Bases for representations, LS-paths and Verma flags

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Abstract. The basic result of this paper is the construction of two kind of bases $\mathbb{B}(\lambda)$ and $\mathbb{M}(\lambda)$ for simple finite dimensional representations $V(\lambda)$ of a semisimple complex Lie algebra $\mathfrak{g}$. The construction combines the combinatorics of LS-paths [**4**], the Weyl group combinatorics related to inclusions of Verma modules [**1**] and the structure of singular vectors in Verma modules [**7**].

Dedicated to Professor Seshadri on his 70th birthday

Introduction

In this paper we describe the construction of two kind of bases $\mathbb{B}(\lambda)$ and $\mathbb{M}(\lambda)$ (Theorems 5.2 and 5.3) for simple finite dimensional representations $V(\lambda)$ of a semisimple complex Lie algebra $\mathfrak{g}$. The construction combines the combinatorics of LS-paths [**4**], the Weyl group combinatorics related to inclusions of Verma modules [**1**] and the structure of singular vectors in Verma modules [**7**].

The development of the basis $\mathbb{B}(\lambda)$ was inspired by the similarity of the combinatorial description of inclusions of Verma modules and the combinatorial description of LS-paths. This similarity is used to attach in a canonical way to each LS-path π of shape λ a vector $v_\pi \in V(\lambda)$. The construction of these vectors uses singular vectors in Verma modules which are naturally associated to the LS-path π. By repeating the procedure for all multiples of λ, this basis globalizes to a basis $\mathbb{B}(\lambda,\infty)$ of $U(\mathfrak{n}^-)$ (for λ regular, for a precise statement in the non-regular case see Theorem 5.2). This basis is compatible with the surjections $\Psi_m : U(\mathfrak{n}^-) \to V(m\lambda)$, $u \mapsto uv_{m\lambda}$, i.e., the kernel has $\operatorname{Ker}\Psi_m \cap \mathbb{B}(\lambda,\infty)$ as basis, and the basis $\mathbb{B}(\lambda,\infty)$ depends only on the ray $\mathbb{R}_{>0}\lambda$ spanned by λ.

The second basis $\mathbb{M}(\lambda)$ is in some sense a simplified version of the first. It is obtained by attaching to an LS-path π of shape λ a monomial $\mathfrak{m}_\pi$ in $U(\mathfrak{n}^-)$ in the Chevalley generators. The vectors $u_\pi = \mathfrak{m}_\pi v_\lambda$, $v_\lambda \in V(\lambda)$ a highest weight vector, provide a basis $\mathbb{M}(\lambda)$ of the $\mathbb{Z}$-lattice $V_{\mathbb{Z}}(\lambda)$. The monomials used for the construction of $\mathbb{M}(\lambda)$ have first been considered by V. Lakshmibai and C. S. Seshadri in [**3**], where they conjecture that the $u_\pi \in V_{\mathbb{Z}}(\lambda)$ form a basis, and they give a proof in some special cases. This

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basis again globalizes nicely to a basis $\mathbb{M}(\lambda,\infty)$ of $U(\mathfrak{n}^-)$ (for λ regular, for a precise statement see Theorem 5.3), and is compatible with the surjections $\Psi_m : U(\mathfrak{n}^-) \to V(m\lambda)$, $u \mapsto uv_{m\lambda}$. The basis $\mathbb{M}(\lambda,\infty)$ has the advantage of being given by monomials, its disadvantage is that its is not anymore canonically defined.

The main tools used in the proof and in the construction are the path model, the quantum Frobenius splitting (to prove linear independence with the same method as in [**5**]), and the theory of Verma modules (mainly its combinatorial part). Since in our construction only real roots are involved in the combinatorics as well as in the computation of singular vectors, all these tools are also available in the case of symmetrizable Kac-Moody algebras. So the construction can be easily adapted to the more general setting of symmetrizable Kac-Moody algebras, and the theorems above hold also in these cases. But to keep the notation simple, we decided to consider in this paper only the case of semisimple Lie algebras.

1. Inclusions of Verma modules

In this section we fix some notation and recall some standard facts on homomorphisms between Verma modules. Let $\mathfrak{g}$ be a semisimple complex Lie algebra and denote $U(\mathfrak{g})$ its universal enveloping algebra. We fix a Cartan subalgebra $\mathfrak{h}$ and a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ such that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$ is the opposite Borel subalgebra. Denote by Φ the root system of $\mathfrak{g}$, let Φ^+ be the corresponding set of positive roots and denote by $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots. For $\beta \in \Phi$ let $H_\beta \in \mathfrak{h}$ be its coroot. If $\lambda \in \mathfrak{h}^*$, then we write $\langle \lambda, \beta \rangle$ for $\lambda(H_\beta)$. We fix a Chevalley basis $\{X_\beta, Y_\beta \mid \beta \in \Phi^+\} \cup \{H_\alpha \mid \alpha \in \Delta\}$ of $\mathfrak{g}$, where $X_\beta \in \mathfrak{g}_\beta$ and $Y_\beta \in \mathfrak{g}_{-\beta}$. We write often just X_i and Y_i instead of X_{α_i} and Y_{α_i} for $\alpha_i \in \Delta$. Let $\lambda \in \mathfrak{h}^*$, the Verma module $M(\lambda)$ is the $U(\mathfrak{g})$-module

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ denotes the one dimensional $U(\mathfrak{b})$-module obtained by extending λ trivially to $\mathfrak{b}$. By definition, $M(\lambda)$ is a cyclic highest weight module of highest weight λ with highest weight vector $1_\lambda = 1 \otimes 1 \in M(\lambda)$.

For an element w of the Weyl group W and $\mu \in \mathfrak{h}^*$ set $w \circ \mu := w(\mu + \rho) - \rho$, here ρ is the sum of the fundamental weights. We summarize the properties of homomorphisms between Verma modules in the following theorem (due to Verma, Bernstein, Gelfand and Gelfand), see [**1**] or [**2**], Chapter 7.

Theorem 1.1. *Let $\lambda, \mu \in \mathfrak{h}^*$, $\lambda \neq \mu$.*
i) The vector space $\mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda))$ *is null or one-dimensional, and every non-zero element of* $\mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda))$ *is injective.*
ii) If $\dim \mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)) = 1$, *then* $\mu \in W \circ \lambda$.
iii) If $\beta \in \Phi^+$, $s_\beta \circ \mu \neq \mu$, then

$$\dim \mathrm{Hom}_{\mathfrak{g}}(M(s_\beta \circ \mu), M(\mu)) = 1 \Leftrightarrow \langle \mu + \rho, \beta \rangle \in \mathbb{N}_+.$$

iv) Every non-trivial homomorphism $\phi : M(\mu) \to M(\lambda)$ factors through maps of the type above: there exists a sequence of positive roots $\beta_1, \ldots, \beta_t$ such that *the weights* $\lambda_0 = \lambda$, $\lambda_1 = s_{\beta_1} \circ \lambda_0$, $\ldots$, $\lambda_t = s_{\beta_t} \circ \lambda_{t-1} = \mu$ *are all pairwise different and*

$$M(\mu) = M(\lambda_t) \hookrightarrow \ldots \hookrightarrow M(\lambda_2) \hookrightarrow M(\lambda_1) \hookrightarrow M(\lambda_0) = M(\lambda).$$

We fix an ordering of the positive roots, the correspondingly ordered monomials in the Chevalley basis form then a PBW-basis for $U(\mathfrak{n}^-)$. We use the standard filtration on $U(\mathfrak{n}^-)$ respectively $U(\mathfrak{b}^-)$: the elements of $\mathfrak{n}^-$ respectively of $\mathfrak{b}^-$ are all of degree one. The following result is due to Shapovalov. For $\beta = \sum_{\alpha\in\Gamma} \ell_\alpha \alpha \in \Phi^+$ and $m \in \mathbb{N}$ set

Proposition 1.2 ([**8**]). *There exists an element* $\theta_{\beta,m} \in U(\mathfrak{n}^- + \mathfrak{h})_{-m\beta}$, *unique modulo the left ideal generated by* $(H_\beta - m + \langle\rho, \beta\rangle)$, *such that*
i) $\theta_{\beta,m} 1_\lambda \in M(\lambda)$ *is a highest weight vector of weight* $s_\beta \circ \lambda$ *for all* $\lambda \in \mathcal{H}^\rho_{\beta,m}$,
ii) $\theta_{\beta,m} = \prod_{\alpha\in\Gamma} Y_\alpha^{m\ell_\alpha} +$ *elements of the form* uh, *where* $h \in U(\mathfrak{h})$ *and* $u \in U(\mathfrak{n}^-)_{-m\beta}$ *is of degree strictly less than* $m \sum \ell_\alpha$.

Note that the condition above on the term of highest degree is independent of the ordering of the product. We call $\theta_{\beta,m}$ a *Shapovalov element* for the pair (β, m). To have uniqueness, it is sometimes more convenient to replace $\theta_{\beta,m}$ by an element $\theta_{\mu,\lambda} \in U(\mathfrak{n}^-)$. By applying $\theta_{\beta,m}$ to 1_λ, one gets by Proposition 1.2:

Corollary 1.3. *Let* $\lambda, \mu \in \mathfrak{h}^*$ *and* $\beta = \sum_{\alpha\in\Gamma} \ell_\alpha \alpha \in \Phi^+$ *be such that* $m := \langle\lambda + \rho, \beta\rangle \in \mathbb{N}$ *and* $\mu = s_\beta \circ \lambda$. *There exists a unique element* $\theta_{\mu,\lambda} \in U(\mathfrak{n}^-)_{-m\beta}$ *such that*
i) $\theta_{\mu,\lambda} 1_\lambda \in M(\lambda)$ *is a highest weight vector of weight* μ,
ii) $\theta_{\mu,\lambda} = \prod_{\alpha\in\Gamma} Y_\alpha^{m\ell_\alpha} +$ *elements of lesser degree.*

2. L-S paths and Verma flags

Let $X \subset \mathfrak{h}^*$ be the lattice of integral weights and denote X^+ the monoid of dominant weights. We discuss now the connection between the combinatorics of L-S paths $\pi = (\underline{\tau}, \underline{a})$ of shape $\lambda \in X^+$ and the combinatorics of inclusions of Verma modules. Let $X_\mathbb{Q} \subset \mathfrak{h}^*$ be the rational span of the weight lattice, we denote $X_\mathbb{Q}^+$ the submonoid $\mathbb{Q}_{\geq 0} X^+$. We define a length function on $X_\mathbb{Q}$ as follows: for $\mu \in X_\mathbb{Q}$ let $\tau \in W$ be the unique element of minimal length such such that $\mu + \rho = \tau(\lambda + \rho)$ for some $\lambda + \rho \in X_Q^+$, we set $\ell(\mu) = \ell(\tau)$. If $\nu \in X_\mathbb{Q}^+$, then we write W_ν for the stabilizer of ν in W. We use the notation "$\geq$" for the Bruhat order on W as well as for the induced order on W/W_ν.

