Relaxed Highest Weight Modules from $\mathcal{D}$-Modules on the Kashiwara Flag Scheme

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November 29, 2016
Relaxed highest weight modules for $\hat{\mathfrak{sl}}_2$ after Feigin, Semikhatov, Sirota, Tipunin

Introduction to localization on the affine flag variety

Setup

Overview of results
Relaxed highest weight modules for $\hat{\mathfrak{sl}}_2$ after Feigin, Semikhatov, Sirota, Tipunin
We start with the Lie algebra \( \mathfrak{sl}_2 = \mathbb{C} e \oplus \mathbb{C} h \oplus \mathbb{C} f \) and define \( \hat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \oplus \mathbb{C} K \), where \( K \) is central and

\[
[X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n} + m\delta_{m+n,0} \text{Tr}(XY)K.
\]

This endows \( \hat{\mathfrak{sl}}_2 \) with the structure of a Lie algebra.
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$$[X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n} + m\delta_{m+n,0} \text{Tr}(XY)K.$$ 

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Up to the derivation element this defines the **affine Kac-Moody algebra** with Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. 

**Definition (Semikhatov-Sirota ’97)**

Let $\mu_1, \mu_2 \in \mathbb{C}$ and $t \in \mathbb{C} \setminus \{0\}$. The relaxed Verma module $R_{\mu_1, \mu_2, t}$ is the $\hat{\mathfrak{sl}}_2$-module generated from a vector $v$ that satisfies the annihilation conditions

$$(e \otimes z^n)v = (h \otimes z^n)v = (f \otimes z^n)v = 0 \quad n \geq 1$$

and the relations

$$(f \otimes 1)(e \otimes 1)v = -\mu_1\mu_2 v \quad (h \otimes 1)v = -(1 + \mu_1 + \mu_2)v$$

$Kv = (t - 2)v$

by a free action of $e \otimes z^n$, $f \otimes z^n$, $h \otimes z^n$, $n \leq -1$. 
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To get a first impression about the structure of $R_{\mu_1, \mu_2, t}$ we can look at the $\mathfrak{sl}_2 \otimes \mathbb{C}1$-submodule generated by $v$. It is a weight module with weights $-(1 + \mu_1 + \mu_2) + 2 \mathbb{Z}$, each of which has multiplicity one. We have $(f \otimes 1)(e \otimes 1)^{\mu_j+1} v = 0$ and $(e \otimes 1)(f \otimes 1)^{-\mu_j} v = 0$ if these expressions are actually defined and $\mu_j \neq 0$ in the second case.
\[
\mu_1 \notin \mathbb{Z}, \mu_2 \notin \mathbb{Z}
\]

\[-(\mu_1 + \mu_2 + 1)\]

\[
\mu_1 \in \mathbb{Z}_{\geq 0}, \mu_2 \notin \mathbb{Z}
\]

\[
\mu_1 - \mu_2 + 1
\]

\[
\mu_1 \in \mathbb{Z}_{< 0}, \mu_2 \notin \mathbb{Z}
\]

\[
\mu_1 - \mu_2 - 1
\]
\( \mu_1 \notin \mathbb{Z}, \mu_2 \notin \mathbb{Z} \) case (0)

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\( \mu_1 \in \mathbb{Z}_{\geq 0}, \mu_2 \notin \mathbb{Z} \) case (1, +)

\( \mu_1 - \mu_2 + 1 \)

\( \mu_1 \in \mathbb{Z}_{< 0}, \mu_2 \notin \mathbb{Z} \) case (1, −)

\( \mu_1 - \mu_2 - 1 \)
$\mu_1 \in \mathbb{Z}_{<0}, \mu_2 \in \mathbb{Z}_{\geq 0}$

$\mu_1 - \mu_2 - 1 \quad \quad \quad \quad \quad \quad \quad \mu_2 - \mu_1 + 1$

$\mu_1 \in \mathbb{Z}_{\geq 0}, \mu_2 \in \mathbb{Z}_{\geq 0}, \mu_1 \geq \mu_2$

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\( \mu_1 \in \mathbb{Z}_{<0}, \mu_2 \in \mathbb{Z}_{\geq 0} \) case (2, \(-+\))

\[
\begin{align*}
\mu_1 - \mu_2 - 1 & \quad \mu_2 - \mu_1 + 1 \\
& \quad \vdots \quad \vdots
\end{align*}
\]

\( \mu_1 \in \mathbb{Z}_{\geq 0}, \mu_2 \in \mathbb{Z}_{\geq 0}, \mu_1 \geq \mu_2 \) case (2, \(++\))

\[
\begin{align*}
\mu_2 - \mu_1 + 1 & \quad \mu_1 - \mu_2 + 1 \\
& \quad \vdots \quad \vdots
\end{align*}
\]

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\end{align*}
\]
Coming back to $R_{\mu_1,\mu_2,t}$, we note that $(e \otimes 1)(f \otimes 1)^{-\mu_1} v = 0$ implies that the submodule generated by $(f \otimes 1)^{-\mu_1} v$ is isomorphic to a Verma module of highest weight $\lambda = \mu_1 - \mu_2 - 1$. We will denote it by $M_{\lambda,t}$. 
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Let’s formulate a similar statement for $(e \otimes 1)^{\mu_1+1} v$. 
Consider the automorphism of $\hat{\mathfrak{sl}_2}$ sending $K \mapsto K$ and

$$e \otimes z^n \mapsto e \otimes z^{n+\theta} \quad f \otimes z^n \mapsto f \otimes z^{n-\theta} \quad h \otimes z^n \mapsto h \otimes z^n + \theta \delta_{n,0} K.$$
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\end{align*}
\]

The vector $w = (e \otimes 1)^{\mu_1 + 1} v$ satisfies

\[
\begin{align*}
(e \otimes z^{\geq 1})w &= (h \otimes z^{\geq 1})w = (f \otimes z^{\geq 0})w = 0 \\
(h \otimes 1 + (t - 2))w &= (t + \mu_1 - \mu_2 - 1)w .
\end{align*}
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Consider the automorphism of $\hat{\mathfrak{sl}}_2$ sending $K \mapsto K$ and

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(h \otimes 1 + (t - 2))w = (t + \mu_1 - \mu_2 - 1)w.
\]

Thus $w$ generates a submodule of $R_{\mu_1,\mu_2,t}$ which is isomorphic to a Verma module twisted by the automorphism for $\theta = 1$. We will denote it by $M_{t+\mu_1-\mu_2-1,t}^{(1)}$. 
So we have the following embeddings

(1, +) \quad R_{\mu_1, \mu_2, t} \leftarrow M_{t+\mu_1-\mu_2-1, t}^{(1)}

(1, -) \quad M_{\mu_1-\mu_2-1, t} \hookrightarrow R_{\mu_1, \mu_2, t}

(2, -+) \quad M_{\mu_1-\mu_2-1, t} \hookrightarrow R_{\mu_1, \mu_2, t} \leftarrow M_{t+\mu_2-\mu_1-1, t}^{(1)}

(2, ++) \quad R_{\mu_1, \mu_2, t} \leftarrow M_{t+\mu_2-\mu_1-1, t}^{(1)} \leftarrow M_{t+\mu_1-\mu_2-1, t}^{(1)}

