q^*}$ is defined as the class $\sigma w_0 \mod W_{Q^*}$. Note if $\tau \in W/W_\lambda$ and $\sigma \equiv \tau \mod W_\lambda$, then $\sigma^* \equiv \tau^* \mod W_{\lambda^*}$. It follows easily that the map: $B(\lambda) \to B(\lambda^*), \pi \mapsto \pi^*$ induces a bijection

$$\sharp\{\pi \in B_\tau(\lambda) \mid i(\pi,\sigma) = \tau\} = \sharp\{\pi \in B_{\sigma^*}(\lambda^*) \mid e(\pi,\sigma^*) = \tau^*\}.$$

To compare the characters denote $^-$ the linear map $\mathbb{Z}[\Delta] \to \mathbb{Z}[\Delta]$ defined by $e^\mu \to e^{-\mu}$. Let $C^\lambda_{\tau,\sigma}$ be the coefficient of $[\mathcal{O}_{X(\sigma)}]_T$ in Corollary 4.4 and let $D^\lambda_{\tau,\sigma}$ be the coefficient of $[\mathcal{I}_{X(\sigma)}]_T$ in Corollary 4.6, then (see Mathieu [M]):

Corollary 4.8. $D^{\lambda^*}_{\sigma^*,\tau^*} = \overline{C^\lambda_{\tau,\sigma}}$.

Suppose that $\tau \in W/W_Q$ is minimal with respect to W_λ, i.e., the projection $W/W_Q \to W/W_\lambda$ is such that the length of τ in W/W_Q coïncides with the length of its image in W/W_λ. In this case $B_\tau(\lambda)^0$ is just the set of L-S paths of shape λ having $\tau = \tau_0$ as first element in the parameterization.

With the same reasoning we obtain for the mixed formula (6):

Corollary 4.9. *If $\tau \in W/W_Q$ is minimal with respect to W_λ, then*

$$\begin{aligned}
[\mathcal{I}_{X(\tau)}(\lambda)]_T &= \sum_{\pi \in B_\tau(\lambda)^0} e^{-\pi(1)}[\mathcal{O}_{X(e(\pi,\tau))}]_T \\
&= \sum_{\substack{\sigma \in W/W_Q \\ \sigma \leq \tau}} \Big(\sum_{\substack{\pi \in B_\tau(\lambda) \\ i(\pi,\sigma)=\tau \\ e(\pi,\tau)=\sigma}} e^{-\pi(1)} \Big)[\mathcal{O}_{X(\sigma)}]_T
\end{aligned}$$

For $\lambda \in \Delta_Q^{++}$ the result can be reformulated as:

Corollary 4.10. *The coefficient of $[\mathcal{O}_{X(\sigma)}]_T$ in the expression of $[\mathcal{I}_{X(\tau)}(\lambda)]_T$ is $\sum e^{-\pi(1)}$, where the sum runs over all L-S paths of shape λ starting in τ and ending in σ.*

5. Filtrations of $\mathcal{O}_{X(\tau)}(\nu)$

As in section 3, let $X(\tau) \subset X = G/Q$ be a Schubert variety. Fix a dominant weight $\lambda \in \Delta_Q^+$. In the equivariant K-group of X, the product

$$[\mathcal{L}_\lambda]_T \cdot [\mathcal{O}_{X(\tau)}]_T = [\mathcal{O}_{X(\tau)}(\lambda)]_T$$

can be expressed as a linear combination of the basis given by the classes of the structure sheaves of the Schubert varieties: $\{[\mathcal{O}_{X(\sigma)}]_T \mid \sigma \in W/W_Q\}$. The aim of this section is to provide an explicit B-equivariant filtration of $\mathcal{O}_{X(\tau)}(\lambda)$ as $\mathcal{O}_X$-modules which proves the formulas for the coefficients announced in the introduction.