Let $\mu \neq \nu \in X_\mathbb{Q}$ be such that $\dim \mathrm{Hom}_\mathfrak{g}(M(\mu), M(\nu)) = 1$. By Theorem 1.1, there exists a sequence of positive roots $\beta_1, \ldots, \beta_t$ and pairwise different weights $\nu_0 = \nu$, $\nu_1 = s_{\beta_1} \circ \nu_0$, $\ldots$, $\nu_t = s_{\beta_t} \circ \nu_{t-1} = \mu$ such that

$$M(\mu) = M(\nu_t) \hookrightarrow \ldots \hookrightarrow M(\nu_1) \hookrightarrow M(\nu_0) = M(\nu), \tag{2.1}$$

and part *iii)* of the theorem implies for the length of the flag: $t \leq \ell(\mu) - \ell(\nu)$.

Definition 2.2. Let $\mu \neq \nu \in X_\mathbb{Q}$. We say that there exists a *maximal Verma flag* for the pair (μ, ν) if $\dim \mathrm{Hom}_\mathfrak{g}(M(\mu), M(\nu)) = 1$ and one can find a flag of length $t = \ell(\mu) - \ell(\nu)$ in (2.1).

We come now to the definition of an L-S path of shape λ. To simplify the notation with the shifts by ρ, we introduce the abbreviation

$$\hat{\mu} = \mu - \rho$$

The definition of an LS-path below is equivalent to the definition given in [**4**], we will comment on the connection in Remark 2.4 and 2.5:

DEFINITION 2.3. An *L-S path* $\pi = (\underline{\tau}, \underline{a})$ *of shape* $\lambda \in X^+$ is a pair of sequences $\underline{\tau} = (\tau_0, \tau_1, \dots, \tau_r)$ and $\underline{a} = (a_1, \dots, a_r)$ satisfying the following conditions:
i) $0 < a_1 < \dots < a_r < 1$ is a strictly increasing sequence of rational numbers,
ii) $\tau_0 > \tau_1 > \dots > \tau_r$ is a strictly decreasing sequence of cosets in W/W_λ;
iii) for all pairs $(\widehat{\nu_j}, \widehat{\mu_j})$ below there exists a maximal Verma flag:

$$\begin{cases} \nu_0 = a_1\tau_0(\lambda) & \nu_1 = a_2\tau_1(\lambda) & \dots & \nu_{r-1} = a_r\tau_{r-1}(\lambda) & \nu_r = \tau_r(\lambda) \\ \mu_0 = a_1\tau_1(\lambda) & \mu_2 = a_2\tau_2(\lambda) & \dots & \mu_{r-1} = a_r\tau_r(\lambda) & \mu_r = \lambda. \end{cases}$$

REMARK 2.4. Note: $s_\beta \circ \hat{\mu} = s_\beta(\hat{\mu}+\rho) - \rho = s_\beta(\mu) - \rho = \widehat{s_\beta(\mu)}$. A simple calculation (using part *iii)* and *iv)* of Theorem 1.1) shows: the existence of a maximal Verma flag for the pairs $(\widehat{\nu_j}, \widehat{\mu_j})$, is equivalent to the existence of an a_j-chain and the integrality conditions on the pairs (τ_j, τ_{j+1}) in the definition of an LS-path of shape λ given in [**4**].

REMARK 2.5. Since $\mu_r = \lambda \in X^+$ and $\nu_r = \tau_r(\lambda) \in X$, there exists always a maximal Verma flag for the pair $(\widehat{\lambda_r}, \widehat{\mu_r})$: Let $\tau_r = s_{i_1} \cdots s_{i_p}$ be a reduced decomposition, then a maximal Verma flag is given by:

$$M(\widehat{\tau_r(\lambda)}) = M((s_{i_1} \cdots s_{i_p}) \circ \widehat{\lambda}) \hookrightarrow \dots \hookrightarrow M(s_{i_p} \circ \widehat{\lambda}) \hookrightarrow M(\widehat{\mu_r}) = M(\widehat{\lambda})$$

and $\theta_{\widehat{\tau_r(\lambda)}, \widehat{\lambda}}$ is of the form $Y_{i_1}^{\langle s_{i_2} \cdots s_{i_p}(\lambda), \alpha_{i_1} \rangle} \cdots Y_{i_{p-1}}^{\langle s_{i_p}(\lambda), \alpha_{i_{p-1}} \rangle} Y_{i_p}^{\langle \lambda, \alpha_{i_p} \rangle}$.

REMARK 2.6. Set $a_0 = 0$ and $a_{r+1} = 1$. We identify sometimes the path with its "visible" path, i.e., the piecewise linear map $\pi : [0,1] \to X_\mathbb{R}$,

$$t \to \sum_{i=1}^{j} (a_i - a_{i-1})\tau_{i-1}(\lambda) + (t - a_j)\tau_j(\lambda) \quad \text{for} \quad a_j \le t \le a_{j+1},\ 0 \le j \le r.$$

Having this identification in mind, it makes sense to talk about the endpoint or the *weight* $\pi(1)$ *of the path.* The *LS-path character formula* in [**4**] states: Let $\lambda \in X$ be a dominant weight, let $V(\lambda)$ be the corresponding irreducible finite dimensional representation of $\mathfrak{g}$ and denote $B(\lambda)$ the set of LS-paths of shape λ. Then

$$\text{Char}V(\lambda) = \sum_{\pi \in B(\lambda)} e^{\pi(1)}. \tag{2.7}$$

3. Verma elements

We associate now to an LS-path a unique element in $U(\mathfrak{n}^-)$. Let $\nu, \eta \in X_\mathbb{Q}$ be a pair such that there exists a *maximal Verma flag* of length $t = \ell(\eta) - \ell(\nu)$:

$$M(\eta) = M(\nu_t) \hookrightarrow \dots \hookrightarrow M(\nu_2) \hookrightarrow M(\nu_1) \hookrightarrow M(\nu_0) = M(\nu) \tag{3.1}$$

and let $\beta_1, \ldots, \beta_t$ be the positive roots such that

$$\eta = \nu_t = s_{\beta_t} \circ \nu_{t-1}, \ \ldots, \ \nu_2 = s_{\beta_2} \circ \nu_1, \ \nu_1 = s_{\beta_1} \circ \nu_0, \ \nu_0 = \nu.$$

We associate to the flag (3.1) the monomial $\mathfrak{m}_{\eta,\nu}$ in $U(\mathfrak{n}^-)$ defined by

$$\mathfrak{m}_{\eta,\nu} = \frac{Y_{\beta_t}^{m_t}}{m_t!} \cdots \frac{Y_{\beta_1}^{m_1}}{m_1!}, \quad \text{where } m_j := \langle \nu_{j-1} + \rho, \beta_j \rangle. \tag{3.2}$$

Note that $\mathfrak{m}_{\eta,\nu}$ depends on the choice of the flag in (3.1). To obtain elements which depend only on η and ν, recall that the image of $i: M(\eta) \hookrightarrow M(\nu)$ is independent of the chosen maximal Verma flag because the space of homomorphisms is one-dimensional. By Corollary 1.3, there exists for all $j = 1, \ldots, t$ a unique element $\theta_{\nu_j,\nu_{j-1}}$ in $U(\mathfrak{n}^-)_{-m_j\beta_j}$ such that $\theta_{\nu_j,\nu_{j-1}} 1_{\nu_{j-1}}$ is a highest weight vector in $M(\nu_{j-1})$ of weight ν_j, and

$$\theta_{\nu_j,\nu_{j-1}} = \prod_{\alpha\in\Gamma} Y_\alpha^{m\ell_\alpha} + \text{ elements of lesser degree,}$$

where $\beta_j = \sum_{\alpha\in\Gamma} \ell_\alpha \alpha$. We set $\Theta_{\eta,\nu} = \theta_{\nu_t,\nu_{t-1}} \cdots \theta_{\nu_2,\nu_1}\theta_{\nu_1,\nu_0}$.

LEMMA 3.3. *Set $\nu - \eta = \sum_{\alpha\in\Gamma} q_\alpha \alpha$. The element $\Theta_{\eta,\nu} \in U(\mathfrak{n}^-)$ is uniquely characterized by the following properties: $\Theta_{\eta,\nu} 1_\nu \in M(\nu)$ is a highest weight vector of weight η, and $\Theta_{\eta,\nu} = \prod_{\alpha\in\Gamma} Y_\alpha^{q_\alpha} +$ elements of lesser degree.*

REMARK 3.4. The uniqueness property is independent on the chosen ordering on the product.

PROOF. By construction, we know that $\Theta_{\eta,\nu} 1_\nu$ is a highest weight vector in $M(\nu)$ of weight η, and $\Theta_{\eta,\nu}$ is, up to scalar multiples, uniquely determined by this property in $U(\mathfrak{n}^-)$. Rewriting a product of two monomials in a PBW-basis as a linear combination of the fixed PBW-basis gives a monomial of degree the sum of the two monomials (the product rewritten in the desired order) plus elements of strictly lesser degree. It follows hence that $\Theta_{\eta,\nu}$ has the form described above, and, by fixing the coefficient of the highest degree term, $\Theta_{\eta,\nu}$ is uniquely determined. □

DEFINITION 3.5. Let $\pi = (\underline{\tau}, \underline{a})$ be an LS-path of shape $\lambda \in X^+$, say $\underline{\tau} = (\tau_0, \ldots, \tau_r)$ and $\underline{a} = (a_1, \ldots, a_r)$. By the *Verma element* Θ_π in $U(\mathfrak{n}^-)$ we mean the product

$$\Theta_\pi = \Theta_{\widehat{\nu_0},\widehat{\mu_0}} \Theta_{\widehat{\nu_1},\widehat{\mu_1}} \cdots \Theta_{\widehat{\nu_r},\widehat{\mu_r}}$$

where the weights $\nu_0, \mu_0, \ldots, \nu_r, \mu_r$ are given as in Definition 2.3 by

$$\begin{cases} \nu_0 = a_1\tau_0(\lambda) & \nu_1 = a_2\tau_1(\lambda) & \ldots & \nu_{r-1} = a_r\tau_{r-1}(\lambda) & \nu_r = \tau_r(\lambda); \\ \mu_0 = a_1\tau_1(\lambda) & \mu_2 = a_2\tau_2(\lambda) & \ldots & \mu_{r-1} = a_r\tau_r(\lambda) & \mu_r = \lambda, \end{cases}$$

The Verma elements Θ_π have in general a rather complicated structure, for a general algorithm to compute the $\theta_{\mu,\nu}$ see [**7**]. For this reason it is also useful to define monomials $\mathfrak{m}_\pi$ in a similar way as above:

DEFINITION 3.6. Let $\pi = (\underline{\tau}, \underline{a})$ be an LS-path of shape λ as above in Definition 3.5. For each pair $(\widehat{\nu_j}, \widehat{\mu_j})$ fix a maximal Verma flag and let $\mathfrak{m}_{\widehat{\nu_j},\widehat{\mu_j}}$ be as in (3.2). By the *path monomial* $\mathfrak{m}_\pi$ in $U(\mathfrak{n}^-)$ we mean the monomial

$$\mathfrak{m}_\pi = \mathfrak{m}_{\widehat{\nu_0},\widehat{\mu_0}} \mathfrak{m}_{\widehat{\nu_1},\widehat{\mu_1}} \cdots \mathfrak{m}_{\widehat{\nu_r},\widehat{\mu_r}}$$

Remark 3.7. $\mathfrak{m}_\pi, \Theta_\pi \in U(\mathfrak{n}^-)$ are $\mathfrak{h}$-eigenvectors of weight $\lambda - \pi(1)$. The Verma element Θ_π depends only on the path π and is hence *canonically* defined, but the path monomial $\mathfrak{m}_\pi$ depends on the choice of the maximal Verma chains necessary for the definition of the $\mathfrak{m}_{\widehat{\nu_j},\widehat{\mu_j}}$.