(2, --) \quad M_{\mu_1-\mu_2-1, t} \hookrightarrow M_{\mu_2-\mu_1-1, t} \hookrightarrow R_{\mu_1, \mu_2, t}
So we have the following embeddings

\[(1, +) \quad R_{\mu_1, \mu_2, t} \hookrightarrow M^{(1)}_{t+\mu_1-\mu_2-1, t}\]

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\[(2, --) \quad M_{\mu_1-\mu_2-1, t} \hookrightarrow M_{\mu_2-\mu_1-1, t} \hookrightarrow R_{\mu_1, \mu_2, t}\]

The **goal** of Semikhatov-Sirota ’97 is to describe which modules $M_{\lambda, t}$, $M^{(1)}_{\lambda, t}$ or $R_{\mu_1', \mu_2', t}$ embed into $R_{\mu_1, \mu_2, t}$. 
From the above we conclude that $\mu_1 \notin \mathbb{Z}$ and $\mu_2 \notin \mathbb{Z}$ is a necessary condition for $R_{\mu_1,\mu_2,t}$ to be simple.
Simplicity of $R_{\mu_1,\mu_2,t}$

From the above we conclude that $\mu_1 \notin \mathbb{Z}$ and $\mu_2 \notin \mathbb{Z}$ is a necessary condition for $R_{\mu_1,\mu_2,t}$ to be simple.

**Theorem (Semikhatov-Sirota ’97)**

$R_{\mu_1,\mu_2,t}$ simple $\iff$ $\mu_1 \notin \mathbb{Z}$ and $\mu_2 \notin \mathbb{Z}$

and $\nexists r, s \in \mathbb{Z}_{>0} \mu_1 - \mu_2 = r - st$ or $\mu_2 - \mu_1 = r - st$
The description of the so-called embedding diagrams for $R_{\mu_1, \mu_2, t}$ is the main result of Semikhatov-Sirota '97. These diagrams are labeled by combining I, II, III_± determined by the row 
(0), (1, +), (1, −), (2, −−), . . . determined by the column.

<table>
<thead>
<tr>
<th>Condition</th>
<th>$\mu_1, \mu_2 \notin \mathbb{Z}$</th>
<th>$\mu_1 \in \mathbb{Z}, \mu_2 \notin \mathbb{Z}$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1 - \mu_2 \notin \mathbb{K}(t),$</td>
<td>I(0), Eq. (3.2)</td>
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<td>$\mu_1 \cdot \mu_2 &gt; 0$</td>
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<tr>
<td>$t \notin \mathbb{Q}$</td>
<td>II(0), Eq. (3.4)</td>
<td>II(1), Eq. (3.4)</td>
<td>$\mu_1 \cdot \mu_2 &lt; 0$</td>
</tr>
<tr>
<td>$\mu_1 - \mu_2 \in \mathbb{K}(t),$</td>
<td>III±(0), Eq. (3.7)</td>
<td>III±(1), Eq. (3.9) and (3.10)</td>
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<tr>
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<tr>
<td>$\mu_1 - \mu_2 \notin \mathbb{Z},$</td>
<td>III±(0), Eq. (3.11)</td>
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<tr>
<td>$(\mu_1 - \mu_2)/t \notin \mathbb{Z}$</td>
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<td>III±(0)</td>
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Example: case $\Pi^0_+(2, -+)$
Introduction to localization on the affine flag variety
Let \( \mathfrak{g} \) be a finite dimensional semisimple Lie algebra over \( \mathbb{C} \). The celebrated theorem of Beilinson-Bernstein ’81 states that the functor of global sections is an exact equivalence of categories between the \( \mathcal{D} \)-modules on the flag variety twisted by the line bundle associated to a regular dominant weight and the \( \mathfrak{g} \)-modules of the corresponding central character.
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At present the full analogue of this statement in the case of affine Kac-Moody algebras is not known. Postponing definitions, let us start by pointing out related theorems in the case of affine Kac-Moody algebras $\mathfrak{g}$. 
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In particular, we need to associate a “flag variety” to $\mathfrak{g}$. 
A first possibility, in case $\mathfrak{g}$ is untwisted, is to consider the thin flag variety defined as a quotient of the loop group by the Iwahori group scheme $X^{\text{thin}} = L\hat{G}/L^{+}I$ (Beilinson-Drinfeld, Pappas-Rapoport ’08 and others). Here $\hat{G}$ is a semisimple algebraic group.
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Beilinson-Drinfeld define a category of twisted right $\mathcal{D}$-modules on $X^{\text{thin}}$ and a functor of global sections $\Gamma(X^{\text{thin}}, \cdot)$ landing in $\mathfrak{g}\text{ mod}$. Theorem (Beilinson-Drinfeld, P. Shan ’11)

Let $\lambda + \rho$ be regular antidominant. The functor $\Gamma(X^{\text{thin}}, \cdot)$ between the $\lambda$-twisted right $\mathcal{D}$-modules on $X^{\text{thin}}$ and $\mathfrak{g}\text{ mod}$ is exact and faithful.

The basic open question is to describe the essential image of this functor in $\mathfrak{g}\text{ mod}$ (conjectural description by Beilinson ’02, I. Shapiro ’09).
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A second possibility is to consider the Kashiwara flag scheme $X$. As we will detail later, it is a scheme, not locally of finite type, but having an open cover by affine spaces of countable dimension. The finite dimensional Schubert cells $X_w$ can be defined as subschemes of $X$ and one again has a notion of twisted $\mathcal{D}$-modules on $X$ and a functor of global sections.
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Recall the notion of the Verma module

$$M(\mu) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\mu$$

of highest weight $\mu \in \mathfrak{h}^*$. 

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**Theorem (Kashiwara-Tanisaki ’95)**

*The global sections of the $\ast$- and $!$-direct image from $X_w$ identify with $M(w \cdot \lambda)^\vee$ and $M(w \cdot \lambda)$ respectively when $\lambda + \rho$ is regular antidominant.*
The last two theorems can be combined into

**Theorem (Frenkel-Gaitsgory ’09)**

Let $\lambda + \rho$ be regular antidominant. $\Gamma(X^{\text{thin}}, \cdot)$ defines an exact equivalence between the category of $\lambda$-twisted right $\mathcal{D}$-modules on $X^{\text{thin}}$ that are equivariant for the pro-unipotent radical of $L^+ I$ and the block of category $\mathcal{O}$ of $\mathfrak{g}$ defined by $\lambda$. 
Setup
Affine Kac-Moody algebras