To define a filtration of the $\mathcal{O}_X$-sheaf $\mathcal{O}_{X(\tau)}(\lambda)$ with the help of standard monomial theory, we use its description as the sheaf associated to the $\mathbb{N}$-graded R-module $M_\tau(\lambda)$ (see section 3). Let

$$\{p_\pi \mid \pi \text{ L-S path of shape } \nu \text{ on } X(\tau)\} = \{p_\pi \mid \pi \in B_\tau(\lambda)\}$$

be the path basis of $H^0(X(\tau), \mathcal{L}_\lambda)$. Recall that the module $M_\tau(\lambda)$ differs from M_τ "just" by a shift in the dominant weights. By standard monomial theory (Theorem 3.12) we know that we can actually write for the homogenous part of degree n:

$$M_\tau^n(\lambda) = H^0(X(\tau), \mathcal{L}_{\lambda+n\rho}) = \sum_{\pi \in B_\tau(\lambda)} p_\pi H^0(X(\tau), \mathcal{L}_{n\rho}) = \sum_{\pi \in B_\tau(\lambda)} p_\pi M_\tau^n.$$

We use this description to define a filtration. Fix a total order on W/W_Q which refines the Bruhat order, and denote the induced lexicographic ordering on the L–S paths "$\succ$" (see section 3). Fix an enumeration $\pi_1, \pi_2, \ldots, \pi_N$ of the L–S paths of shape λ on $X(\tau)$ such that $\pi_1 \succ \pi_2 \succ \ldots \succ \pi_N$ and set

$$M_\tau(\lambda)_k := p_{\pi_1} M_\tau + p_{\pi_2} M_\tau + \ldots + p_{\pi_k} M_\tau.$$

Lemma 5.1. *The filtration $\mathcal{F}$ is a filtration by B–R-submodules:*

$$\mathcal{F} : 0 = M_\tau(\lambda)_0 \subset M_\tau(\lambda)_1 \subset M_\tau(\lambda)_2 \subset \ldots \subset M_\tau(\lambda)_N = M_\tau(\lambda).$$

Proof. These are clearly R-submodules, and Theorem 3.7 implies that this is a filtration by B-submodules. ∎

For a weight μ let $\chi(\mu) : B \to k^*$ be the corresponding B-character obtained by trivial extension. Let $\pi_k = (\underline{\tau}, \underline{a}) \in B_\tau(\lambda)$ be an L-S path of shape λ on $X(\tau)$. Recall the notion $e(\pi, \tau)$ for the "final direction" of the path π (see Definition 3.9). This notion is important to check standardness of a monomial: The product $p_\pi p_\eta$, $\eta \in B_\tau(n\rho)$ is standard on $X(\tau)$ if and only if $e(\pi, \tau) \geq \sigma_0$, where σ_0 is the initial element of the path $\eta = (\underline{\sigma}, \underline{b})$.

Theorem 5.2. *The subquotient $M_\tau(\lambda)_k / M_\tau(\lambda)_{k-1}$ of the filtration $\mathcal{F}$ is isomorphic to M_κ as a R-module and isomorphic to $M_\kappa \otimes \chi(\mu)$ as B-module, where $\kappa = e(\pi_k, \tau)$ is the last direction of the path π_k and μ is the weight of the path basis element p_{π_k}.*

By translating the result into the language of sheaves one gets:

Theorem 5.3. *The ordering $p_{\pi_1}, \ldots, p_{\pi_N}$ on the path basis of $H^0(X(\tau), \mathcal{L}_\lambda)$ induces a B-stable filtration $\mathcal{F} = \{\mathcal{F}^j\}$ of the $\mathcal{O}_X$–sheaf $\mathcal{O}_{X(\tau)}(\lambda)$ such that the subquotients are as B-equivariant $\mathcal{O}_X$–sheaves isomorphic to structure sheaves $\mathcal{O}_{X(\kappa_k)}$ twisted by a B-character $-\pi_k(1)$, where $\kappa_k = e(\pi_k, \tau)$ is the final direction of π_k with respect to τ:*

$$\operatorname{gr} \mathcal{O}_{X(\tau)}(\lambda) = \bigoplus \mathcal{O}_{X(e(\pi,\tau))} \otimes \chi_{-\pi(1)},$$

where the sum runs over all L–S paths π of shape λ on $X(\tau)$.

