Relaxed Highest Weight Modules from $\mathcal{D}$-Modules on the Kashiwara Flag Scheme

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Relaxed highest weight modules for $\widehat{\mathfrak{sl}_2}$ after Feigin, Semikhatov, Sirota,Tipunin

Introduction to localization on the affine flag variety

Setup

Overview of results

Relaxed highest weight modules for $\widehat{\mathfrak{sl}_2}$ after Feigin, Semikhatov, Sirota, Tipunin

We start with the Lie algebra $\mathfrak{sl}_2 = \mathbb{C}\, e \oplus \mathbb{C}\, h \oplus \mathbb{C}\, f$ and define $\widehat{\mathfrak{sl}_2} = \mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}\, K$, where K is central and

$$[X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n} + m\delta_{m+n,0}\, \mathrm{Tr}(XY)K\ .$$

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Up to the derivation element this defines the **affine Kac-Moody algebra with Cartan matrix** $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

Definition (Semikhatov-Sirota '97)

Let $\mu_1, \mu_2 \in \mathbb{C}$ and $t \in \mathbb{C} \setminus \{0\}$. The **relaxed Verma module** *$R_{\mu_1,\mu_2,t}$ is the $\widehat{\mathfrak{sl}_2}$-module generated from a vector v that satisfies the annihilation conditions*

$$(e \otimes z^n)v = (h \otimes z^n)v = (f \otimes z^n)v = 0 \quad n \geq 1$$

and the relations

$$(f \otimes 1)(e \otimes 1)v = -\mu_1\mu_2 v \quad (h \otimes 1)v = -(1 + \mu_1 + \mu_2)v$$
$$Kv = (t-2)v$$

by a free action of $e \otimes z^n$, $f \otimes z^n$, $h \otimes z^n$, $n \leq -1$.

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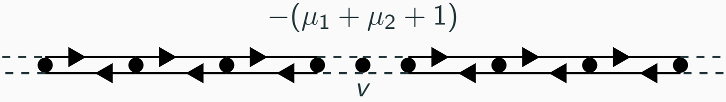
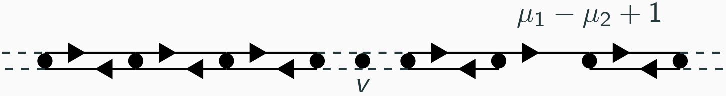
To get a first impression about the structure of $R_{\mu_1,\mu_2,t}$ we can look at the $\mathfrak{sl}_2 \otimes \mathbb{C}\,1$-submodule generated by v. It is a weight module with weights $-(1+\mu_1+\mu_2)+2\,\mathbb{Z}$, each of which has multiplicity one. We have $(f \otimes 1)(e \otimes 1)^{\mu_j+1} v = 0$ and $(e \otimes 1)(f \otimes 1)^{-\mu_j} v = 0$ if these expressions are actually **defined** and $\mu_j \neq 0$ in the second case.

$\mu_1 \notin \mathbb{Z}$, $\mu_2 \notin \mathbb{Z}$



$\mu_1 \in \mathbb{Z}_{\geq 0}$, $\mu_2 \notin \mathbb{Z}$



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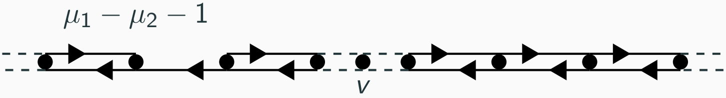
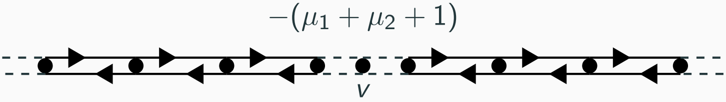
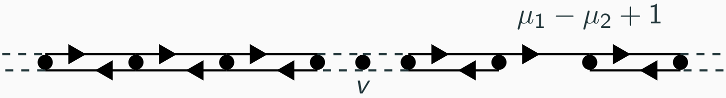