- $(\mathfrak{h}, (\alpha_i)_{i \in I} \subseteq \mathfrak{h}^*, (h_i)_{i \in I} \subseteq \mathfrak{h})$ affine Kac-Moody data
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- \(\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}\) triangular decomposition into positive and negative part \(\mathfrak{n}\) and \(\mathfrak{n}^-\) and the Cartan subalgebra \(\mathfrak{h}\)
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- \(\mathfrak{h}^{(-)} = \mathfrak{n}^{(-)} \oplus \mathfrak{h}\) Borel and opposite Borel subalgebra
- \(e_i \in \mathfrak{n}, f_i \in \mathfrak{n}^-\) simple generators, \(i \in I\)
- \(\Phi = \Phi^0 \sqcup \Phi^0\) positive and negative roots of \(\mathfrak{g}\)
• $n_l^-$ Lie ideal of $n^-$ given by the root spaces associated to the negative roots of height $\geq l$, $l \in \mathbb{Z}_{\geq 0}$, similarly $n_i$.
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• $p_i^- = \mathbb{C} e_i \oplus b^-$ and $n_i \oplus p_i^- = g$, $i \in I$
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• $p_i = \mathbb{C}f_i \oplus b$ and $n^-_i \oplus p_i = g$, $i \in I$

• $W$ Weyl group of $g$. It acts linearly on $\mathfrak{h}^*$. For $i \in I$ there is a simple reflection $s_i \in W$. 
• $n_l^-$ Lie ideal of $n^-$ given by the root spaces associated to the negative roots of height $\geq l$, $l \in \mathbb{Z}_{\geq 0}$, similarly $n_l$

• $g_i = \mathbb{C} f_i \oplus \mathfrak{h} \oplus \mathbb{C} e_i$, $i \in I$

• $p_i^- = \mathbb{C} e_i \oplus b^-$ and $n_i \oplus p_i^- = g$, $i \in I$

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- $w \cdot \lambda = w(\lambda + \rho) - \rho$ dot-action of $W$ on $\mathfrak{h}^*$

- lattice $P \subseteq \mathfrak{h}^*$ such that $\alpha_i \in P$ and $P(h_i) \subseteq \mathbb{Z}$ for all $i \in I$
Group schemes

- Pro-unipotent group scheme $U = \varprojlim_i \exp(n/n_i)$
• Pro-unipotent group scheme $U = \varprojlim \mathbb{exp}(n/n_l)$
• Pro-unipotent group scheme $U(\Psi) \subseteq U$, where $\Psi \subseteq \Phi^{>0}$ satisfies $(\Psi + \Psi) \cap \Phi^{>0} \subseteq \Psi$, similarly $U^-(\Psi) \subseteq U^-$. 
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• Pro-unipotent group scheme $U_i^- = U^-(\Phi_i^<0)$, where $\Phi_i^<0 \subseteq \Phi^<0$ is the subset of negative roots of height $\geq i$. 

T = Spec $\mathbb{C}[P]$ ~ = $\text{G_{mdim}}$, algebraic torus

$B(\cdot)$ = $T \bowtie U(\cdot)$, Borel and opposite Borel group scheme

$G_i$ = reductive group determined by $g_i$ and $P_i$ = $G_i \bowtie U(\Phi^>0 \{-\alpha_i\})$ and similarly $P_i^-$.
Group schemes

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- Pro-unipotent group scheme $U(\Psi) \subseteq U$, where $\Psi \subseteq \Phi^{>0}$ satisfies $(\Psi + \Psi) \cap \Phi^{>0} \subseteq \Psi$, similarly $U^-(\Psi) \subseteq U^-$.  
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- $T = \text{Spec } \mathbb{C}[P] \cong \mathbb{G}_m^{\dim \mathfrak{h}}$ algebraic torus
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Out of the affine Kac-Moody data and the lattice $P$ Kashiwara ’90 constructs a scheme $G$ with a distinguished point $1 \in G$ and commuting left and right actions of $P_i^-$ and $P_i$ respectively, for all $i \in I$. 
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The Kashiwara flag scheme is defined as $X = G/B$ (quotient by a locally free action). The so-called Tits extension $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ acts on $G$ and $X$. On $T$-invariant subsets of $X$, the action of $\widetilde{\mathcal{W}}$ factors through $\mathcal{W}$. 
Basic properties

- \( U^\sim 1B \simeq \mathbb{A}^\infty \) is an open subscheme of \( X \) called big cell.
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- $X = \bigcup_{w \in W} X^{\leq w}$, where $X^{\leq w} = \bigcup_{v \leq w} N(X_v)$ is $B^-$-invariant and quasi-compact and $\leq$ is the Bruhat partial order
- There is a line bundle $\mathcal{O}_X(\lambda)$ on $X$ associated to $\lambda \in P$. 
For fixed $w$ and all $l \in \mathbb{Z}_{>0}$ large enough $U_l^-$ acts locally freely on $X_{\leq w}$. The quotient $X_{l}^{\leq w} = U_l^- \backslash X_{\leq w}$ is a smooth quasi-projective variety (Shan-Varagnolo-Vasserot ’14).
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We have for $l_1 \geq l_2$ large enough a commutative diagram

\[
\begin{array}{ccc}
X_{l_1}^{\leq w} & \xrightarrow{p_{l_2}^{l_1}} & X_{l_2}^{\leq w} \\
\downarrow & & \downarrow \\
X_w & \rightarrow & X_{l_2}^{\leq w} \\
\end{array}
\]

The fibers of the projection $p_{l_2}^{l_1}$ are affine spaces and $\hookrightarrow$ are closed embeddings.
For fixed $w$ and all $l \in \mathbb{Z}_{>0}$ large enough $U_l^-$ acts locally freely on $X^{\leq w}$. The quotient $X_i^{\leq w} = U_l^- \backslash X^{\leq w}$ is a smooth quasi-projective variety (Shan-Varagnolo-Vasserot ’14).

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The fibers of the projection $p_{l_2}^{l_1}$ are affine spaces and $\hookrightarrow$ are closed embeddings. Subsequently we will always assume that $l$ is large enough.
Twisted $\mathcal{D}$-modules on $X$

$\text{Hol}(\mathcal{D}_{X_{\lambda}^{\leq w}}, \overline{X_w})$ Category of holonomic right $\mathcal{D}$-modules on $X_{\lambda}^{\leq w}$ twisted by $\mathcal{O}_{X_{\lambda}^{\leq w}}$, $\lambda \in P$, with support in $\overline{X_w}$
Twisted $\mathcal{D}$-modules on $X$

$\text{Hol}(\mathcal{D}_{X_{\leq w}(\lambda), X_w})$ Category of holonomic right $\mathcal{D}$-modules on $X_{\leq w}$ twisted by $\mathcal{O}_{X_{\leq w}(\lambda)}$, $\lambda \in P$, with support in $X_w$

- It is an abelian category and every object has finite length.
Twisted $\mathcal{D}$-modules on $X$

$\text{Hol}(\mathcal{D}_{X^{\leq w}_{\lambda}}(\lambda), \overline{X_w})$ Category of holonomic right $\mathcal{D}$-modules on $X^{\leq w}_{\lambda}$ twisted by $\mathcal{O}_{X^{\leq w}_{\lambda}}(\lambda)$, $\lambda \in P$, with support in $\overline{X_w}$

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Twisted $\mathcal{D}$-modules on $X$

$\text{Hol}(\mathcal{D}_{X^w}^\leq (\lambda), \overline{X_w})$ Category of holonomic right $\mathcal{D}$-modules on $X^w$ twisted by $\mathcal{O}_{X^w}^\leq (\lambda)$, $\lambda \in P$, with support in $\overline{X_w}$