Proof. Let f be an element of $M_\tau(\lambda)_k$, so one can write

$$f = \sum_{1 \leq i \leq k} p_{\pi_i} q_i,$$

where the q_i are elements of M_τ. Note that the q_i are not determined by f, so such an expression is in general far from being unique. Let $\kappa = e(\pi_k, \tau)$ be the last direction of the path π_k. We define a map

$$M_\tau(\lambda)_k \to M_\kappa$$

in the following way: For $f \in M_\tau(\lambda)_k$ let q_k be the coefficient of p_{π_k} and consider its restriction $q_k|_{X(\kappa)}$ to the Schubert variety $X(\kappa)$, this is an element of M_κ. We will show that, despite the non-uniqueness of the q_i, this map is well–defined:

$$M_\tau(\lambda)_k \to M_\kappa, \quad f = \sum_{1 \leq i \leq k} p_{\eta_i} q_i \mapsto q_k|_{X(\kappa)},$$

and induces an isomorphism of R-modules $M_\tau(\lambda)_k / M_\tau(\lambda)_{k-1} \to M_\kappa$.

To prove this we show first that $M_\tau(\lambda)_k$ has a basis given by certain standard monomials. The proof is by induction. For $\tau \in W/W_Q$ write $\overline{\tau}$ for its class in W/W_λ.

By the definition of the ordering, $(\overline{\tau})$ is the maximal L-S path of shape λ on $X(\tau)$ and hence $\pi_1 = (\overline{\tau})$ is such that its finial direction is $e(\pi_1, \tau) = \tau$. Now M_τ has as basis the standard monomials p_η, $\eta \in B_\tau(n\rho)$, $n \in \mathbb{N}$. Since the product $p_{(\overline{\tau})} p_\eta$ is standard on $X(\tau)$ for all such η, it follows that

$M_\tau(\lambda)_1 = p_{\overline{\tau}} M_\tau$ has as basis the standard monomials of the form $p_{(\overline{\tau})}p_\eta$, which proves the claim in this case.

We assume now that for all $j < k$, $M_\tau(\lambda)_j$ has as basis the standard monomials on $X(\tau)$ starting with $p_{\pi_1}, p_{\pi_2}\ldots$, or p_{π_j}. Let f be an element of $M_\tau(\lambda)_k$, so one can write

$$f = \sum_{i\leq k} p_{\eta_i} q_i,$$

where the q_i are elements of M_τ. By induction, the summand $p_{\eta_i}q_i$ can be expressed for $i < k$ as a linear combination of standard monomials on $X(\tau)$ of the form $p_{\pi_j}p_\eta$, $1 \leq j \leq k-1$ and $\eta \in B_\tau(n\rho)$ for some $n \in \mathbb{N}$.

Consider the summand $p_{\pi_k}q_k$. Without loss of generality (Theorem 3.6) we assume $q_k = p_\eta$ is an element of the path basis. If $p_{\pi_k}p_\eta$ is a standard monomial, then we have a description of f in the desired form.

If $p_{\pi_k}p_\eta$ is not standard on $X(\tau)$, then we have by Theorem 3.13:

$$p_{\pi_k}p_\eta = \sum a_{\delta,\delta'} p_\delta p_{\delta'}, \text{ where } (\delta,\delta') \geq (\pi_k,\eta) \geq^r (\delta,\delta'),$$

and the $p_\delta p_{\delta'}$ are standard. Since $(\delta,\delta') \geq (\pi_k,\eta)$, we have in particular $\delta > \pi_k$ because: obviously is $\delta \geq \pi_k$, so suppose $\delta = \pi_k$, then $\delta' \geq \eta$. In particular, for the initial elements we have $\delta'_0 \geq \kappa_0$, where $\delta' = (\underline{\delta}', \underline{b}'), \eta = (\underline{\kappa}, \underline{c})$. But this is not possible since $p_{\pi_k}p_\eta$ is not standard but $p_\delta p_{\delta'}$ is standard.