4. LS-concatenations of shape $m\lambda$

We consider now the LS-paths of shape $m\lambda$ for all $m \in \mathbb{N}$ at once. To "compare" the LS-paths of shape $p\lambda$ with those of shape $q\lambda$, it is convenient to use the following definition which formalizes the usual concatenation of the "visible" paths in Remark 2.6 (see [**4**]):

Definition 4.1. An *L-S concatenation* $\pi=(\underline{\tau},\underline{a})$ *of shape* $m\lambda \in X^+$ is a pair $\underline{\tau}=(\tau_0,\dots,\tau_r)$, $\underline{a}=(a_1,\dots,a_r)$, satisfying the following conditions:
i) $0<a_1<\dots<a_r<m$ is a strictly increasing sequence of rational numbers;
ii) $\tau_0>\tau_1>\dots>\tau_r$ is a strictly decreasing sequence of cosets in W/W_λ;
iii) for all pairs $(\widehat{\nu_j},\widehat{\mu_j})$ below there exists a maximal Verma flag:

$$\begin{cases} \nu_0 = a_1\tau_0(\lambda) & \nu_1 = a_2\tau_1(\lambda) & \dots & \nu_{r-1} = a_r\tau_{r-1}(\lambda) & \nu_r = m\tau_r(\lambda); \\ \mu_0 = a_1\tau_1(\lambda) & \mu_2 = a_2\tau_2(\lambda) & \dots & \mu_{r-1} = a_r\tau_r(\lambda) & \mu_r = m\lambda, \end{cases}$$

Set $a_0 = 0$ and $a_{r+1} = m$, the weight of π is defined as

$$\pi(1) = \sum_{j=0}^{r}(a_{j+1} - a_j)\tau_j(\lambda)$$

Denote by $B(m\lambda)$ the set of all LS-paths of shape $m\lambda$ and let $C(m\lambda)$ be the set of all LS-concatenations of shape $m\lambda$. We have obviously $C(\lambda) = B(\lambda)$. The next lemma follows immediately from the definitions:

Lemma 4.2. *The following map defines a weight preserving bijection:*

$$\begin{array}{rcl} \phi : B(m\lambda) & \longrightarrow & C(m\lambda) \\ (\tau_0,\dots,\tau_r; a_1,\dots,a_r) & \mapsto & (\tau_0,\dots,\tau_r; ma_1,\dots,ma_r). \end{array} \tag{4.3}$$

The advantage of the notation of LS-concatenations is that for $q > p$ we can easily regard $C(p\lambda)$ as a subset of $C(q\lambda)$. The inclusion $i : C(p\lambda) \hookrightarrow C(q\lambda)$ is given by

$$(\tau_0,\dots,\tau_r; a_1,\dots,a_r) \mapsto \begin{cases} (\tau_0,\dots,\tau_r,\mathrm{id}; a_1,\dots,a_r,p) & \text{if } \tau_r \neq \mathrm{id} \\ (\tau_0,\dots,\tau_r; a_1,\dots,a_r) & \text{if } \tau_r = \mathrm{id} \end{cases}$$

and the image is characterized as

$$i(C(p\lambda)) = \Big\{(\kappa_0,\dots,\kappa_r; b_1,\dots,b_r) \in C(q\lambda) \Big| \kappa_r = \mathrm{id} \text{ and } b_r \leq p\Big\} \tag{4.4}$$

The inclusion $i : C(p\lambda) \hookrightarrow C(q\lambda)$ itself is not weight preserving, but note that $p\lambda - \pi(1) = q\lambda - i(\pi)(1)$ for $\pi \in C(p\lambda)$. By identifying the sets with their images, we get a sequence of inclusions

$$C(\lambda) \subset C(2\lambda) \subset C(3\lambda) \subset C(4\lambda) \subset \dots \subset C(m\lambda) \subset \dots \tag{4.5}$$

So it makes sense to talk about $C(\lambda,\infty) = \bigcup_{m\in\mathbb{N}} C(\lambda, m)$.

Definition 4.6. We say that $\pi \in C(\lambda,\infty)$ *is of shape* $m\lambda$ if π is an element of $C(m\lambda)$.

Lemma 4.7. *If $\pi \in B(m\lambda)$ and $\pi' \in B(n\lambda)$ are such that $\pi = \pi'$ in $C(\lambda, \infty)$ (via the inclusions in (4.3) and (4.5)), then $\mathfrak{m}_\pi = \mathfrak{m}_{\pi'}$ and $\Theta_\pi = \Theta_{\pi'}$.*

So in the following we will only write π and not make a difference between LS-paths of shape $m\lambda$ and LS-concatenations of shape $m\lambda$. Further, by Lemma 4.7 it makes sense to consider the collection of elements in $U(\mathfrak{m}^-)$:

$$\mathbb{B}(\lambda, \infty) = \{\Theta_\pi \mid \pi \in C(\lambda, \infty)\} \text{ and } \mathbb{M}(\lambda, \infty) = \{\mathfrak{m}_\pi \mid \pi \in C(\lambda, \infty)\}.$$

Proof (Lemma 4.7). Let $\pi \in C(\lambda, \infty)$, say $\pi = (\underline{\tau}, \underline{a}) \in C(m\lambda)$. We can associate to π a Verma element and a monomial as in Definition 3.5 and in Definition 3.6, we set

$$\Theta_\pi = \Theta_{\widehat{\nu_0}, \widehat{\mu_0}} \Theta_{\widehat{\nu_1}, \widehat{\mu_1}} \cdots \Theta_{\widehat{\nu_r}, \widehat{\mu_r}} \quad \text{and} \quad \mathfrak{m}_\pi = \mathfrak{m}_{\widehat{\nu_0}, \widehat{\mu_0}} \mathfrak{m}_{\widehat{\nu_1}, \widehat{\mu_1}} \cdots \mathfrak{m}_{\widehat{\nu_r}, \widehat{\mu_r}}$$

where the weights $\nu_0, \mu_0, \ldots, \nu_r, \mu_r$ are given as in Definition 4.1 by $\nu_0 = a_1\tau_0(\lambda)$, $\mu_0 = a_1\tau_1(\lambda)$, ... It is evident from the definition that Θ_π and $\mathfrak{m}_\pi$ are independent of the fact whether we regard π as an element of $C(m\lambda)$ or $C(n\lambda)$ for $n \geq m$.

Let now $\pi \in B(\lambda)$ be an LS-path of shape $m\lambda$. One verifies easily $\Theta_\pi = \Theta_{\phi(\pi)}$ and $\mathfrak{m}_\pi = \mathfrak{m}_{\phi(\pi)}$ for the map $\phi : B(m\lambda) \to C(m\lambda)$ in (4.3), which finishes the proof. □

5. The basis given by Verma elements

For simplicity, let first λ be a regular integral dominant weight.

Theorem 5.1. i) *The Verma elements $\mathbb{B}(\lambda, \infty)$ form a basis of $U(\mathfrak{n}^-)$.*

ii) *$\mathbb{B}(\lambda, \infty)$ is compatible with all irreducible representations $V(m\lambda)$ of highest weight $m\lambda$, i.e., $\{\Theta_\pi \mid \pi \in C(\lambda, \infty), \pi \notin C(m\lambda)\}$ is a basis for the kernel of the surjective map*

$$\Psi : U(\mathfrak{n}^-) \to V(m\lambda), \quad u \mapsto uv_{m\lambda},$$

and the set of vectors $\{\Theta_\pi v_{m\lambda} \mid \pi \in C(m\lambda)\}$ is a basis of $V(m\lambda)$.

The formulation for a not necessarily regular $\lambda \in X^+$ has to be slightly adapted. Let $\mathfrak{p}$ be the parabolic subalgebra associated to λ, i.e., $\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{n}_\lambda^-$ where $\mathfrak{n}_\lambda^- = \oplus_{\beta \perp \lambda} \mathfrak{g}_{-\beta}$. Here the sum runs over all positive roots orthogonal to λ. Denote by $\mathfrak{n}_\mathfrak{p}^-$ the nilpotent radical of the parabolic subgroup opposite to $\mathfrak{p}$, so $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}_\mathfrak{p}^-$.

Let $\mathcal{I}_\lambda \subset U(\mathfrak{n}^-)$ be the left ideal generated by $\mathfrak{n}_\lambda^-$, then the canonical map $U(\mathfrak{n}_\mathfrak{p}^-) \to U(\mathfrak{n}^-)/\mathcal{I}_\lambda$ is an isomorphism of left $U(\mathfrak{n}_\mathfrak{p}^-)$-modules. Denote by

$$\overline{\mathbb{B}}(\lambda, \infty) \subset U(\mathfrak{n}^-)/\mathcal{I}_\lambda$$

the images $\overline{\Theta}_\pi$ of the Verma elements Θ_π in the quotient module $U(\mathfrak{n}^-)/\mathcal{I}_\lambda$.

Theorem 5.2. i) *The Verma elements $\overline{\mathbb{B}}(\lambda, \infty)$ form a basis of $U(\mathfrak{n}^-)/\mathcal{I}_\lambda$.*

ii) $\overline{\mathbb{B}}(\lambda,\infty)$ *is compatible with all irreducible representations* $V(m\lambda)$ *of highest weight* $m\lambda$, *i.e.,* $\{\overline{\Theta}_\pi \mid \pi\in C(\lambda,\infty),\ \pi\notin C(m\lambda)\}$ *is a basis of the kernel of the surjective map*

$$\Psi : U(\mathfrak{n}^-)/\mathcal{I}_\lambda \to V(m\lambda),\quad u\mapsto uv_{m\lambda},$$

and the set of vectors $\{\overline{\Theta}_\pi v_{m\lambda}\mid \pi\in C(m\lambda)\}$ *is a basis of* $V(m\lambda)$.