- It is an abelian category and every object has finite length.
- It has a contravariant exact auto-equivalence $\mathbb{D}$, the holonomic duality.
- $p_{l_2}^{l_1} : \text{Hol}(\mathcal{D}_{X^w}^\leq (\lambda), \overline{X_w}) \rightarrow \text{Hol}(\mathcal{D}_{X^w}^\leq (\lambda), \overline{X_w})$ exact equivalence
\[ \text{Hol}(\lambda, \overline{X_w}, X^{\leq w}, l_0) \ni M = \left( (M_l)_{l\geq l_0}, (\gamma_{l_2}^l)_{l_1 \geq l_2 \geq l_0} \right) \]

\[ \text{Hol}(\mathcal{D}_{X_l^{\leq w}}(\lambda), \overline{X_w}) \ni M_l \]

\[ \gamma_{l_2}^{l_1} : p_{l_2*} M_{l_1} \xrightarrow{\sim} M_{l_2} \quad \gamma_{l_3}^{l_1} = \gamma_{l_3}^{l_2} \circ p_{l_3*} \gamma_{l_2}^{l_1}, \quad l_1 \geq l_2 \geq l_3 \geq l_0 \]
\[
\text{Hol}(\lambda, \overline{X_w}, X^{\leq w}, l_0) \ni \mathcal{M} = \left( (\mathcal{M}_l)_{l \geq l_0}, (\gamma_{l_2}^{l_1})_{l_1 \geq l_2 \geq l_0} \right)
\]

\[
\text{Hol}(\mathcal{D}_{X_1^{\leq w}}(\lambda), \overline{X_w}) \ni \mathcal{M}_l
\]

\[
\gamma_{l_2}^{l_1} : p_{l_2}^{l_1}_* \mathcal{M}_{l_1} \xrightarrow{\cong} \mathcal{M}_{l_2} \quad \gamma_{l_3}^{l_1} = \gamma_{l_3}^{l_2} \circ p_{l_3}^{l_2}_* \gamma_{l_2}^{l_1}, \quad l_1 \geq l_2 \geq l_3 \geq l_0
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For any \( l \geq l_0 \), \( \mathcal{M} \mapsto \mathcal{M}_l \) is an exact equivalence.
\[ \text{Hol}(\lambda, \overline{X_w}, \underline{X}^{\leq w}, l_0) \ni \mathcal{M} = \left( (\mathcal{M}_l)_{l \geq l_0}, (\gamma_{l_2}^l)_{l_1 \geq l_2 \geq l_0} \right) \]

\[ \text{Hol}(\mathcal{D}_{X_{\leq w}}(\lambda), \overline{X_w}) \ni \mathcal{M}_l \]

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For any \( l \geq l_0 \), \( \mathcal{M} \mapsto \mathcal{M}_l \) is an exact equivalence.

Taking limits we get rid of the auxiliary choices \( \overline{X_w}, \underline{X}^{\leq w} \) and \( l_0 \) and define the category \( \text{Hol}(\lambda) \) of \( \lambda \)-twisted holonomic right \( \mathcal{D} \)-modules on \( X \).
For $\mathcal{M} \in \text{Hol}(\lambda)$ and $j \in \mathbb{Z}_{\geq 0}$ define

$$H^j(X, \mathcal{M}) = \lim_{\leftarrow l} H^j(X_{\leq w}^l, \mathcal{M}_l),$$

where $H^j(X_{\leq w}^l, \mathcal{M}_l)$ are the sheaf cohomology groups.
For $\mathcal{M} \in \text{Hol}(\lambda)$ and $j \in \mathbb{Z}_{\geq 0}$ define

$$H^j(X, \mathcal{M}) = \lim_{\leftarrow l} H^j(X_{i}^{\leq w}, \mathcal{M}_{l}),$$

where $H^j(X_{i}^{\leq w}, \mathcal{M}_{l})$ are the sheaf cohomology groups. If $\nu \in \mathfrak{g}$ there is a $m \in \mathbb{Z}_{\geq 0}$ such that $[\nu, \mathfrak{n}_{i+m}] \subseteq \mathfrak{n}_{i}$ for all $l$. Then $\nu$ defines a $\mathbb{C}$-linear map $H^j(X_{i+m}^{\leq w}, \mathcal{M}_{l+m}) \rightarrow H^j(X_{i}^{\leq w}, \mathcal{M}_{l})$. 


For $\mathcal{M} \in \text{Hol}(\lambda)$ and $j \in \mathbb{Z}_{\geq 0}$ define

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where $H^i(X_i^{\leq w}, \mathcal{M}_l)$ are the sheaf cohomology groups. If $\nu \in \mathfrak{g}$ there is a $m \in \mathbb{Z}_{\geq 0}$ such that $[\nu, \mathfrak{n}^-] \subseteq \mathfrak{n}^-$ for all $l$. Then $\nu$ defines a $\mathbb{C}$-linear map $H^i(X_{l+m}^{\leq w}, \mathcal{M}_{l+m}) \to H^i(X_{l}^{\leq w}, \mathcal{M}_{l})$. In this way $H^i(X, \mathcal{M})$ becomes a $\mathfrak{g}$-module.
Cohomology groups

For $\mathcal{M} \in \text{Hol}(\lambda)$ and $j \in \mathbb{Z}_{\geq 0}$ define

$$H^j(X, \mathcal{M}) = \lim_{\leftarrow l} H^j(X^{\leq w}_l, \mathcal{M}_l),$$

where $H^j(X^{\leq w}_l, \mathcal{M}_l)$ are the sheaf cohomology groups. If $\nu \in \mathfrak{g}$ there is a $m \in \mathbb{Z}_{\geq 0}$ such that $[\nu, \mathfrak{n}_{l+m}^-] \subseteq \mathfrak{n}_l^-$ for all $l$. Then $\nu$ defines a $\mathbb{C}$-linear map $H^j(X^{\leq w}_{l+m}, \mathcal{M}_{l+m}) \to H^j(X^{\leq w}_l, \mathcal{M}_l)$. In this way $H^j(X, \mathcal{M})$ becomes a $\mathfrak{g}$-module.

Define

$$\overline{H}^j(X, \mathcal{M}) = \bigoplus_{\mu \in \mathfrak{h}^*} H^j(X, \mathcal{M})_{\mu},$$

where $(\cdot)_{\mu}$ denotes the generalized weight space associated to $\mu$. This is a $\mathfrak{g}$-submodule of $H^j(X, \mathcal{M})$. 

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Coordinates on $N(X_w)$

For $w \in W$ abbreviate

$$U_w^- = U^- (\Phi^< 0 \cap w\Phi^< 0) \subseteq U^-$$
$$U_w = U (\Phi^> 0 \cap w\Phi^< 0) \subseteq U.$$

The map $(u_1, u_2) \mapsto u_1 u_2 w 1 B$ defines a $T$-equivariant isomorphism of schemes

$$U_w^- \times U_w \xrightarrow{\cong} N(X_w).$$

The image of $1 \times U_w$ is the (finite dimensional) Schubert cell $X_w$ in $X$.  

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Let \( w \) be such that \( s_iw < w \).