It follows that either $p_{\pi_k}p_\eta$ is standard or $p_{\pi_k}p_\eta \in M_\tau(\lambda)_{k-1}$, and hence $M_\tau(\lambda)_k$ has a basis given by the desired standard monomials.

Further, suppose $f \in M_\tau(\lambda)_k$, where $f = \sum_{i\leq k} p_{\pi_i}q_i$. Write $q_k = \sum a_\eta p_\eta$ as a linear combination of standard monomials. Rewrite $q_k = q_k^1 + q_k^2$, where

$$q_k^1 = \sum_{p_\eta \text{ standard on } X(\kappa)} a_\eta p_\eta \quad \text{and} \quad q_k^2 = \sum_{p_\eta \text{ not standard on } X(\kappa)} a_\eta p_\eta$$

and $\kappa_k = e(\pi_k,\tau)$. By the definition of standardness, $q_k^2|_{X(\kappa)} \equiv 0$, and hence $q_k|_{X(\kappa)} = q_k^1|_{X(\kappa)}$. The discussion above shows:

$$f = \sum_{i\leq k} p_{\eta_i}q_i = \sum_{i<k} p_{\eta_i}q_i + p_{\pi_k}q_k^1 + p_{\pi_k}q_k^2 = \sum_{i<k}(\sum_\eta a_\eta p_{\pi_i}p_\eta) + p_{\pi_k}q_k^1,$$

where the $p_{\pi_i}p_\eta$ are standard. This expression is unique by the basis property of the standard monomials. It follows that the map

$$M_\tau(\lambda)_k \to M_\kappa, \text{ defined by } f = \sum_{i\leq k} p_{\eta_i}q_i \mapsto q_k|_{X(\kappa)} = q_k^1|_{X(\kappa)},$$

is well defined, and it is a morphism of graded R-modules. The standard monomial basis of $M_\tau(\lambda)_k$ shows that the morphism is surjective with kernel $M_\tau(\lambda)_{k-1}$. We get an induced isomorphism of graded R-modules:

$$M_\tau(\lambda)_k / M_\tau(\lambda)_{k-1} \to M_\kappa,$$

which is obviously B-equivariant up to a twist by a character corresponding to the weight of p_{π_k}. □

Remark 5.4. In the same way one proves the corresponding effective version of Corollary 4.6. The analysis can also be extended to provide a combinatorial formula and its effective version to calculate the class $[diag(X(\tau))]_T$ of a diagonally embedded Schubert variety in $K_T(G/B \times G/B)$, this will be discussed in a subsequent paper. See also [B].

6. Filtrations in ([G/P-V]) and remarks

We shall now briefly indicate the role of filtrations in ([G/P-V]) similar to those of (8) and Theorem 5.2 and Theorem 5.3, and explain more precisely the remark in the introduction where we said that (8) is nearly equivalent to SMT.

Let P be a maximal parabolic subgroup containing the Borel subgroup B of G. We have Pic G/P $\simeq \mathbb{Z}$. We denote by $\mathcal{O}_{G/P}(1)$ the ample (in fact very ample) generator of Pic G/P. The W-translates of the lowest weight vector of the G-module $H^0(G/P, \mathcal{O}_{G/P}(1))$ are the "extremal" weight vectors $\{p(\phi)\}$ indexed by W/W_P (which is also the indexing set for the Schubert varieties in G/P). Then one knows that the "hyperplane" $\{p(\phi) = 0\}$ in G/P intersects the Schubert variety $X(\phi)$ properly, in fact the scheme theoretic intersection $H(\phi) = X(\phi) \cap \{p(\phi) = 0\}$ is B-stable and $H(\phi)_{red}$ is the union of all the Schubert varieties of codimension one in $X(\phi)$. We refer to $H(\phi)$ (resp. $H(\phi)_{red}$) as the "hyperplane" (resp. "reduced hyperplane") section of $X(\phi)$. The intersection multiplicity (of the hyperplane section) along a Schubert variety $X(\theta)$ of codimension in $X(\phi)$ is called the "multiplicity" of $X(\theta)$ in $X(\phi)$.