Denote by $\overline{\mathbb{M}}(\lambda,\infty)\subset U(\mathfrak{n}^-)/\mathcal{I}_\lambda$ the images $\overline{\mathfrak{m}}_\pi$ of the path monomials $\mathfrak{m}_\pi$ in $U(\mathfrak{n}^-)/\mathcal{I}_\lambda$. The disadvantage of the monomials $\mathfrak{m}_\pi$ is that they are not attached canonically to a path, certain choices have to be made. The advantage is the simpler structure (the elements in $\mathbb{B}(\lambda,\infty)$ are in general not monomials) and they provide a basis already for the Kostant lattice $U_\mathbb{Z}(\mathfrak{n}^-)$ and the corresponding lattice $V_\mathbb{Z}(m\lambda)$ in the representation. The following theorem has been conjectured by V. Lakshmibai and C. S. Seshadri in [**3**], section 2.

THEOREM 5.3. *i) The path monomials* $\overline{\mathbb{M}}(\lambda,\infty)$ *form a* $\mathbb{Z}$*-basis of the quotient* $U_\mathbb{Z}(\mathfrak{n}^-)/\mathcal{I}_\lambda$.
ii) $\overline{\mathbb{M}}(\lambda,\infty)$ *is compatible with all irreducible representations* $V(m\lambda)$ *of highest weight* $m\lambda$, *i.e.,* $\{\overline{\mathfrak{m}}_\pi\mid\pi\in C(\lambda,\infty),\ \pi\notin C(m\lambda)\}$ *is a basis of the kernel of the surjective map*

$$\Psi : U(\mathfrak{n}^-)/\mathcal{I}_\lambda \to V(m\lambda),\quad u\mapsto uv_{m\lambda},$$

and the elements $\{\overline{\mathfrak{m}}_\pi v_{m\lambda}\mid\pi\in C(m\lambda)\}$ *form a* $\mathbb{Z}$*-basis of* $V_\mathbb{Z}(m\lambda)$.

We give only the proof of Theorem 5.1 and the proof of Theorem 5.3 in the case of a regular dominant weight, the proof of Theorem 5.2 as well as the proof of Theorem 5.3 in the non-regular case is on the same lines and is left to the reader.

PROOF (Theorem 5.1 and 5.3). The proof is split into several parts, the most difficult part being the proof of the linear independence. In the rest of this section we show how the linear independence implies the rest of the theorems over the complex numbers, the proof of the linear independence (and generating over $\mathbb{Z}$) will be given in section 7.

Let Q be the root lattice and let Q^+ be the submonoide generated by the positive roots. Then $U(\mathfrak{n}^-)$ is a $\mathfrak{h}$-module with weight space decomposition

$$U(\mathfrak{n}^-)=\bigoplus_{\eta\in Q^+}U(\mathfrak{n}^-)_{-\eta}.$$

The map $\Psi : U(\mathfrak{n}^-)\longrightarrow V(m\lambda)$ is surjective, and for a fixed $\eta\in Q^-$ the restriction of Ψ to a weight space is in fact an isomorphism for all $m\gg 0$:

$$\Psi|_{U(\mathfrak{n}^-)_{-\eta}} : U(\mathfrak{n}^-)_{-\eta}\xrightarrow{\sim}V(m\lambda)_{m\lambda-\eta}.$$

By the character formula (Remark 2.6), the dimension of $V(m\lambda)_{m\lambda-\eta}$ is equal to the number of LS-paths of shape $m\lambda$ ending in $m\lambda-\eta$. So for $m\gg 0$:

$$\dim U(\mathfrak{n}^-)_{-\eta}=\dim V(m\lambda)_{m\lambda-\eta}=\sharp\{\pi\in C(m\lambda)\mid \pi(1)=m\lambda-\eta\}.$$

This shows by the (assumed) linear independence that the corresponding Θ_π (respectively $\mathfrak{m}_\pi$) span $U(\mathfrak{n}^-)_{-\eta}$ and hence form a basis for $U(\mathfrak{n}^-)_{-\eta}$.

A similar counting argument proves *ii)* once one has shown that all the Θ_π (respectively $\mathfrak{m}_\pi$) with $\pi \notin C(m\lambda)$ are in the kernel of the map Ψ. So let $\pi \in C(\lambda,\infty)$ and let p be minimal such that $\pi = (\underline{\tau}, \underline{a}) \in C(p\lambda)$. We suppose first $\underline{\tau} = (\tau_0, \ldots, \tau_r)$ is such that $\tau_r \neq \mathrm{id}$. Let $\tau_r = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition. By Remark 2.5 we know that Θ_π (respectively $\mathfrak{m}_\pi$) is of the form $\mathfrak{m}Y_{i_k}^{p\langle\lambda,\alpha_{i_k}\rangle}$ for some element $\mathfrak{m} \in U(\mathfrak{n}^-)$. Now $\pi \notin C(m\lambda)$ can only be if $p > m$, but in this case we have

$$Y_{i_k}^{p\langle\lambda,\alpha_{i_k}\rangle} v_{m\lambda} = 0 \quad \Rightarrow \quad \Theta_\pi v_{m\lambda} = 0 \text{ respectively } \mathfrak{m}_\pi v_{m\lambda} = 0.$$

Next suppose $\tau_r = \mathrm{id}$, then $a_r > m$, otherwise π would be of shape $m\lambda$, see (4.4). Any maximal Verma flag for the pair $(\widehat{a_r\tau_r(\lambda)}, \widehat{a_r\lambda})$ has to be of the form

$$M(\widehat{a_r\tau_r(\lambda)}) \hookrightarrow \ldots \hookrightarrow M(s_\alpha \circ \widehat{a_r\lambda}) \hookrightarrow M(\widehat{a_r\lambda})$$

for some simple root α. So Θ_π is of the form $\mathfrak{m}Y_\alpha^{a_r\langle\lambda,\alpha\rangle}$ and hence $\Theta_\pi v_{m\lambda} = 0$ because $a_r > m$. The arguments for $\mathfrak{m}_\pi$ are again similar: fix a reduced decomposition $\tau_{r-1} = s_{i_1} \ldots s_{i_t}$, then $\mathfrak{m}_\pi$ is of the form $\mathfrak{m}Y_{i_t}^{a_r\langle\lambda,\alpha_{i_t}\rangle}$ and hence $\mathfrak{m}_\pi v_{m\lambda} = 0$ because $a_r > m$. □

6. The elements $\theta_{\mu,\nu}$

We describe in this section in more detail the elements $\theta_{\mu,\nu}$ (Corollary 1.3). Let $\nu \in X_\mathbb{Q}$ be a rational weight and let $\beta \in \Phi^+$ be such that $m = \langle \nu + \rho, \beta \rangle$ is a positive integer, set $\mu = s_\beta \circ \nu$. We assume *in addition* that $\ell(\mu) - \ell(\nu) = 1$; in other words: $M(\mu) \subset M(\nu)$ is a maximal Verma flag.

Consider the associated element $\theta_{\mu,\nu}$ of weight $-m\beta$, so $\theta_{\mu,\nu}1_\nu$ is a highest weight vector in $M(\nu)$. We will show that for an appropriate choice of the ordering of the positive roots, the *coefficient of Y_β^m is nonzero* in the expression of $\theta_{\mu,\nu}$ as a linear combination of elements of the associated PBW-basis.

Denote $X_\mathbb{Q}^+$ the dominant Weyl chamber and set $\phi = X_\mathbb{Q}^+ \cap W(\nu + \rho)$. Denote W_ϕ the stabilizer of ϕ and let $W^\phi \subset W$ be the unique representatives of minimal length of the cosets W/W_ϕ. Fix $\kappa \in W^\phi$ such that

$$\kappa(\phi) = \nu + \rho, \text{ set } s_\beta\kappa = \tau, \text{ so } \tau(\phi) = \nu + \rho - m\beta = \mu + \rho. \tag{6.1}$$

Now $\beta \in N(\tau) = \{\delta \succ 0 \mid \tau^{-1}(\delta) \prec 0\}$ by construction. For a reduced decomposition $\tau = s_{i_1} \cdots s_{i_r}$ one has

$$N(\tau) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), s_{\alpha_{i_1}} s_{\alpha_{i_2}}(\alpha_{i_3}), \ldots, s_{\alpha_{i_1}} \cdots s_{\alpha_{i_{r-1}}}(\alpha_{i_r})\}, \tag{6.2}$$

so there exists a j such that $\beta = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$. Complete the reduced decomposition of τ to a reduced decomposition of the longest word

$$w_0 = s_{i_1} \cdot \ldots \cdot s_{i_N}. \tag{6.3}$$

The reduced decomposition defines a convex enumeration of the positive roots (i.e., if $\beta_i + \beta_\ell = \beta_k$, then either $i > k > \ell$ or $\ell > k > i$), we set:

(6.4)

$$\beta_1 = \alpha_{i_1},\ \beta_2 = s_{i_1}(\alpha_{i_2}),\ \beta_3 = s_{i_1}s_{i_2}(\alpha_{i_3}),\ \ldots,\ \beta_N = s_{i_1}s_{i_2}\ldots s_{i_{N-1}}(\alpha_{i_N})$$

Let $\mathbb{B}$ be the PBW-basis of $U(\mathfrak{n}^-)$ given by the monomials:

$$\mathbb{B} := \{Y_{\beta_N}^{n_N} \dots Y_{\beta_{j+1}}^{n_{j+1}} Y_{\beta}^{n_j} Y_{\beta_{j-1}}^{n_{j-1}} \dots Y_{\beta_2}^{n_2} Y_{\beta_1}^{n_1} \mid n_1, \dots, n_N \geq 0\} \tag{6.5}$$

PROPOSITION 6.6. *In the expression of $\theta_{\mu,\nu} = \sum_{\mathfrak{m}\in\mathbb{B}} a_{\mathfrak{m}}\mathfrak{m}$ as a linear combination of elements of $\mathbb{B}$, the coefficient of $\mathfrak{m} = Y_\beta^m$ is a non-zero rational number.*

REMARK 6.7. Since the weights are rational, all coefficients are rational numbers. The important point for the later application is the fact that the coefficient of Y_β^m is non-zero.

The proof of the proposition needs some preparation, it will be given at the end of this section.

Let λ be a dominant integral weight, let $W_\lambda \subset W$ be the stabilizer and let $V(\lambda)$ be the corresponding simple complex $\mathfrak{g}$-module. The weight space $V(\lambda)_{w(\lambda)}$ of weight $w(\lambda)$ is of dimension one for all $w \in W^\lambda$, and a nonzero generator $v_w \in V(\lambda)_{w(\lambda)}$ is called an *extremal weight vector*. The *Demazure submodule* $V(\lambda)_w \subseteq V(\lambda)$ is the $U(\mathfrak{n})$-submodule $U(\mathfrak{n})v_w$, and the *opposite Demazure module* $V(\lambda)_w^- \subseteq V(\lambda)$ is the $U(\mathfrak{n}^-)$-submodule $U(\mathfrak{n}^-)v_w$.