**Lemma**

We have \( X_w \cap s_iX_w = s_iX_w \setminus X_{s_iw} = X_w \setminus s_iX_{s_iw} \). In the above coordinates on \( N(X_w) \) and \( s_iN(X_w) = N(X_{s_iw}) \) the identity map \( s_iX_w \setminus X_{s_iw} \to X_w \setminus s_iX_{s_iw} \) is the isomorphism

\[
(U^\alpha_i \setminus 1) \times U_{s_iw} \to (U(\alpha_i) \setminus 1) \times U_{s_iw}
\]

\[
(e^{zf_i}, h_i(z)^{-1}uh_i(z)) \mapsto (e^{z^{-1}e_i}, \dot{s}_i^{-1}\tilde{u}(z)\dot{s}_i).
\]

Here \( z \in \mathbb{G}_m \) and \( \dot{s}_i = e^{e_i}e^{-f_i}e^{e_i} \). \( h_i \) is considered as a group homomorphism \( \mathbb{G}_m \to T \). Given \( u \) and \( z \), \( \tilde{u}(z) \in U_{s_iw} \) is uniquely determined by the condition \( e^{ze_i}u \in \tilde{u}(z)U(\Phi^0 \cap s_iw\Phi^0) \).
For $s_iw < w$ consider the locally closed affine embedding

$$i_{w,l} : X_w \cap s_iX_w \hookrightarrow X_{\leq w}.$$
For $s_i w < w$ consider the locally closed affine embedding

$$i_{w,l} : X_w \cap s_i X_w \hookrightarrow X_{\leq w}.$$ 

**Definition**

Define the right $D_{X_{\leq w}}(\lambda)$-module for $\lambda \in P$ and $\alpha \in \mathbb{C}$

$$\mathcal{R}_{w}(\lambda, \alpha)_l = i_{w,l} \left( \left( \Omega_{\mathbb{C}^X}^{(\alpha)} \boxtimes \Omega_{s_iU_{s_iw}} \right) \otimes i_{w,l}^* \mathcal{O}_{X_{\leq w}}(\lambda) \right) \quad ? \in \{*, !\}.$$
For $s_i w < w$ consider the locally closed affine embedding

$$i_{w,l} : X_w \cap s_i X_w \hookrightarrow X^{\leq w}_l.$$

**Definition**

*Define the right $\mathcal{D}_{X^{\leq w}_l}(\lambda)$-module for $\lambda \in P$ and $\alpha \in \mathbb{C}$*

$$\mathcal{R}_{w}(\lambda, \alpha)_l = i_{w,l？} \left( \left( \Omega^{(\alpha)}_{\mathbb{C}^x} \boxtimes \Omega^{s_i U_{s_i w}} ight) \otimes i^*_w, l \mathcal{O}_{X^{\leq w}_l(\lambda)} \right) \quad ? \in \{*, !\}.$$  

Here we introduced the right $\mathcal{D}_{\mathbb{C}^x}$-module

$$\Omega^{(\alpha)}_{\mathbb{C}^x} = \mathcal{D}_{\mathbb{C}^x}/(x \partial_x - \alpha)\mathcal{D}_{\mathbb{C}^x}.$$  

The coordinate $x$ on $\mathbb{C}^x$ is the one of $U(\alpha_i)$. Thus $x = \infty$ corresponds to $X_{s_i w}$. 

29
For $s_i w < w$ consider the locally closed affine embedding

$$i_{w, l} : X_w \cap s_i X_w \hookrightarrow X_{l \leq w}.$$ 

**Definition**

Define the right $\mathcal{D}_{X_{\leq w}^i}(\lambda)$-module for $\lambda \in P$ and $\alpha \in \mathbb{C}$

$$\mathcal{R}_{w}(\lambda, \alpha)_l = i_{w, l} \left( \left( \Omega^{(\alpha)}_{\mathbb{C}^x} \boxtimes \Omega_{s_i u_{s_i w}}^i \right) \otimes i_{w, l}^* \mathcal{O}_{X_{l \leq w}^i}(\lambda) \right) \ ? \in \{*, !\}.$$ 

Here we introduced the right $\mathcal{D}_{\mathbb{C}^x}$-module

$$\Omega^{(\alpha)}_{\mathbb{C}^x} = \mathcal{D}_{\mathbb{C}^x} / (x \partial_x - \alpha) \mathcal{D}_{\mathbb{C}^x}. \text{ The coordinate } x \text{ on } \mathbb{C}^x \text{ is the one of } U(\alpha_i). \text{ Thus } x = \infty \text{ corresponds to } X_{s_i w}.$$ 

Then $\mathcal{R}_{w}(\lambda, \alpha) = (\mathcal{R}_{w}(\lambda, \alpha)_l)_{l \geq l_0} \in \text{Hol}(\lambda)$, where the $\gamma^{l_1}_{l_2}$ are induced.
Before describing the cohomology of these $\mathcal{D}$-modules, let us pause briefly and explain that $X_w \cap s_i X_w$ can be understood as orbits for the subgroup $L^+ I \cap s_i L^+ I$ of the Iwahori group $L^+ I$ acting on $X^{\text{thin}}$. Indeed, for $s_i w > w$ the $L^+ I \cap s_i L^+ I$-orbit $X_w$ is also a $L^+ I \cap s_i L^+ I$-orbit, as is $s_i X_w$. For $s_i w < w$ the $L^+ I$-orbit $X_w$ splits into two $L^+ I \cap s_i L^+ I$-orbits $X_w = (X_w \cap s_i X_w) \sqcup s_i X_{s_i w}$. 
Before describing the cohomology of these $\mathcal{D}$-modules, let us pause briefly and explain that $X_w \cap s_iX_w$ can be understood as orbits for the subgroup $L^+/I \cap s_iL^+/I$ of the Iwahori group $L^+/I$ acting on $X^{\text{thin}}$.

Indeed, for $s_iw > w$ the $L^+/I$-orbit $X_w$ is also a $L^+/I \cap s_iL^+/I$-orbit, as is $s_iX_w$. For $s_iw < w$ the $L^+/I$-orbit $X_w$ splits into two $L^+/I \cap s_iL^+/I$-orbits

$$X_w = (X_w \cap s_iX_w) \sqcup s_iX_{s_iw}.$$
The case of $\mathfrak{sl}_2$

The arrows indicate the closure relations.
Overview of results
We will identify the **global sections of** $\mathcal{R}_w(\lambda, \alpha)$ as a $g$-module for some choices of the parameters $\omega, w, \lambda, \alpha$ following the methods of Kashiwara-Tanisaki '95. We thereby extend several of their results to a class of non-highest weight representations which we call relaxed highest weight because they generalize the $\widehat{sl}_2$-representations that we have seen earlier.
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We will start by discussing the $\mathfrak{h}$-module structure.
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We then introduce **candidate $g$-modules**.
We will identify the **global sections of** $\mathcal{R}_{w}(\lambda, \alpha)$ as a $\mathfrak{g}$-module for some choices of the parameters $?, w, \lambda, \alpha$ following the methods of Kashiwara-Tanisaki '95. We thereby extend several of their results to a class of non-highest weight representations which we call relaxed highest weight because they generalize the $\widehat{\mathfrak{sl}}_2$-representations that we have seen earlier.