If the multiplicities are ≤ 2, we say that P is of "classical type" (if the multiplicities are all 1, P is associated to a "minuscule" fundamental weight). If G is a classical group every maximal parabolic subgroup is of classical type. Let us suppose that P is of classical type. Then an L-S path π of shape $\mathcal{O}_{G/P}(1)$ on $X(\phi)$ is written in the form $\pi = (\tau, \sigma)$; $\tau, \sigma \in W/W_P$ such that either $\tau = \sigma$ or there exist elements $\tau = \tau_1, \tau_2, \dots, \tau_r = \sigma$ such that $X(\tau_i)$ is of multiplicity two in $X(\tau_{i+1})$ (see Example 3.4). In ([G/P-V]) π is called an "admissible pair". The general L-S paths of Definition 2.1 can be similarly defined in a geometric manner.

Let R be the homogeneous coordinate ring of G/P with $R = \bigoplus_{n \geq 0} R_n$, $R_n = H^0(\mathcal{O}_{G/P}(n))$. Similarly, we write $R(\phi)$ for the homogeneous coordinate ring of $X(\phi)$ so that $R(\phi)_n = H^0(\mathcal{O}_{X(\phi)}(n))$. Then the basic theorems of SMT in ([G/P-V]) on G/P are the following First and Second Basis Theorems. They are of course particular cases of [Li$_2$]:

(I) There is a canonical basis $\{p(\theta, \lambda)\}$ of weight vectors of $R(\phi)_1$, indexed by admissible pairs (θ, λ) such that $\phi \geq \theta$ and weight of $p(\theta, \lambda) = -\frac{1}{2}(\theta(\omega) + \lambda(\omega))$, where ω is the fundamental weight defining canonically associated to P (when $\theta = \lambda$, it follows that $p(\theta, \lambda)$ is the extremal weight vector $p(\theta)$).

(II) We call $p(\theta_1,\lambda_1)\cdots p(\theta_n,\lambda_n)$ a standard monomial of length n on $X(\phi)$ if $\phi \geq \theta_1 \geq \lambda_1 \geq \theta_2 \geq \lambda_2 \geq \cdots \geq \theta_n \geq \lambda_n$. Then $R(\phi)_n$ has a basis consisting of standard monomials of length n on $X(\phi)$.

Similar statements hold for unions of Schubert varieties, in particular on $H(\phi)_{red}$.

In ([G/P-V]), there is a more general SMT, namely on G/Q, where Q is a parabolic subgroup ($Q \supset B$) of "classical type" (i.e. every maximal subgroup containing Q is of classical type). For simplicity we have restricted to the case when Q is a maximal parabolic subgroup P.

In ([G/P-V]) one proves first the First Basis Theorem and then (II) by induction on the dimension of Schubert varieties:

To understand the inductive argument, consider the exact sequence

$$(13)\quad 0 \longrightarrow \mathcal{O}_{X(\phi)}(-1) \underset{\substack{\text{multiplication by}\\ p(\phi)}}{\longrightarrow} \mathcal{O}_{X(\phi)} \longrightarrow \mathcal{O}_{H(\phi)} \longrightarrow 0(13)$$

which implies the exact sequence

$$0 \longrightarrow \mathcal{O}_{X(\phi)}(n-1) \longrightarrow \mathcal{O}_{X(\phi)}(n) \longrightarrow \mathcal{O}_{H(\phi)}(n) \longrightarrow 0$$

in particular:

$$(14) \qquad 0 \longrightarrow \mathcal{O}_{X(\phi)} \longrightarrow \mathcal{O}_{X(\phi)}(1) \longrightarrow \mathcal{O}_{H(\phi)}(1) \longrightarrow 0$$

By the induction argument we can supose that SMT holds on $H(\phi)_{red}$ and then for the study of $H^0(\mathcal{O}_{X(\phi)}(n))$ we are led to the understanding of the scheme $H(\phi)$ which may not be reduced (we use also induction on n). Suppose that SMT holds on $X(\phi)$. Then we see that the ideal $I = I(H(\phi)_{red})$ in $R(\phi)$ of elements vanishing on $H(\phi)_{red}$ is generated by all the elements of the form $p(\phi,\lambda)$. Take a total order on the set of all admissible pairs of the form (ϕ,λ) so that

$$ord(\phi,\lambda_1) \geq ord(\phi,\lambda_2) \text{ if } \lambda_2 \leq \lambda_1.$$

