HALL ALGEBRAS AND QUANTUM GROUPS

READING SEMINAR WS 15/16

The Hall algebra of a finitary category encodes its extension structure. The story starts from the work of Steinitz on the module category of an abelian p-group, where the Hall algebra is the algebra of symmetric functions. The theory of Hall algebras is highlighted by Ringel around the 90's in his seminal work realizing a half of a quantum group via the Hall algebra of quiver representations. Further developments include Lusztig's canonical bases, cluster categories (Caldero-Keller), higher genus quantum algebras (Burban-Schiffmann, Schiffmann-Vasserot) – just to name a few. One of the goals of the seminar is to introduce Bridgeland's construction of the quantized enveloping algebras (quantum groups) associated to a symmetric Kac-Moody Lie algebra via Hall algebras of $\mathbb{Z}/2$-graded complexes of quiver representations and several recent progresses around it.

In the last part of this seminar, we try to open the window to some applications of Hall algebras to mathematical physics via Hall algebra of curves: for example, they play an important rôle in the proof of AGT conjecture concerning pure $N = 2$ gauge theory for the group $\mathrm{SU}(r)$ ([SV]).

Organization of talks:

(1) (*Xin*) Hopf algebras and quantum groups, panorama of the subject ([Kas], [KRT])
(2) (*Valentin*) Hall algebra of a finitary category: definition of the product, sketch of proof of the associativity, definition of the coproduct, statement of Green's theorem on bialgebra structure, example if time permits ([Hubery], Chapter 2.1-2.4)
(3) (*Lara*) Quiver and their representations: quivers and their path algebras, statement of Krull-Remak-Schmidt theorem, statement of Gabriel theorem, examples, if time permits, reflection functors ([Krause], Section 1-5, [Sch], Section 3.1-3.2)
(4) (*Bea*) Auslander-Reiten theory
(5) (*Robin*) Ringel's construction of the negative part a quantum group ([Ringel], [Sch], Section 3.3)
(6) (*Andreas*) A brief introduction to derived categories, example of tilting
(7) (*Lennart or Sebastian*) Hall algebra of category of complexes ([Bri], Section 3)
(8) (*Christian*) Bialgebra structure on the Hall algebra of category of complexes ([Yana1], [Yana2])
(9) (*Lennart or Sebastian*) Bridgeland's realization of the quantum group ([Bri], Section 4)
(10) Coherent sheaves on curves: category $\mathbf{Coh}(X)$ for X a curve, example: $\mathbf{Coh}(\mathbb{P}^1)$, Harder-Narasimhan filtration ([Burban], [Sch], Section 4.1,4.2,4.7)
(11) Hall algebra of the Kronecker quiver and quantum affine $\mathfrak{sl}_2$ ([BK], [BS1])

(12) (*Igor*) Weighted projective lines and Hall algebras: definition of weighted projective lines, structure of the category of coherent sheaves, identification of the Hall algebra ([BS2], [Sch], Section 4.4-4.6)

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[Ringel] C. M. Ringel, *Hall algebras and quantum groups.* Invent. Math. 101 (1990), no. 3, 583-591.

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[SV] O. Schiffmann, E. Vasserot, *Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on $\mathbb{A}^2$*, Publications mathmatiques de l'IHÉS, November 2013, Volume 118, Issue 1, pp 213-342.

[Yana1] S. Yanagida, *Bialgebra structure on Bridgeland's Hall algebra of two-periodic complexes*, arXiv:1304.6970.

[Yana2] S. Yanagida, *A note on Bridgeland's Hall algebra of two-periodic complexes*, arXiv:1207.0905.