Let $w, u \in W^\lambda$ and $\beta \in \Phi^+$ be such that $s_\beta w = u$ and $\ell(u) - \ell(w) = 1$. Fix $u = s_{i_1} \cdots s_{i_r}$ a reduced decomposition and enumerate correspondingly the positive roots as in (6.4) (so $\beta = \beta_j$ for some $j \leq r$), and let $\mathbb{B}$ as in (6.5).

LEMMA 6.8. *Let $\mathfrak{m} \in \mathbb{B}$ be an element of weight $-k\beta_i$, $i \leq j$. Then either $\mathfrak{m} = Y_\beta^k$ or $\mathfrak{m} = Y_{\beta_N}^{n_N} \dots Y_{\beta_\ell}^{n_\ell}$ is such that $n_\ell > 0$ for some $\ell < j$. Further, let v_w be an extremal weight vector in $V(\lambda)_{w(\lambda)}$. Then $\mathfrak{m}v_w = 0$ unless $\mathfrak{m} = Y_\beta^k$ for some $0 \leq k \leq \langle w(\lambda), \beta\rangle$.*

As an immediate consequence we get:

COROLLARY 6.9. *Suppose $1 \leq i < j$ and $r \geq 1$ or $i = j$ and $r > \langle w(\lambda), \beta\rangle$. Then $w(\lambda) - r\beta_i$ is not a weight in $V(\lambda)_w^-$, and for $0 \leq r \leq \langle w(\lambda), \beta\rangle$, the weight space of weight $w(\lambda) - r\beta$ is of dimension one in $V(\lambda)_w^-$.*

PROOF (of Lemma 6.8). The proof is by induction on the j such that $\beta_j = \beta$. If $j = 1$, then $\beta = \alpha$ is a simple reflection and the proof is obvious.

Suppose $j > 1$ and that the lemma holds for all $j' < j$. If $\mathfrak{m} = Y_\beta^k$, then $\beta \in N(u)$ implies $\langle u(\lambda), \beta\rangle < 0$ and hence $\langle w(\lambda), \beta\rangle > 0$. It follows by $\mathfrak{sl}_2$-representation theory that $Y_\beta^k v_w = 0$ for $k > \langle w(\lambda), \beta\rangle$ and $Y_\beta^k v_w \neq 0$ for $k \leq \langle w(\lambda), \beta\rangle$. Let $\mathfrak{m} = Y_{\beta_N}^{n_N} \dots Y_{\beta_\ell}^{n_\ell}$ be such that $n_\ell > 0$ for some $\ell < j$. Recall that w has a reduced decomposition of the form $w = s_{i_1} \dots s_{i_{j-1}} s_{i_{j+1}} \dots s_r$. By the choice of the enumeration of the roots, $\ell < j$ implies $\beta_\ell \in N(w)$ and hence $\langle w(\lambda), \beta_\ell\rangle < 0$, so $Y_{\beta_\ell} v_w = 0$. But this implies also $\mathfrak{m}v_w = 0$.

Let $\mathfrak{m} = Y_{\beta_N}^{n_N} \dots Y_{\beta_\ell}^{n_\ell}$ be of weight $-k\beta_i$ for some $i \leq j$, but $\ell \geq j > 1$ and $\mathfrak{m}$ is not a power of Y_β. Let $\tilde{s}_{i_1} \in \mathrm{Nor}_G(T)$ be a representative of s_{i_1}, set $\delta := s_{i_1}(\beta)$ and $u' := s_{i_1}u$. Note that $\ell(s_\delta u') = \ell(u') - 1$. Consider the reduced expression $w_0 = s_{i_2} \cdot \ldots \cdot s_{i_N} s_{-w_0(\alpha_{i_1})}$ and the associated ordering of the positive roots (compare with (6.4)):

$$\beta_1' := s_{i_1}(\beta_2),\ \beta_2' := s_{i_1}(\beta_3),\ \dots,\ \beta_{N-1}' := s_{i_1}(\beta_N),\ \beta_N' := -w_0(\alpha_{i_1}),$$

so $\delta = \beta'_{j-1}$. Denote $\mathbb{B}'$ the corresponding PBW–basis given by the monomials of the form $\mathfrak{m}' = Y_{\beta'_N}^{n'_N} \dots Y_{\beta'_1}^{n'_1}$. By induction we know that an element of weight $-k\beta'_q$, $1 \le q \le j-1$ of $\mathbb{B}'$ is either equal to Y_δ^k or of the form $Y_{\beta'_N}^{n'_N} \dots Y_{\beta'_t}^{n'_t}$ with $n'_t > 0$ for some $t < j-1$. Consider now the element $\mathfrak{m} = Y_{\beta_N}^{n_N} \dots Y_{\beta_\ell}^{n_\ell} \in \mathbb{B}$ above. Since $\ell \ge j > 1$, the monomial

$$\mathfrak{m}' := \mathbf{Ad}\tilde{s}_{i_1}(\mathfrak{m}) = \mathbf{Ad}\tilde{s}_{i_1}(Y_{\beta_N})^{n_N} \dots \mathbf{Ad}\tilde{s}_{i_1}(Y_{\beta_\ell}^{n_\ell}) = cY_{\beta'_{N-1}}^{n_N} \dots Y_{\beta'_{\ell-1}}^{n_\ell}$$

for some $c \in \mathbb{C}^*$. Since $\mathfrak{m}'$ is of weight $-ks_{i_1}(\beta_i) = -k\beta'_{i-1}$ and not a power of Y_δ and an element of $\mathbb{B}'$, we know by induction that $\ell - 1 < j - 1$ and hence $\ell < j$, a contradiction to the assumption. □

Let $\sigma = s_{i_1} \dots s_{i_r}$ be a reduced decomposition of an element in the Weyl group and set $\beta = s_{i_1} \dots s_{i_{r-1}}(\alpha_{i_r}) \in \Phi^+$. Let C be the dominant Weyl chamber and for a natural number m let $\mathcal{H}^\rho_{\beta,m}$ be the affine hyperplane

$$\mathcal{H}^\rho_{\beta,m} := \{\nu \in X_\mathbb{R} \mid \langle \nu + \rho, \beta \rangle = m\}.$$

Lemma 6.10. *$(s_\beta\sigma \circ C) \cap X \cap \mathcal{H}^\rho_{\beta,m}$ is Zariski dense in $\mathcal{H}^\rho_{\beta,m}$.*

Proof. We have $s_\beta\sigma = \sigma s_\gamma$, where $\gamma = \alpha_{i_r}$. For a simple root $\alpha \in \Delta$ denote ω_α the corresponding fundamental weight. Now γ is a simple root, so

$$C \cap X \cap \mathcal{H}^\rho_{\gamma,m} = \{\lambda \in X^+ \mid \lambda = (m-1)\omega_\gamma + \sum_{\substack{\alpha \in \Delta - \{\gamma\} \\ n_\alpha \ge 0}} n_\alpha \omega_\alpha\}, \tag{6.11}$$

and the latter is obviously Zariski dense in $\mathcal{H}^\rho_{\gamma,m}$. Further, the equation

$$\langle w \circ \nu + \rho, w(\gamma) \rangle = \langle w(\nu + \rho), w(\gamma) \rangle = \langle \nu + \rho, \beta \rangle$$

shows that for all $w \in W$ we have:

$$w \circ \mathcal{H}^\rho_{\gamma,m} = \mathcal{H}^\rho_{w(\gamma),m} \tag{6.12}$$

The lemma follows immediately from (6.11) and (6.12) □

We will use this Zariski dense subset to get an recursive description of the coefficient in question. Let again $\tau, \kappa \in W$ and $\beta \in \Phi^+$ be such that $\tau = s_\beta\kappa$ and $\ell(\tau) = \ell(\kappa) + 1$. Fix a reduced decomposition $\tau = s_{i_1} \cdots s_{i_r}$ and let j be such that $\beta = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$. We assume from *now on that* $j > 1$. Set

$$\sigma = s_{i_1} \cdots s_{i_j},\ \alpha = \alpha_{i_1},\ \delta := s_\alpha(\beta) \text{ and } \sigma' := s_\alpha\sigma.$$

Make the decomposition of τ complete to a decomposition of w_0, and let $\mathbb{B}$ be the corresponding PBW–basis of $U(\mathfrak{n}^-)$ (as in (6.3) and (6.5)), and let $\mathbb{B}'$ be the PBW–basis of $U(\mathfrak{n}^-)$ associated to the reduced decomposition of the longest word given by $w_0 = s_{i_2} \dots s_{i_N} s_{i_0}$, where $\alpha_{i_0} = -w_0(\alpha_{i_1})$, so

$$\mathbb{B}' := \{Y_{\beta'_N}^{n_N} \dots Y_{\beta'_2}^{n_2} Y_{\beta'_1}^{n_1} \mid n_1, \dots, n_N \ge 0\}. \tag{6.13}$$

Let m be a positive natural number and consider the Shapovalov elements $\theta_{\beta,m}, \theta_{\delta,m}$ in $U(\mathfrak{n}^- \oplus \mathfrak{h})$. We write the first one as a linear combination of elements in $\mathbb{B}$ (and coefficients in $U(\mathfrak{h})$), the second as a linear combination of elements in $\mathbb{B}'$:

$$\theta_{\beta,m} = \sum_{\mathfrak{m}\in\mathbb{B}, h_\mathfrak{m}\in U(\mathfrak{h})} \mathfrak{m} h_\mathfrak{m} \quad \text{and} \quad \theta_{\delta,m} = \sum_{\mathfrak{m}'\in\mathbb{B}', h_{\mathfrak{m}'}\in U(\mathfrak{h})} \mathfrak{m}' h_{\mathfrak{m}'}.$$

Let $h_{m\beta} \in U(\mathfrak{h})$ be the coefficient of Y_β^m in $\theta_{\beta,m}$ and let $h_{m\delta} \in U(\mathfrak{h})$ be the coefficient of Y_δ^m in $\theta_{\delta,m}$. Recall that the coefficients are only determined up to the left ideal generated by $(H_\beta - m + \langle\rho,\beta\rangle)$ respectively $(H_\delta - m + \langle\rho,\delta\rangle)$, see Proposition 1.2, and that $s_\alpha \circ \eta \in \mathcal{H}^\rho_{\delta,m}$ for $\eta \in \mathcal{H}^\rho_{\beta,m}$, see (6.12).