Let I_j be the ideal in $R(\phi)$ generated by elements of the form $p(\phi,\lambda)$ such that $ord(\phi,\lambda) \leq j$:

$$I_j = \langle p(\phi,\lambda) \mid ord(\phi,\lambda) \leq j \rangle \subset R(\phi).$$

Then we have a filtration of I (by ideals in $R(\phi)$) as follows:

$$0 = I_{-1} \subset I_0 \subset \cdots \subset I_N$$

where $I_N = I(H(\phi)_{red})$ and $I_0 = I(H(\phi)) = p(\phi)R(\phi) \simeq R(\phi)(-1)$.

Assuming SMT on $X(\phi)$ we have the following (§7, G/P - V):

$$(15)\quad \begin{cases} \text{The filtration is } B\text{-equivariant and we have isomorphisms:} \\ f_j : I_j/I_{j-1} \simeq R(\lambda_j)(-1)e^{wt\ p(\phi,\lambda_j)}, ord(\phi,\lambda_j) = j,\ f_j \text{ is} \\ \text{a } B\text{-equivariant isomorphism of } R(\phi) \text{ modules i.e. } f_j \text{ is} \\ \text{a } B\text{-equivariant } R(\phi) \text{ isomorphism of } I_j/I_{j-1} \text{ onto } R(\lambda_j) \\ \text{upto the character twist } e^{wt\ p(\phi,\lambda_j)} \text{ (recall wt. } p(\phi,\lambda_j) = \\ -\frac{1}{2}(\phi(w)+\lambda(w_j)). \end{cases}$$

The filtration (15) figures in (G/P - IV, Th. 9.3) without the mention of B-equivariance. Then we have the following:

$$(16)\quad \begin{cases} F = \{F^j\},\ -1 \le j \le N+1, \text{ is a } B\text{-equivariant filtration of} \\ R(\phi)(1) \text{ by } R(\phi) \text{ submodules and we have a } B\text{-equivariant} \\ \text{isomorphism of } R(\phi) \text{ modules:} \\ gr\ F \simeq \left(\bigoplus_{j,ord(\phi,\lambda_j)=j} R(\lambda_j)e^{wt\ p(\phi,\lambda_j)}\right) \ \oplus\ R(H(\phi)_{red})(1). \end{cases}$$

This implies that

$$(17)\quad \begin{cases} \text{We have a } B\text{-equivariant filtration } \mathcal{F} = \{\mathcal{F}^j\},\ -1 \le \\ j \le N+1, \text{ of } \mathcal{O}_{X(\phi)}(1) \text{ by } \mathcal{O}_{X(\phi)}\text{-submodules and a } B\text{-} \\ \text{equivariant } \mathcal{O}_{X(\phi)}\text{-isomorphism} \\ gr\ \mathcal{F} \simeq \left(\bigoplus_{j,ord(\phi,\lambda_j)=j} \mathcal{O}_{X(\lambda_j)}e^{wt\ p(\phi,\lambda_j)}\right) \oplus \mathcal{O}_{H(\phi)_{red}}(1). \end{cases}$$

We see that this closely resembles the filtration (8) (which can be considered as a refinement of (17)). We see that this implies that in $K(G/P)_T$, we have

$$(18)\quad [\mathcal{O}_{X(\phi)}(1)]_T = \sum_{j,ord(\phi,\lambda_j)=j} [\mathcal{O}_{X(\lambda_j)}]_T e^{wt\ p(\phi,\lambda_j)} + [\mathcal{O}_{H(\phi)}(1)]_T.$$

Now iterating (18) for the Schubert varieties in $H(\phi)_{red}$ (and using good properties of unions and intersections), we get

$$(19)\quad \begin{cases} [\mathcal{O}_{X(\phi)}(1)]_T = \sum_\lambda [\mathcal{O}_{X(\lambda)}]e^{wt\ p(\theta,\lambda)} \text{ where } \lambda \text{ runs over the} \\ \text{end elements of admissible pairs } (\theta,\lambda) \text{ such that } \phi \ge \theta. \end{cases}$$

Now (19) is just the formula (8) for our particular case. In particular we get also (7) (for our case).