We will start by discussing the $\mathfrak{h}$-module structure.

We then introduce **candidate $\mathfrak{g}$-modules**.

We identify these candidate $\mathfrak{g}$-modules with the (dual of the) $\mathfrak{g}$-module of global sections.
Theorem

We have isomorphisms of $\mathfrak{h}$-modules

1. $H^0(X^{\leq w}, R_{*w}(\lambda, \alpha)_l) \cong \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} S(n_i^-/n_i^-) \otimes_{\mathbb{C}} \mathbb{C}_{s_i w \cdot \lambda + \alpha \alpha_i}$

2. $H^0(X, R_{*w}(\lambda, \alpha)) \cong \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} S n_i^- \otimes_{\mathbb{C}} \mathbb{C}_{s_i w \cdot \lambda + \alpha \alpha_i}$

Here $z$ has weight $\alpha_i$. 
Consider the following factorization of $i_{w,l} : X_w \cap s_i X_w \hookrightarrow X_l^{\leq w}$

$$X_w \cap s_i X_w \hookrightarrow X_w \hookrightarrow N(X_w)_l \cong (U_l^- \setminus U_w^-) \times U_w \hookrightarrow X_l^{\leq w}.$$
Consider the following factorization of $i_{w,l} : X_w \cap s_i X_w \hookrightarrow X_{l^{\leq w}}$

$$X_w \cap s_i X_w \hookrightarrow X_w \hookrightarrow N(X_w)_I \cong (U_l^- \setminus U_w^-) \times U_w \hookrightarrow X_{l^{\leq w}}.$$  

The first and third embedding are open and affine, while the second embedding is closed.
Consider the following factorization of $i_{w,l} : X_w \cap s_i X_w \hookrightarrow X^\leq_w$

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The first and third embedding are open and affine, while the second embedding is closed.

By definition $H^0(X^\leq_w, \mathcal{R}_{*w}(\lambda, \alpha)_l) \cong H^0(N(X_w)_l, \mathcal{R}_{*w}(\lambda, \alpha)_l)$

and there is the explicit description of the $\ast$-direct image w.r.t. the second embedding $\kappa_{w,l} : X_w \hookrightarrow N(X_w)_l$

$$\kappa_{w,l} \ast \mathcal{M} = \kappa_{w,l} \cdot \mathcal{M} \otimes \mathbb{C}[\partial_1, \ldots, \partial_r],$$

where $\mathcal{M}$ is any right $\mathcal{D}$-module on $X_w$ and $r = \dim U_l^- \setminus U_w^-$. 
Lemma

\[ H^j(X_{i \leq w}^\leq w, R_*w(\lambda, \alpha)_l) = 0 \] and consequently
\[ H^j(X, R_*w(\lambda, \alpha)) = 0 \] for \( j > 0 \).
Cohomology vanishing

**Lemma**

\[ H^j(X_{\leq w}, R_{*w}(\lambda, \alpha)_l) = 0 \text{ and consequently } H^j(X, R_{*w}(\lambda, \alpha)) = 0 \text{ for } j > 0. \]

This is again proven using the fact that

\[ H^j(X_{\leq w}, R_{*w}(\lambda, \alpha)_l) \cong H^j(N(X_w)_l, R_{*w}(\lambda, \alpha)_l) \] and the above explicit description of the \(*\)-direct image.
Definition

Define the $\mathfrak{sl}_2$-module for $\Lambda, \alpha \in \mathbb{C}$

$$R^{\mathfrak{sl}_2}(\Lambda, \alpha) = U \mathfrak{sl}_2 / (h + 2\alpha + \Lambda, ef + (\alpha + \Lambda)(\alpha + 1)).$$
Definition

Define the $\mathfrak{sl}_2$-module for $\Lambda, \alpha \in \mathbb{C}$

$$R^{\mathfrak{sl}_2}(\Lambda, \alpha) = \mathcal{U}\mathfrak{sl}_2/(h + 2\alpha + \Lambda, ef + (\alpha + \Lambda)(\alpha + 1)).$$

When $\Lambda \in \mathbb{Z}_{\geq 2}, \alpha \in \mathbb{Z}$ and $1 - \Lambda \leq \alpha \leq -1$ this is a single isomorphism class denoted by $R^{\mathfrak{sl}_2}(\Lambda)$ (case (2, $-+$)). Its weight diagram is

$$\begin{array}{c}
\cdots \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \\
-\Lambda & -\Lambda + 2 & \Lambda - 2 & \Lambda \\
\end{array}$$
Relaxed Verma modules

The generalization of $R_{\mu_1, \mu_2, t}$ for arbitrary $g$ is

**Definition**

Define the $g$-module for $\lambda \in P$, $\alpha \in \mathbb{C}$

$$R(\lambda, \alpha) = \mathcal{U} \mathfrak{g} \otimes_{\mathcal{U} p_i} \left( \mathbb{C}_\lambda \otimes_{\mathbb{C}} R^{sl_2}(\lambda(h_i), \alpha) \right).$$

Here $p_i$ acts on $\mathbb{C}_\lambda \otimes_{\mathbb{C}} R^{sl_2}(\lambda(h_i), \alpha)$ via the projection

$$p_i \mapsto p_i/n_i = g_i = \{ h \in \mathfrak{h} \mid \alpha_i(h) = 0 \} \oplus g_i'.$$
Relaxed Verma modules

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**Definition**

Define the $g$-module for $\lambda \in P$, $\alpha \in \mathbb{C}$

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$$p_i \mapsto p_i/n_i = g_i = \{ h \in \mathfrak{h} | \alpha_i(h) = 0 \} \oplus g'_i.$$

Put $R^{sl_2}(\lambda(h_i))$ to get the isomorphism class $R(\lambda)$. 
The first observation one can make about this definition is that the underlying \( \mathfrak{h} \)-module of \( R(w \cdot \lambda, \alpha) \) coincides with the \( \mathfrak{h} \)-module \( H^0(X, \mathcal{R}_{*w}(\lambda, \alpha)) \) described earlier.
The first observation one can make about this definition is that the underlying \( \mathfrak{h} \)-module of \( R(w \cdot \lambda, \alpha) \) coincides with the \( \mathfrak{h} \)-module \( H^0(X, R_{\lambda-\alpha}(\lambda, \alpha)) \) described earlier.

In the rest of the presentation we will explain the cases in which we can prove that this induced module is indeed the (dual of the) \( \mathfrak{g} \)-module of global sections.
Let \( \mathcal{M} \) be a holonomic right \( \mathcal{D} \)-module on \( \overline{X_{s_i}} \cong \mathbb{P}^1 \) twisted by \( \mathcal{O}_{\mathbb{P}^1}(-\lambda(h_i)) \). Let \( i'_{s_i,l} : \overline{X_{s_i}} \hookrightarrow X_{l^{\leq s_i}} \) be the closed embedding. Then \( i'_{s_i,*}\mathcal{M} = (i'_{s_i,l,*}\mathcal{M})_{l \geq l_0} \in \text{Hol}(\lambda) \).
The case $w = s_i$

Let $\mathcal{M}$ be a holonomic right $\mathcal{D}$-module on $\overline{X_{s_i}} \cong \mathbb{P}^1$ twisted by $\mathcal{O}_{\mathbb{P}^1}(-\lambda(h_i))$. Let $i'_{s_i, l} : \overline{X_{s_i}} \hookrightarrow X^{<s_i}$ be the closed embedding. Then $i'_{s_i*}\mathcal{M} = (i'_{s_i,l*}\mathcal{M})_{l \geq l_0} \in \text{Hol}(\lambda)$.