Lemma 6.14. *For all $\eta \in \mathcal{H}^\rho_{\beta,m}$ we have:*

$$\begin{cases} \eta(h_{m\beta}) &= \pm(s_\alpha\circ\eta)(h_{m\delta}\prod_{s=1}^{m\langle\beta,\alpha\rangle}(H_\alpha+1+s)) & \text{if } \langle\beta,\alpha\rangle \geq 0 \\ (s_\alpha\circ\eta)(h_{m\delta}) &= \pm\eta(h_{m\beta}\prod_{s=1}^{-m\langle\beta,\alpha\rangle}(-H_\alpha-s)) & \text{if } \langle\beta,\alpha\rangle < 0 \end{cases}$$

Proof. We view the elements of $U(\mathfrak{h})$ as polynomial functions on $\mathcal{H}^\rho_{\beta,m}$ respectively $\mathcal{H}^\rho_{\delta,m}$. By Lemma 6.10, $S = (s_\beta\sigma)\circ C \cap X \cap \mathcal{H}^\rho_{\beta,m}$ is Zariski dense in $\mathcal{H}^\rho_{\beta,m}$. Consider $\eta \in S$ and set $\varphi = s_\beta\circ\eta = \eta - m\beta \in \mathcal{H}^\rho_{\delta,m}$. By the choice of η, there exists an integral dominant weight λ such that $\eta = s_\beta\sigma\circ\lambda$. Now $\lambda+\rho$ is a regular integral dominant weight and $\ell(s_\alpha s_\beta\sigma) < \ell(s_\beta\sigma)$ as well as $\ell(s_\alpha\sigma) < \ell(\sigma)$, so it follows that

$$\langle\eta+\rho,\alpha\rangle = \langle s_\beta\sigma(\lambda+\rho),\alpha\rangle = -r < 0, \quad \langle\varphi+\rho,\alpha\rangle = \langle\sigma(\lambda+\rho),\alpha\rangle = -p < 0.$$

We have the following square of embeddings of Verma modules:

$$\begin{array}{ccccc} & & M(\eta) & & \\ & \nearrow^{\theta_{\varphi,\eta}}_{\text{weight}-m\beta} & & \searrow^{\theta_{\eta,s_\alpha\circ\eta}}_{\text{weight}-r\alpha} & \\ M(\varphi) & & & & M(s_\alpha\circ\eta) \\ & \searrow^{\theta_{\varphi,s_\alpha\circ\varphi}}_{\text{weight}-p\alpha} & & \nearrow^{\theta_{s_\alpha\circ\varphi,s_\alpha\circ\eta}}_{\text{weight}-m\delta} & \\ & & M(s_\alpha\circ\varphi) & & \end{array}$$

Since α is a simple root and because of the normalization we know that (6.15)

$$\theta_{\varphi,s_\alpha\circ\varphi} = Y_\alpha^p, \quad \theta_{\eta,s_\alpha\circ\eta} = Y_\alpha^r \quad \text{and} \quad \theta_{\varphi,s_\alpha\circ\varphi}\theta_{s_\alpha\circ\varphi,s_\alpha\circ\eta} = \theta_{\varphi,\eta}\theta_{\eta,s_\alpha\circ\eta}.$$

Set $\nu = \lambda+\rho$ and $w = s_\alpha s_\beta\sigma$, then we get for

$$\nu_0 = w(\nu),\ \nu_1 = s_\alpha w(\nu) = \eta+\rho,\ \nu_2 = s_\delta w(\nu),\ \nu_3 = s_\beta s_\alpha w(\nu) = \varphi+\rho$$

the following diagram of weights:

$$\begin{array}{ccccc} & & \nu_0 = w(\nu) & & \\ & \swarrow r\alpha & & \searrow m\delta & \\ \nu_1 = s_\alpha w(\nu) & & & & \nu_2 = s_\delta w(\nu) \\ & \searrow m\beta & & \swarrow p\alpha & \\ & & \nu_3 = s_\beta s_\alpha w(\nu) = s_\alpha s_\delta w(\nu) & & \end{array}$$

Consider the simple module $V(\nu)$ of highest weight ν. We fix a highest weight vector v_ν. For an extremal weight $\zeta = w(\nu)$, $w \in W$, denote v_ζ the extremal weight vector of weight ζ obtained from v_ν by applying a sequence of maximal divided powers $Y_{\alpha_{j_k}}^{(n_k)}$ according to a reduced decomposition $w = s_{j_1}\cdots s_{j_t}$. Note that this vector is independent of the choice of the reduced decomposition because of the Verma relations. We get by equation (6.15) in the finite dimensional representation $V(\nu)$:

$$Y_\alpha^p \theta_{s_\alpha\circ\varphi, s_\alpha\circ\eta} v_{\nu_0} = \theta_{\varphi,\eta} Y_\alpha^r v_{\nu_0}.$$

Now the triple $(w, s_\delta w, \delta)$ together with the highest weight ν and the PBW-basis $\mathbb{B}'$ as well as the triple $(s_\alpha w, s_\beta s_\alpha w, \beta)$ together with the highest weight ν and the PBW-basis $\mathbb{B}$ satisfy the conditions of Lemma 6.8. So in the expression of $\theta_{\varphi,\eta}$ with respect to $\mathbb{B}$ and the expression $\theta_{s_\alpha\circ\varphi,s_\alpha\circ\eta}$ with respect to $\mathbb{B}'$, only the terms which are a power of Y_β respectively a power of Y_δ give a possibly non-zero contribution when applied to the extremal weight vector v_{ν_1} respectively v_{ν_0}, and hence:

$$(s_\alpha \circ \eta)(h_{m\delta}) Y_\alpha^p Y_\delta^m v_{\nu_0} \quad = \quad \eta(h_{m\beta}) Y_\beta^m Y_\alpha^r v_{\nu_0}$$

Now $Y_\alpha^p Y_\delta^m v_{\nu_0} = \pm(m!)(p!) v_{\nu_2}$ and $Y_\beta^m Y_\alpha^r v_{\nu_0} = \pm(m!)(r!) v_{\nu_2}$, so

$$\eta(h_{m\beta}) = \pm(s_\alpha \circ \eta)(h_{m\delta}) p!/r!. \tag{6.16}$$

Note that $p - r = -\langle \varphi + \rho, \alpha\rangle + \langle \eta + \rho, \alpha\rangle = \langle \eta - \varphi, \alpha\rangle = m\langle \beta, \alpha\rangle$. So if $\langle \beta, \alpha\rangle > 0$, then we can write (6.16) also as

$$\begin{aligned} \eta(h_{m\beta}) &= \pm(s_\alpha \circ \eta)(h_{m\delta}) p(p-1)\cdots(r+1) \\ &= \pm(s_\alpha \circ \eta)(h_{m\delta} \textstyle\prod_{s=1}^{m\langle\beta,\alpha\rangle} (H_\alpha + 1 + s)). \end{aligned}$$

If $\langle \beta, \alpha\rangle < 0$, then we can write (6.16) also as

$$\begin{aligned} (s_\alpha \circ \eta)(h_{m\delta}) &= \pm r(r-1)\cdots(p+1)\eta(h_{m\beta}) \\ &= \pm\eta(h_{m\beta} \textstyle\prod_{s=1}^{-m\langle\beta,\alpha\rangle} (-H_\alpha - s)). \end{aligned}$$

If $\langle \beta, \alpha\rangle = 0$, then we have $\eta(h_{m\beta}) = \pm(s_\alpha \circ \eta)(h_{m\delta})$.

Each of the equalities (for either $+$ or $-$) holds for a Zariski dense subset $S' \subset \mathcal{H}_{\beta+\rho,m}$, and hence it holds for all elements η in $\mathcal{H}_{\beta+\rho,m}$. □

Proof (Proposition 6.6). Let λ, ν, μ and $\tau, \kappa, \sigma, \beta, \alpha = \alpha_{i_1}, \mathbb{B}$ etc. be as in (6.1)–(6.5). The proof of the proposition is by induction on $\ell(\sigma)$. If $\ell(\sigma) = 1$, then β is a simple root and $\theta_{\mu,\nu} = Y_\beta^m$ has obviously the desired properties.

If $\ell(\sigma) \geq 2$, then $\ell(\tau) \geq 2$. Set $\sigma' := s_\alpha\sigma$ and $\delta = s_\alpha(\beta)$, and let $\mathbb{B}'$ be the PBW-basis in (6.13). Now $\ell(\sigma') < \ell(\sigma)$, and $\lambda, s_\alpha\nu, s_\alpha\mu$ and $s_\alpha\tau, s_\alpha\kappa, s_\alpha\sigma, \delta = s_\alpha(\beta)$ etc. are again as in (6.1)–(6.3), so we can apply our induction procedure:

The coefficient $(s_\alpha \circ \nu)(h_m)$ of Y_δ^m in $\theta_{s_\alpha\circ\mu, s_\alpha\circ\nu} = \sum_{\mathfrak{m}'\in\mathbb{B}'} a_{\mathfrak{m}'}\mathfrak{m}'$ is nonzero.

Choose Shapovalov elements

$$\theta_{\delta,m} = \sum_{\mathfrak{m}'\in\mathbb{B}'} \mathfrak{m}' h_{\mathfrak{m}'} \in U(\mathfrak{n}^- \oplus \mathfrak{h}) \quad \text{and} \quad \theta_{\beta,m} = \sum_{\mathfrak{m}\in\mathbb{B}} \mathfrak{m} h_{\mathfrak{m}} \in U(\mathfrak{n}^- \oplus \mathfrak{h})$$

for $\theta_{s_\alpha\circ\mu, s_\alpha\circ\nu}$ respectively $\theta_{\mu,\nu}$ and let $h_{m\delta} \in U(\mathfrak{h})$ be the coefficient of $\mathfrak{m}' = Y_\delta^m$. Since $(s_\alpha \circ \nu)(h_{m\delta})$ is the coefficient of Y_δ^m in $\theta_{s_\alpha\circ\nu, s_\alpha\circ\mu}$, the latter is nonzero.

Let $h_{m\beta} \in U(\mathfrak{h})$ be the coefficient of $\mathfrak{m} = Y_\beta^m$. To prove the proposition, it is sufficient to prove that $\nu(h_{m\beta}) \neq 0$. If $\langle \beta, \alpha\rangle < 0$, then Lemma 6.14 immediately implies $\nu(h_{m\beta}) \neq 0$ because $(s_\alpha \circ \nu)(h_{m\delta}) \neq 0$.

If $\langle\beta,\alpha\rangle \geq 0$, then we know by Lemma 6.14:

$$\nu(h_{m\beta}) = \pm(s_\alpha \circ \nu)(h_{m\delta} \prod_{s=1}^{m\langle\beta,\alpha\rangle} (H_\alpha + 1 + s))$$

Since $(s_\alpha \circ \nu)(h_{m\delta}) \neq 0$, and

$$\begin{aligned}(s_\alpha \circ \nu)(H_\alpha + 1 + s) &= 1 + s + \langle s_\alpha(\nu+\rho) - \rho, \alpha\rangle \\ &= s + \langle s_\alpha \kappa(\lambda), \alpha\rangle > 0\end{aligned}$$

because $\langle s_\alpha\kappa(\lambda), \alpha\rangle \geq 0$. It follows: $\nu(h_{m\beta}) \neq 0$, which finishes the proof. □

7. The linear independence

To prove the linear independence of the Θ_π respectively the $\mathfrak{m}_\pi$, we use the quantum Frobenius splitting. We recall quickly the method developed in [**5**]. Let $A = (a_{i,j})_{1\leq i,j\leq n}$ be the Cartan matrix of $\mathfrak{g}$ and denote by $A^t = (\overline{a}_{i,j})$, $\overline{a}_{i,j} := a_{j,i}$, the transposed matrix. Let $\underline{d} = (d_1, \ldots, d_n)$, $d_i \in \mathbb{N}$, be minimal such that the matrix $(d_i a_{i,j})$ is symmetric. We denote by d the smallest common multiple of the d_j, and we set $\overline{d} = (\overline{d}_1, \ldots, \overline{d}_n)$, where $\overline{d}_i := d/d_i$. We assume throughout the rest of section 7 that $\ell \in \mathbb{N}$ is divisible by $2d$.