As we have remarked before (16) is deduced as a consequence of SMT in ([G/P-V]). On the other hand observe that (16) or (17) is equivalent to SMT (on $X(\phi)$) in our case (i.e. for classical type). To see this let us assume the validity of the vanishing theorems for line bundles on Schubert varieties and their unions contained in $X(\phi)$ associated to Δ_P^+ (these can be deduced by induction). The First Basis Theorem ((I) above) is a consequence of the isomorphism f_j in (15) (or the isomorphism in (17)). Now f_j induces an isomorphism

$$H^0(\mathcal{F}^j/\mathcal{F}^{j-1}) \longrightarrow H^0(\mathcal{O}_{X(\lambda_j)})$$

and the element 1 of $H^0(\mathcal{O}_{X(\lambda_j)})$ corresponds to a weight vector y_j of $H^0(\mathcal{F}^j/\mathcal{F}^{j-1})$ of weight $-\frac{1}{2}(\phi(w)+\lambda_j(w))$. Then using the vanishing theorems, y_j lifts to an element $H^0(\mathcal{O}_X(1))$ which is a weight vector of weight $-\frac{1}{2}(\phi(w)+\lambda_j(w))$ and we call this $p(\phi,\lambda_j)$. To construct basis elements of the form $p(\theta,\lambda)$, $\phi > \theta$, observe that they live in $H^0(H(\phi)_{red}(1))$ by induction and they lift to elements of $(H^0(\mathcal{O}_{X(\phi)}(1))$. To see the generation and linear independence of standard monomials, we split the filtration $\{\mathcal{F}^j(n)\}$ into short exact sequences, we use the cohomology of these exact sequence as well as induction on the dimension of Schubert varieties, induction on n etc.

The proof of SMT in ([G/P-V]) can be viewed as proving (17) by induction on the dimension of Schubert varieties. This proof runs along the following lines: One assumes (17) on $X(\phi)$ and then it suffices to show that it holds on $X(\tau)$ with $\tau = s_\alpha\phi$, $\tau > \phi$ and s_α - the reflection with respect to a simple root α. Let $Z_\phi \longrightarrow \mathbb{P}^1$ be the fibre space with fibre $X(\phi)$ associated to the principal filtration $SL(2) \longrightarrow \mathbb{P}^1$ (or $P_\alpha \longrightarrow P_\alpha/B$, P_α being the minimal parabolic subgroup containing B, associated to α) and $\psi : Z_\phi \longrightarrow X(\tau)$ the canonical birational morphism. Taking $Z_\phi = Z$ we denote by $\mathcal{O}_Z(1)$ the line bundle $\psi^*(\mathcal{O}_{X(\tau)}(1))$.

Now we take the filtration $\widetilde{\mathcal{F}}$ on Z_ϕ "associated to $\mathcal{F}$" in the language of fibre spaces ($Z_\phi \longrightarrow \mathbb{P}^1$ fibre space with fibre $X(\phi)$). The properties of the filtration $\mathcal{F}$ carry over to $\widetilde{\mathcal{F}}$ and then one gets a nice filtration $\mathcal{K}$ for the sheaf $\mathcal{O}_Z(1)$ (in [G/P-V], $\mathcal{K}$ is given by the sheaves $K_i(1)$). The required filtration on $X(\tau)$ can be obtained by taking $\psi_*\mathcal{K}$. The proof in ([G/P-V]) is a little different. Using the filtration $\mathcal{K}$, one shows that $\dim H^0(\mathcal{O}_Z(n))$ is equal to the expected number of standard monomials of length n on $X(\tau)$, which easily implies that SMT holds on $X(\tau)$, which of course implies that (17) holds on $X(\tau)$. We observe also that in ([G/P-V]) the assertion I (First Basis Theorem) is given a separate proof, whereas a better presentation would have been to include it as proving (17) by induction on the dimension of Schubert varieties.