**Theorem**

\[ \overline{\mathcal{H}}^j(X, i'_{s_i*}\mathcal{M}) \cong \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{p}_i} \mathcal{H}^j(\mathbb{P}^1, \mathcal{M}) \text{ as } \mathfrak{g}\text{-module} \]
The case $w = s_i$

Let $\mathcal{M}$ be a holonomic right $\mathcal{D}$-module on $\overline{X_{s_i}} \cong \mathbb{P}^1$ twisted by $\mathcal{O}_{\mathbb{P}^1}(-\lambda(h_i))$. Let $i'_{s_i,l} : \overline{X_{s_i}} \hookrightarrow X_{\leq s_i}$ be the closed embedding. Then $i'_{s_i*}\mathcal{M} = (i'_{s_i,l*}\mathcal{M})_{l \geq l_0} \in \text{Hol}(\lambda)$.

Theorem

\[ \overline{H^j}(X, i'_{s_i*}\mathcal{M}) \cong \mathcal{U}_g \otimes \mathcal{U}_{\mathbb{P}_1} H^j(\mathbb{P}^1, \mathcal{M}) \text{ as } \mathfrak{g}\text{-module} \]

Together with the description of the cohomology of twisted $\mathcal{D}$-modules obtained as direct images from $X_{s_i} \cap s_i X_{s_i} \cong \mathbb{C}^\times \hookrightarrow \mathbb{P}^1$ (arXiv:1509.05299 [math.RT])


The case $w = s_i$

Let $\mathcal{M}$ be a holonomic right $\mathcal{D}$-module on $\bar{X}_{s_i} \cong \mathbb{P}^1$ twisted by $\mathcal{O}_{\mathbb{P}^1}(-\lambda(h_i))$. Let $i_{s_i,l}': \bar{X}_{s_i} \hookrightarrow X_{l \leq s_i}$ be the closed embedding. Then $i_{s_i,l}^* \mathcal{M} = (i_{s_i,l}^* \mathcal{M})_{l \geq l_0} \in \text{Hol}(\lambda)$.

**Theorem**

\[
\overline{H}^j(X, i_{s_i,*}^* \mathcal{M}) \cong \mathcal{U} \mathfrak{g} \otimes \mathcal{U}_{\mathfrak{p}_i} H^j(\mathbb{P}^1, \mathcal{M}) \text{ as } \mathfrak{g}\text{-module}
\]

Together with the description of the cohomology of twisted $\mathcal{D}$-modules obtained as direct images from $X_{s_i} \cap s_i X_{s_i} \cong \mathbb{C}^\times \hookrightarrow \mathbb{P}^1$ (arXiv:1509.05299 [math.RT]) this gives a description of the $\mathfrak{g}$-modules $\overline{H}^j(X, \mathcal{R}_{s_i} \lambda, \alpha)$ for $j \in \{0, 1\}$ and all values of $?, \lambda, \alpha$ in terms of the above $\mathcal{R}(\lambda, \alpha)$ and obvious modifications thereof.
The automorphism $s := \tilde{s}_i = e^{e_i} e^{-f_i} e^{e_i}$ of $X$ descends to affine morphisms $s_i^{l+\Delta} : X_{l+\Delta}^{\leq w} \to X_{l}^{\leq w}$ for $w$ such that $s_i w < w$ and $\Delta \geq 4$. 
The automorphism \( s := \tilde{s}_i = e^{e_i} e^{-f_i} e^{e_i} \) of \( X \) descends to affine morphisms \( s_{i}^l + \Delta : X_{i}^{l, w} \rightarrow X_{i}^{l, w} \) for \( w \) such that \( s_i w < w \) and \( \Delta \geq 4 \). The functor \( s_{i}^l + \Delta \) is an exact equivalence

\[
\text{Hol} \left( D_{X_{i}^{l, w}}^{\mathcal{O}_{X_{i}^{l, w}}(\lambda)} , X_{w} \right) \rightarrow \text{Hol} \left( D_{X_{i}^{l, w}}^{\mathcal{O}_{X_{i}^{l, w}}(\lambda)} , X_{w} \right).
\]
The automorphism \( s := \tilde{s}_i = e^{e_i} e^{-f_i} e^{e_i} \) of \( X \) descends to affine morphisms \( s_i^{l+\Delta} : \underline{X}_{i+\Delta}^{\leq w} \to \underline{X}_i^{\leq w} \) for \( w \) such that \( s_i w < w \) and \( \Delta \geq 4 \). The functor \( s_i^{l+\Delta} \) is an exact equivalence

\[
\text{Hol} \left( \mathcal{D}_{\underline{X}_i^{\leq w}}^{(s_i^{l+\Delta}) \ast \mathcal{O}_{\underline{X}_i^{\leq w}}(\lambda)}, \underline{X}_w \right) \to \text{Hol} \left( \mathcal{D}_{\underline{X}_i^{\leq w}}^{\mathcal{O}_{\underline{X}_i^{\leq w}}(\lambda)}, \underline{X}_w \right).
\]

Identifying \( (s_i^{l+\Delta}) \ast \mathcal{O}_{\underline{X}_i^{\leq w}}(\lambda) = \mathcal{O}_{\underline{X}_{i+\Delta}^{\leq w}}(\lambda) \) we get an exact auto-equivalence of \( \text{Hol}(\lambda) \).
Exact auto-equivalence $\tilde{s}_i\ast$ of $\text{Hol}(\lambda)$

The automorphism $s := \tilde{s}_i = e^{e_i} e^{-f_i} e^{e_i}$ of $X$ descends to affine morphisms $s_{i}^{l+\Delta} : X_{i}^{\leq w} \rightarrow X_{i}^{\leq w}$ for $w$ such that $s_i w < w$ and $\Delta \geq 4$. The functor $s_{i}^{l+\Delta}\ast$ is an exact equivalence

$$\text{Hol} \left( \mathcal{D}_{X_{i}^{\leq w}}^{(s_i^{l+\Delta})\ast \mathcal{O}_{X_{i}^{\leq w}}(\lambda)} , \overline{X_{w}} \right) \rightarrow \text{Hol} \left( \mathcal{D}_{X_{i}^{\leq w}}^{\mathcal{O}_{X_{i}^{\leq w}}(\lambda)} , \overline{X_{w}} \right).$$

Identifying $(s_{i}^{l+\Delta})\ast \mathcal{O}_{X_{i}^{\leq w}}(\lambda) = \mathcal{O}_{X_{i+\Delta}^{\leq w}}(\lambda)$ we get an exact auto-equivalence of $\text{Hol}(\lambda)$.