Let Φ^t be the corresponding root system, with basis $\Delta^t = \{\gamma_1, \ldots, \gamma_n\}$, and let $\mathfrak{g}^t$ be the corresponding semisimple Lie algebra. Denote $U_q(\mathfrak{g}^t)$ the quantum group associated to $\mathfrak{g}^t$ over the field $\mathbb{Q}(q)$, with generators $E_{\gamma_i}, F_{\gamma_i}, K_{\gamma_i}$ and $K_{\gamma_i}^{-1}$. We will sometimes just write $E_i, K_i, \ldots$ for the generators $E_{\gamma_i}, K_{\gamma_i}, \ldots$. In addition, we use the following abbreviations:

$$q_i := q^{\overline{d}_i} = q^{\frac{(\gamma_i,\gamma_i)^t}{2}}, \ [n]_i := \frac{q^{\overline{d}_i n} - q^{-\overline{d}_i n}}{q^{\overline{d}_i} - q^{-\overline{d}_i}}, \ \begin{bmatrix} n \\ m \end{bmatrix}_i := \frac{[n]_i!}{[m]_i![n-m]_i!},$$

and $E_i^{(k)}$, $F_i^{(k)}$ for the divided powers.

Let $U_{q,A}$ be the Lusztig form of $U_q(\mathfrak{g}^t)$ defined over the ring of Laurent polynomials $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$. We denote by R the ring $\mathcal{A}/I$, where I is the ideal generated by the 2ℓ-th cyclotomic polynomial, and set $U_\xi := U_{q,A} \otimes_A R$.

We can view Φ^t in $\mathfrak{h}^*$ by setting $\gamma_i := \alpha_i/d_i \in \mathfrak{h}^*$. For the weight lattice X^t of $\mathfrak{g}^t$ we get:

$$X^t = \{\lambda \in \mathfrak{h}^* \mid \forall i : \langle\lambda, \gamma_i\rangle \in \mathbb{Z}\} = \{\lambda \in \mathfrak{h}^* \mid \forall i : d_i\langle\lambda, \alpha_i\rangle \in \mathbb{Z}\}.$$

It follows immediately from the definition: $dX^t \subset X \subset X^t$.

Let $\lambda \in X^+$ $(\subset X^t)$ be a dominant weight and denote $V_\xi(\lambda)$ the Weyl module for U_ξ of highest weight λ. Fix ℓ such that for all LS-paths $\pi = (\underline{\tau}, \underline{a})$ of shape λ one has $\frac{\ell}{d} a_i \in \mathbb{N}$ for all i. Let m be a positive integer, the module we are interested is the U_ξ-module

$$V = \underbrace{V_\xi(\lambda) \otimes V_\xi(\lambda) \otimes \ldots \otimes V_\xi(\lambda)}_{\frac{m\ell}{d}-\text{times}}$$

and its subspace given by the weight spaces V_μ, $\mu \in X^t$, such that $\frac{d}{\ell}\mu \in X$:

$$V^{\frac{\ell}{d}} = \bigoplus_{\mu \in X^t,\ \frac{d}{\ell}\mu \in X} V_\mu.$$

Set $\ell_i := \frac{\ell d_i}{d}$, the conditions imposed on ℓ imply that we have a Frobenius homomorphism [**6**] between the classical algebra and the quantum enveloping algebra at a 2ℓ-th root of unity:

$$\mathrm{Fr}' : U(\mathfrak{n}^+) \longrightarrow U_\xi(\mathfrak{n}^{t,+}), \quad X_i^{(k)} \mapsto E_i^{(\ell_i k)}.$$

where $X_i^{(k)} = X_i^k/k!$ is the divided power of the generator. We have a similar map $\mathrm{Fr}' : U(\mathfrak{n}^-) \longrightarrow U_\xi(\mathfrak{n}^{t,-})$ for the generators $Y_i^{(k)} \mapsto F_i^{(\ell_i k)}$ of $\mathfrak{n}^-$.

The subspace $V^{\frac{\ell}{d}} \subset V$ becomes via Fr' a $U(\mathfrak{n}^+)$ and a $U(\mathfrak{n}^-)$-submodule. In [**5**] we have shown that these actions can be glued together to make $V^{\frac{\ell}{d}}$ into a $U(\mathfrak{g})$-module. Let $u_\lambda \in V_\xi(\lambda)$ be a highest weight vector (for the action of U_ξ) and let $v_{m\lambda} \in V(m\lambda)$ be a highest weight vector (for the action of $U(\mathfrak{g})$). The map $v_{m\lambda} \mapsto (u_\lambda)^{\otimes \frac{m\ell}{d}}$ extends to a $U(\mathfrak{g})$-equivariant monomorphism

$$V(m\lambda) \hookrightarrow V^{\frac{\ell}{d}}.$$

To prove that the Θ_π respectively the $\mathfrak{m}_\pi$ are linearly independent, we show that for $\pi \in C(m\lambda)$ the vectors $\Theta_\pi \cdot (u_\lambda)^{\otimes \frac{m\ell}{d}} \in V^{\frac{\ell}{d}}$ (respectively the vectors $\mathfrak{m}_\pi \cdot (u_\lambda)^{\otimes \frac{m\ell}{d}}$) are linearly independent.

To do so, we need some more notation. Let $\succ$ be the usual ordering on the weights in X^t, i.e., $\mu \succ \nu$ if the difference is a sum of positive roots. We define an induced partial order on weight vectors in $V_\xi(\lambda)$ by saying $u_\mu \succ u_\nu$ if $\mu \succ \nu$. We extend this to a partial order on weight vectors in $V^{\frac{\ell}{d}}$. Set

$$t = \frac{m\ell}{d}, \quad \underline{u}_\mu = u_{\mu_1} \otimes \ldots \otimes u_{\mu_t}, \text{ and } \underline{u}_\nu = u_{\nu_1} \otimes \ldots \otimes u_{\nu_t},$$

then we say

$$\underline{u}_\mu \succ \underline{u}_\nu \Leftrightarrow \exists\ j : \mu_j \succ \nu_j \text{ and } \mu_{j+1} = \nu_{j+1}, \ldots, \mu_t = \nu_t.$$

We say more precisely that $\underline{u}_\mu$ is greater than $\underline{u}_\nu$ *after* k:

$$\underline{u}_\mu \succ_{\mathbf{k}} \underline{u}_\nu \Leftrightarrow \exists\ j > k : \mu_j \succ \nu_j \text{ and } \mu_{j+1} = \nu_{j+1}, \ldots, \mu_t = \nu_t.$$

For all $\tau \in W/W_\lambda$ fix an extremal weight vector $u_\tau \in V_\xi(\lambda)$ of weight $\tau(\lambda)$. Let $\pi = (\underline{\tau}, \underline{a}) \in C(m\lambda)$ be an LS-concatenation of shape $m\lambda$. We denote $u_\pi \in V^{\frac{\ell}{d}}$ the vector:

$$u_\pi = \underbrace{u_{\tau_0} \otimes \ldots \otimes u_{\tau_0}}_{\frac{a_1\ell}{d} \text{ times}} \otimes \underbrace{u_{\tau_1} \otimes \ldots \otimes u_{\tau_1}}_{\frac{(a_2-a_1)\ell}{d} \text{ times}} \otimes \ldots \otimes \underbrace{u_{\tau_r} \otimes \ldots \otimes u_{\tau_r}}_{\frac{(m-a_r)\ell}{d} \text{ times}}$$

Proposition 7.1. *Let $\pi \in C(m\lambda)$ be a LS-concatenation of shape $m\lambda$. Then there exists a non-zero complex number c such that*

$$\Theta_\pi(u_\lambda)^{\otimes \frac{m\ell}{d}} = cu_\pi + \sum(\text{ terms } \underline{u} \prec\ u_\pi).$$

Since the u_π, $\pi \in C(m\lambda)$ are linearly independent, as an immediate consequence we see that the $\Theta_\pi v_{m\lambda} \in V(m\lambda) \hookrightarrow V^{\frac{\ell}{d}}$ are linearly independent. So we get by the LS-path character formula (Remark 2.6):

Corollary 7.2. *$\{\Theta_\pi v_{m\lambda} \in V(m\lambda) \mid \pi \in C(m\lambda)\}$ is a basis of $V(m\lambda)$.*

Proof (Theorem 5.1, end). Since any weight space in $U(\mathfrak{n}^-)$ maps under $U(\mathfrak{n}^-) \to V(m\lambda)$ isomorphically onto a weight space in $V(m\lambda)$ for $m \gg 0$, to prove the linear independence of the Θ_π it is sufficient to prove that the $\Theta_\pi v_{m\lambda} \in V(m\lambda)$ are linearly independent for $\pi \in C(m\lambda)$ and $m \gg 0$. But this follows from Corollary 7.2. □

Proof (Proposition 7.1). Let $\pi = (\underline{\tau}, \underline{a})$ be an LS-concatenation of shape $m\lambda$, where $\underline{\tau} = (\tau_0, \ldots, \tau_r)$, $\underline{a} = (a_1, \ldots, a_r)$. The proof is by induction on the length $\ell(\tau_0)$ of the longest element τ_0. To make the induction procedure work, we will prove the following more precise statement:

$$\Theta_\pi(u_\lambda)^{\otimes \frac{m\ell}{d}} = cu_\pi + \sum(\text{ terms } \underline{u} \text{ such that } u_\pi \succ_{\ell\frac{\mathbf{a_1}}{\mathbf{d}}} \underline{u}),$$

If $\ell(\tau_0) = 0$, then $\tau_0 = \mathrm{id}$ and hence $\pi = (\mathrm{id})$, $\Theta_\pi = 1$, and the proposition holds obviously in this case.

Let $\ell(\tau_0) > 0$. If $\ell(\tau_0) - \ell(\tau_1) = 1$, then $\tau_0 = s_\beta \tau_1$ for some positive root, and the pair $\eta = (\tau_1, \ldots, \tau_r; a_2, \ldots, a_r)$ is again a LS-concatenation of shape $m\lambda$ and $\Theta_\pi = \theta_{\widehat{\nu_0}, \widehat{\mu_0}} \Theta_\eta$, where $\mu_0 = a_1\tau_0(\lambda)$ and $\nu_0 = a_1\tau_1(\lambda)$.