$\mu_1 \notin \mathbb{Z}$, $\mu_2 \notin \mathbb{Z}$ case (0)



$\mu_1 \in \mathbb{Z}_{\geq 0}$, $\mu_2 \notin \mathbb{Z}$ case $(1, +)$



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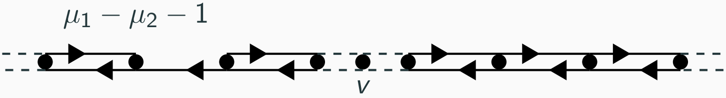
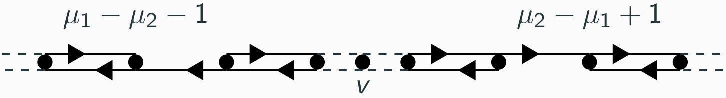


$\mu_1 \in \mathbb{Z}_{<0}$, $\mu_2 \in \mathbb{Z}_{\geq 0}$



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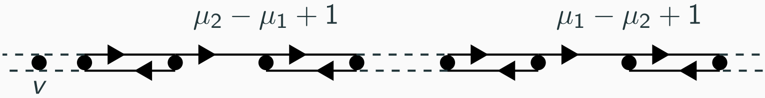

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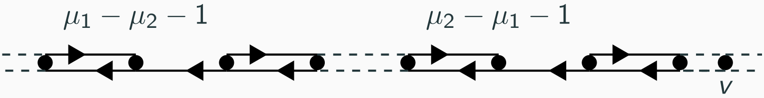
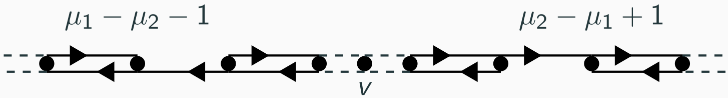


$\mu_1 \in \mathbb{Z}_{<0}$, $\mu_2 \in \mathbb{Z}_{\geq 0}$ case $(2, -+)$



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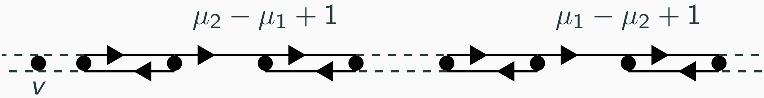

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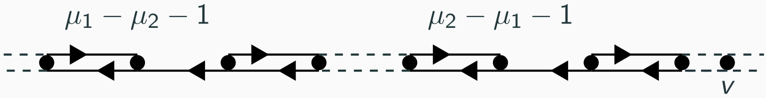

Coming back to $R_{\mu_1,\mu_2,t}$, we note that $(e \otimes 1)(f \otimes 1)^{-\mu_1} v = 0$ implies that the submodule generated by $(f \otimes 1)^{-\mu_1} v$ is isomorphic to a **Verma module** of highest weight $\lambda = \mu_1 - \mu_2 - 1$. We will denote it by $M_{\lambda,t}$.

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Let's formulate a similar statement for $(e \otimes 1)^{\mu_1+1} v$.

Consider the automorphism of $\widehat{\mathfrak{sl}_2}$ sending $K \mapsto K$ and

$$e \otimes z^n \mapsto e \otimes z^{n+\theta} \quad f \otimes z^n \mapsto f \otimes z^{n-\theta} \quad h \otimes z^n \mapsto h \otimes z^n + \theta \delta_{n,0} K \ .$$

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The vector $w = (e \otimes 1)^{\mu_1+1} v$ satisfies

$$\begin{aligned} &(e \otimes z^{\geq 1}) w = (h \otimes z^{\geq 1}) w = (f \otimes z^{\geq 0}) w = 0 \\ &(h \otimes 1 + (t-2)) w = (t + \mu_1 - \mu_2 - 1) w \ . \end{aligned}$$

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\end{aligned}$$

Thus w generates a submodule of $R_{\mu_1,\mu_2,t}$ which is isomorphic to a **Verma module twisted by the automorphism for** $\theta = 1$. We will denote it by $M^{(1)}_{t+\mu_1-\mu_2-1,t}$.