**Theorem**

*Let $\mathcal{M} \in \text{Hol}(\lambda)$. Then $\text{H}^j(X, \tilde{s}_i\ast \mathcal{M}) \cong \text{H}^j(X, \mathcal{M})^{\tilde{s}_i}$, where $(\cdot)^{\tilde{s}_i}$ is the twist of the $\mathfrak{g}$-module by the automorphism $\tilde{s}_i = e^{e_i} e^{-f_i} e^{e_i}$ of $\mathfrak{g}$.*
The case of $\mathcal{R}_w(\lambda)$

Let us abbreviate the isomorphism class $\mathcal{R}_w(\lambda) = \mathcal{R}_w(\lambda, \alpha)$ when $\alpha \in \mathbb{Z}$ (trivial monodromy).
The case of $\mathcal{R}_{*w}(\lambda)$

Let us abbreviate the isomorphism class $\mathcal{R}_{*w}(\lambda) = \mathcal{R}_{*w}(\lambda, \alpha)$ when $\alpha \in \mathbb{Z}$ (trivial monodromy).

**Theorem**

Let $\lambda + \rho$ be regular antidominant. Then

$$\overline{H^0(X, \mathcal{R}_{*w}(\lambda))} \cong R(w \cdot \lambda)^\vee$$

as $\mathfrak{g}$-module.
Sketch of proof

**Lemma**

We have an isomorphism of $g_i$-modules

\[
\bigoplus_{\mu \in \mathbb{Z} \alpha_i + w \cdot \lambda} H^0(X, \mathcal{R} \ast w(\lambda))_{\mu}^\vee \cong \mathbb{C}_{w \cdot \lambda} \otimes \mathbb{R}^{s_{12}}((w \cdot \lambda)(h_i))
\]
Lemma

We have an isomorphism of $\mathfrak{g}_i$-modules

$$\bigoplus_{\mu \in \mathbb{Z} \alpha_i + w \cdot \lambda} \overline{H^0(X, \mathcal{R}_{w\cdot R}(\lambda))}_{\mu} \cong \mathbb{C}_{w\cdot \lambda} \otimes R^{s_{l_2}}((w \cdot \lambda)(h_i))$$

Thus we have an induced morphism

$\phi : R(w \cdot \lambda) \to \overline{H^0(X, \mathcal{R}_{w\cdot R}(\lambda))}^\vee$ of $\mathfrak{g}$-modules. Source and target coincide as $\mathfrak{h}$-modules. In order to prove that $\phi$ is an isomorphism it suffices to prove that it injects.
We have a short exact sequence

$$0 \to \tilde{s}_i^* B_w(\lambda) \to R_{*w}(\lambda) \to B_{s_iw}(\lambda) \to 0$$

in Hol(\lambda). Here $B_w(\lambda)$ is the $*$-direct image from the Schubert cell $X_w$. 
We have a short exact sequence

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in Hol(\lambda). Here \( B_w(\lambda) \) is the \(*\)-direct image from the Schubert cell \( X_w \).

We have \( \bar{H}^1(X, \tilde{s}_i^* B_w(\lambda)) \cong \bar{H}^1(X, B_w(\lambda)) \tilde{s}_i = 0 \). We get a surjection \( \bar{H}^0(X, R_{*w}(\lambda)) \to \bar{H}^0(X, B_{s_iw}(\lambda)) \) and hence an injection

\[ \psi : \bar{H}^0(X, B_{s_iw}(\lambda))^\vee \hookrightarrow \bar{H}^0(X, R_{*w}(\lambda))^\vee. \]
We have a short exact sequence

\[ 0 \to \tilde{s}_i^*\mathcal{B}_w(\lambda) \to \mathcal{R}_{*w}(\lambda) \to \mathcal{B}_{s_iw}(\lambda) \to 0 \]

in $\text{Hol}(\lambda)$. Here $\mathcal{B}_w(\lambda)$ is the $\ast$-direct image from the Schubert cell $X_w$.

We have $\overline{H}^1(\text{X}, \tilde{s}_i^*\mathcal{B}_w(\lambda)) \cong \overline{H}^1(\text{X}, \mathcal{B}_w(\lambda))\tilde{s}_i = 0$. We get a surjection $\overline{H}^0(\text{X}, \mathcal{R}_{*w}(\lambda)) \to \overline{H}^0(\text{X}, \mathcal{B}_{s_iw}(\lambda))$ and hence an injection

\[ \psi : \overline{H}^0(\text{X}, \mathcal{B}_{s_iw}(\lambda))^\vee \leftarrow \overline{H}^0(\text{X}, \mathcal{R}_{*w}(\lambda))^\vee. \]

By Kashiwara-Tanisaki ’95 $\overline{H}^0(\text{X}, \mathcal{B}_{s_iw}(\lambda))^\vee \cong M(s_iw \cdot \lambda)$. 
We have a short exact sequence

\[ 0 \to \tilde{s}_i^*B_w(\lambda) \to R_{\ast w}(\lambda) \to B_{s_iw}(\lambda) \to 0 \]

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\[ \psi : \overline{H}^0(X, B_{s_iw}(\lambda))^\vee \hookrightarrow \overline{H}^0(X, R_{\ast w}(\lambda))^\vee. \]

By Kashiwara-Tanisaki '95 \( \overline{H}^0(X, B_{s_iw}(\lambda))^\vee \cong M(s_iw \cdot \lambda) \).

Similarly, we get an injection \( \psi\tilde{s}_i : M(s_iw \cdot \lambda)\tilde{s}_i \hookrightarrow \overline{H}^0(X, R_{\ast w}(\lambda))^\vee. \)
Lemma

$R(w \cdot \lambda)$ does not have nonzero $g'_i$-finite vectors.

This lemma implies

Proposition

Any nonzero submodule of $R(w \cdot \lambda)$ intersects the submodule $M(s_i w \cdot \lambda) \oplus M(s_i w \cdot \lambda)$ nontrivially.

Apply the proposition to $\ker \phi$. Note that $\phi|_{M(s_i w \cdot \lambda)}$ is a nonzero multiple of $\psi$ and similarly for $\psi$ to conclude $\ker \phi = 0$.
Lemma

\( R(w \cdot \lambda) \) does not have nonzero \( g_i' \)-finite vectors.

This lemma implies

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Any nonzero submodule of \( R(w \cdot \lambda) \) intersects the submodule \( M(s_i w \cdot \lambda) \oplus M(s_i w \cdot \lambda)^{\sim_i} \) nontrivially.
Lemma

\( R(w \cdot \lambda) \) does not have nonzero \( g'_i \)-finite vectors.

This lemma implies

Proposition

Any nonzero submodule of \( R(w \cdot \lambda) \) intersects the submodule \( M(s_i w \cdot \lambda) \oplus M(s_i w \cdot \lambda) \tilde{s}_i \) nontrivially.

Apply the proposition to \( \ker \phi \). Note that \( \phi | M(s_i w \cdot \lambda) \) is a nonzero multiple of \( \psi \) and similarly for \( \psi \tilde{s}_i \) to conclude \( \ker \phi = 0 \).