Otherwise, there exists for (ν_0, μ_0), a maximal Verma flag:

$$M(\hat{\nu}_0) = M((s_{\beta_t} \cdots s_{\beta_1}) \circ \hat{\mu}_0) \hookrightarrow \ldots \hookrightarrow M(s_{\beta_1} \circ \hat{\mu}_0) \hookrightarrow M(\hat{\mu}_0).$$

Let $\kappa = s_{\beta_t}\tau_0$, then $\eta = (\kappa, \tau_1, \ldots, \tau_r; a_1, a_2, \ldots, a_r)$ is an LS-concatenation of shape $m\lambda$ and $\Theta_\pi = \theta_{\widehat{\nu_0}, \widehat{\mu}'_0}\Theta_\eta$, where $\mu'_0 = a_1\kappa(\lambda)$.

Set $\nu = \widehat{\nu_0}$ and $\mu = \widehat{\mu}_0$ in the first case and $\mu = \widehat{\mu_0}'$ in the second case. Then in both cases, the induction hypothesis applies to Θ_η, and $\Theta_\pi = \theta_{\nu,\mu}\Theta_\eta$, and $\nu = s_\beta \circ \mu$, $\langle \mu + \rho, \beta \rangle = p > 0$, and $\ell(\mu) - \ell(\nu) = 1$.

To discuss both cases uniformly, set $\eta = (\kappa_0, \ldots, \kappa_r; b_0, \ldots, b_r)$. We know by induction

$$\Theta_\eta(u_\lambda)^{\otimes \frac{m\ell}{d}} = cu_\eta + \sum(\text{ terms } \underline{u} \text{ such that } u_\eta \succ_{\ell\frac{\mathbf{b_0}}{\mathbf{d}}} \underline{u}).$$

By construction we have $\mu = \widehat{a_0\kappa_0(\lambda)}$, $\nu = \widehat{a_0\tau_0(\lambda)}$, where $a_0 \leq b_0$, $\ell(\tau_0) - \ell(\kappa_0) = 1$ and $s_\beta\tau_0 = \kappa_0$.

Since the tensor components of u_π and u_η coincide for $k > b_0\frac{m\ell}{d}$, i.e., for

$$u_\pi = u_1^\pi \otimes \cdots \otimes u_{\ell m/d}^\pi \text{ and } u_\eta = u_1^\eta \otimes \cdots \otimes u_{\ell m/d}^\eta$$

one has $u_k^\pi = u_k^\eta$ for $k > b_0\frac{m\ell}{d}$, it follows that

$$u_\eta \succ_{\mathbf{b_0}\frac{\ell}{\mathbf{d}}} \underline{u} \Rightarrow \underline{u}_\pi \succ_{\mathbf{b_0}\frac{\ell}{\mathbf{d}}} \underline{u} \Rightarrow \underline{u}_\pi \succ_{\mathbf{b_0}\frac{\ell}{\mathbf{d}}} \theta_{\nu,\mu}\underline{u}$$

To prove the proposition, it suffices hence to prove:

$$\theta_{\nu,\mu}u_\eta = cu_\pi + \sum(\text{ terms } \underline{u} \text{ such that } u_\pi \succ_{\ell\frac{\mathbf{a_0}}{\mathbf{d}}} \underline{u})$$

for some complex number $c \neq 0$. By choosing an appropriate ordering of the positive roots, we know by Proposition 6.6 that in the expression of $\theta_{\nu,\mu} =$

$\sum_{\mathfrak{m}\in\mathbb{B}} a_{\mathfrak{m}}\mathfrak{m}$ as linear combination of elements of the corresponding PBW-basis $\mathbb{B}$, the coefficient of $\mathfrak{m} = Y_\beta^p$ is a non-zero rational number. Further, by Corollary 6.9, if $\mathfrak{m} \neq Y_\beta^p$, then $\mathfrak{m} = Y_{\beta_N}^{n_N} \dots Y_{\beta_x}^{n_x}$ $(n_x > 0)$ is such that in the opposite Demazure module $V(m\lambda)^-_{\kappa_0} = U(\mathfrak{n}^-)v_{\kappa_0}$, $\kappa_0(m\lambda) - s\beta_x$ is never a weight for $s > 0$ and hence $\mathrm{Fr}'(Y_{\beta_x})$ applied to $u_{\kappa_0}^{\otimes \ell/d} \in V(m\lambda)_{\kappa_0(m\lambda)} \subset V^{\frac{\ell}{d}}$ is zero. But this implies of course also $\mathrm{Fr}'(Y_{\beta_x})u_{\kappa_0}^{a_0\otimes\ell/d} = 0$ Since $u_\eta = u_{\kappa_0}^{\otimes a_0\ell/d} \otimes \underline{u}'$, it follows that

$$\mathrm{Fr}'(Y_{\beta_x})u_\eta = \underbrace{\left(\mathrm{Fr}'(Y_{\beta_x})u_{\kappa_0}^{\otimes a_0\ell/d}\right) \otimes \underline{u}'}_{=0} + \sum(\mathfrak{m}_1 u_{\kappa_0}^{\otimes a_0\ell/d}) \otimes (\mathfrak{m}_2\underline{u}')$$

where $\mathfrak{m}_1, \mathfrak{m}_2 \in U_\xi(\mathfrak{n}^{t,-})$ are such that $\mathfrak{m}_2$ has no constant term. Obviously we have $\underline{u}' \succ \mathfrak{m}_2\underline{u}'$, so if $\mathfrak{m}$ is a monomial in the expression of $\theta_{\nu,\mu}$ different from Y_β^p, then

$$\mathfrak{m}u_\eta = \sum(\text{ terms } \underline{u} \text{ such that } u_\pi \succ_{\ell\frac{\mathbf{a_0}}{\mathbf{d}}} \underline{u}).$$

So to finish the proof of the proposition, it remains to show:

$$\mathrm{Fr}'(Y_\beta^p)u_\eta = cu_\pi + \sum(\text{ terms } \underline{u} \text{ such that } u_\pi \succ_{\ell\frac{\mathbf{a_0}}{\mathbf{d}}} \underline{u})$$

for some $c \in \mathbb{C}^*$. As above, since $u_\eta = u_{\kappa_0}^{\otimes a_0\ell/d} \otimes \underline{u}'$, it follows that

$$\mathrm{Fr}'(Y_\beta^p)u_\eta = \left(\mathrm{Fr}'(Y_{\beta_x}^p)u_{\kappa_0}^{\otimes a_0\ell/d}\right) \otimes \underline{u}' + \sum(\mathfrak{m}_1 u_{\kappa_0}^{\otimes a_0\ell/d}) \otimes (\mathfrak{m}_2\underline{u}')$$

where $\mathfrak{m}_1, \mathfrak{m}_2 \in U_\xi(\mathfrak{n}^{t,-})$ are such that $\mathfrak{m}_2$ has no constant term. As above, since $\underline{u}' \succ \mathfrak{m}_2\underline{u}'$, we get:

$$\mathrm{Fr}'(Y_\beta^p)u_\eta = \left(\mathrm{Fr}'(Y_{\beta_x}^p)u_{\kappa_0}^{\otimes a_0\ell/d}\right) \otimes \underline{u}' + \sum(\text{ terms } \underline{u} \text{ such that } u_\pi \succ_{\ell\frac{\mathbf{a_0}}{\mathbf{d}}} \underline{u}).$$

So the proof is finished once we show: $\mathrm{Fr}'(Y_{\beta_x}^p)u_{\kappa_0}^{\otimes a_0\ell/d} = cu_{\tau_0}^{\otimes a_0\ell/d}$ for some nonzero complex number. Since the weight space (for U_ξ) is one-dimensional, it suffices to show that the first is nonzero. But this follows immediately from $\mathfrak{sl}_2$-theory and the following observation. Set

$$W = \underbrace{V_\xi(\lambda) \otimes \dots \otimes V_\xi(\lambda)}_{\frac{\mathbf{a_0}\ell}{\mathbf{d}}-\text{times}},$$

and let $\bigoplus_{\epsilon\in X^t} W_\epsilon$ be the weight space decomposition. Denote $W(\beta)$ the sum of all U_ξ-weight spaces for weights integral for β:

$$W(\beta) = \bigoplus_{\substack{\epsilon\in X^t \\ \langle\epsilon,\beta\rangle\in\mathbb{Z}}} W_\epsilon.$$

Then $W(\beta)$ is actually a module for the subalgebra of $\mathfrak{g}$ generated by Y_β, X_β via the Frobenius maps Fr'. This can be proved in the same way as $V^{\frac{\ell}{d}}$ becomes a $\mathfrak{g}$-module in [**5**]. Further, $u_{\kappa_0}^{\otimes a_0\ell/d} \in W(\beta)$ is a highest weight vector for this operation, and a lowest weight vector for the simple module generated by this vector is $u_{\tau_0}^{\otimes a_0\ell/d}$. □

PROOF (Theorem 5.3, end). The proof that the

$$\{\mathfrak{m}_\pi v_{m\lambda} \mid \pi \in C(m\lambda)\} \subset V(m\lambda)\}$$

form a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}(\lambda)$ uses the same arguments as the proof of Proposition 7.1 above, so we give only a sketch of the proof. Again, we embed $V(m\lambda) \hookrightarrow V^{\frac{\ell}{d}}$ in the quantum group representation for $V = V_\xi(\lambda)^{\otimes m\ell/d}$. Let $A = \mathbb{Z}[\mathtt{x}, \mathtt{x}^{-1}]$ be the ring of Laurent polynomials and let R be ring R/I, where I is the ideal generated by the 2ℓ-th cyclotomic polynomial. Let V_R, $U_R(\mathfrak{g})$, $U_{q,R}(\mathfrak{g}^t)$ etc. be the forms obtained from the corresponding $\mathbb{Z}$-form in the classical case and the corresponding A-form in the quantum group case by base change (as in [**5**]). The corresponding Frobenius map and the embedding of representations above is then already defined over R. By applying the same kind of inductive arguments as above, by fixing the u_π appropriately and by applying $\mathfrak{sl}_2$-theory a little more rigorously, one gets

$$\mathfrak{m}_\pi(u_\lambda)^{\otimes \frac{m\ell}{d}} = \xi^t u_\pi + \sum(\text{ terms } \underline{u} \text{ such that } u_\pi \succ_{\ell \frac{\mathbf{a_1}}{\mathbf{d}}} \underline{u}),$$

for some power of the fixed primitive 2ℓ-th root of unity ξ. By the linear independence of the u_π, this proves the linear independence of the $\mathfrak{m}_\pi(u_\lambda)^{\otimes \frac{m\ell}{d}}$, and by the character formula this proves that they span a R-lattice of maximal rank in $V_R(\lambda)$. Now the u_π can be completed to an R-basis of V_R. Since the u_π are "leading terms" for the $\mathfrak{m}_\pi(u_\lambda)^{\otimes \frac{m\ell}{d}}$, it follows that they form actually a basis for $V_R(\lambda)$. Since they are by definition actually elements in $V_{\mathbb{Z}}(\lambda)$, they form hence a basis of the $\mathbb{Z}$-lattice $V_{\mathbb{Z}}(\lambda)$. □

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