Thus in retrospect the proof of SMT in ([G/P-V]) could be termed as K-theoretic. It would indeed be very nice to have a similar proof for the general SMT established in [Li$_2$]. As stated in ([S$_1$]) a good understanding of SMT would be via a "cellular Riemann–Roch formula" as the definition of L–S paths could be formulated geometrically in terms of the canonical cellular decomposition in G/B. The formulations via B-filtrations and Grothendieck rings seem to provide this approach.

References

[B] Brion, M.: The restriction of homogeneous line bundles to Schubert varieties. *preprint* (2001).

[Ch] Chevalley, C.: Sur les decompositions cellulaires des espaces G/B (with a foreword by A. Borel), *Proc. Sympos., Pure Math.*, Vol. 56, Part 1, Algebraic Groups and their Generalisations: Classical Methods (University Park, PA; 1991), *Amer. Math. Soc.*, Providences, RI, 1994, 1–23.

[D] Deodhar, V.: A splitting criterion for Bruhat orderings on Coxeter groups. *Comm. Alg.* 15, 1889-1894 (1987).

[FL] Fulton, W. and Lascoux, A.: A Pieri formula in the Grothendieck ring of a flag bundle. *Duke Math. J.* 76, 711–729 (1994).

[G/P-IV] Lakshmibai, V., Musili, C. and Seshadri, C.S.: Geometry of G/P-IV, *Proc. Ind. Acad. Sci.*, 88A, (1979), 279–362.

[G/P-V] Lakshmibai, V. and Seshadri, C.S.: Geometry of G/P-V, *J. Algebra*, 100, (1986), 462–557.

[KK] Kostant, B, Kumar, S.: T-equivariant K-theory of generalized flag varieties *Proc. Nat. Acad. Sci. USA*. 84, 4351-4354 (1987).

[Li_1] Littelmann, P.: A Littlewood-Richardson formula for symmetrizable Kac-Moody algebras. *Invent. Math.* 116, 329-346 (1994).

[Li_2] Littelmann, P.: Contracting modules and standard monomial theory. *JAMS* 11, 551–567 (1998)

[LLM] Lakshmibai, V., Littelmann, P. and Magyar, P.: Standard monomial theory and applications, Notes by Rupert Yu, in: *Representation Theories and Algebraic Geometry* Nato Adv. Sci. Inst. Ser. C Math. Phys. Sc.i 514 (A. Broer, ed.), Kluwer Academic Publishers (1998), pp. 319–364.

[LS] Lakshmibai, V. and Seshadri, C.S.: Standard monomial theory, in: *Proc. Hyderabad Conference on Algebraic Groups*, (S. Ramanan et al.,eds.), Manoj Prakashan, Madras, 1991, 279-323.

[M] Mathieu, O.: Positivity of some intersections in $K_0(G/B)$. To appear in *J. Pure Appl. Alg.* 152, (2000)

[PR] Pittie, H. and Ram, A.: A Pieri-Chevalley formula for $K(G/B)$. *Electron. Res. Announc. Amer. Math. Soc.* 5, 102–107 (1999).

[RR] Ramanan, S. and Ramanathan, A.: Projective normality of flag varieties and Schubert varieties. *Invent. Math.* 79, 217–224 (1985).

[S_1] Seshadri, C.S.: The work of Littelmann and Standard Monomial Theory, Current Trends in Mathematics and Physics, A Tribute to Harish-Chandra, Naraso Publishing House, 6, Community Centre, Panchsheel Park, New Delhi-110 017.

[S_2] Seshadri, C.S.: Line bundles on Schubert varieties, in: Vector bundles on algebraic varieties, *Tata Inst. Fund. Res. Stud. Math.* 11, 499–528 (1987).

P.L: Fachbereich Mathematik, Universität Wuppertal, Gauß-Straße 20, 42097 Wuppertal, Germany

C.S.S.: Chennai Mathematical Institute, 92 G. N. Chetty Road, T Nagar, Chennai 600 017, India