So we have the following embeddings

$$
\begin{array}{ll}
(1,+) & R_{\mu_1,\mu_2,t} \hookleftarrow M^{(1)}_{t+\mu_1-\mu_2-1,t} \\[1em]
(1,-) & M_{\mu_1-\mu_2-1,t} \hookrightarrow R_{\mu_1,\mu_2,t} \\[1em]
(2,-+) & M_{\mu_1-\mu_2-1,t} \hookrightarrow R_{\mu_1,\mu_2,t} \hookleftarrow M^{(1)}_{t+\mu_2-\mu_1-1,t} \\[1em]
(2,++) & R_{\mu_1,\mu_2,t} \hookleftarrow M^{(1)}_{t+\mu_2-\mu_1-1,t} \hookleftarrow M^{(1)}_{t+\mu_1-\mu_2-1,t} \\[1em]
(2,--) & M_{\mu_1-\mu_2-1,t} \hookrightarrow M_{\mu_2-\mu_1-1,t} \hookrightarrow R_{\mu_1,\mu_2,t}
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\end{array}
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The **goal** of Semikhatov-Sirota '97 is to describe which modules $M_{\lambda,t}$, $M^{(1)}_{\lambda,t}$ or $R_{\mu'_1,\mu'_2,t}$ embed into $R_{\mu_1,\mu_2,t}$.

Simplicity of $R_{\mu_1,\mu_2,t}$

From the above we conclude that $\mu_1 \notin \mathbb{Z}$ and $\mu_2 \notin \mathbb{Z}$ is a necessary condition for $R_{\mu_1,\mu_2,t}$ to be simple.

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Theorem (Semikhatov-Sirota '97)

$$R_{\mu_1,\mu_2,t} \quad \textit{simple} \quad \Leftrightarrow \quad \mu_1 \notin \mathbb{Z} \ \textit{and}\ \mu_2 \notin \mathbb{Z}$$
$$\textit{and}\ \nexists r, s \in \mathbb{Z}_{>0}\ \ \mu_1 - \mu_2 = r - st \ \textit{or}\ \mu_2 - \mu_1 = r - st$$

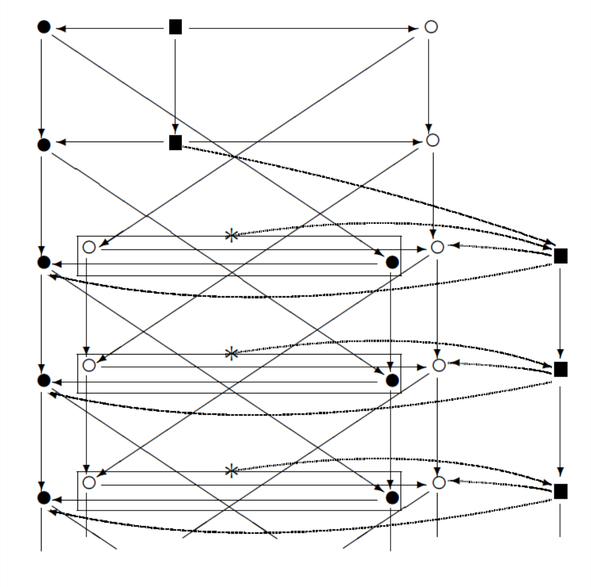
The description of the so-called embedding diagrams for $R_{\mu_1,\mu_2,t}$ is the main result of Semikhatov-Sirota '97. These diagrams are labeled by combining

I, II, $\text{III}_\pm$ determined by the row

(0), $(1,+)$, $(1,-)$, $(2,--)$, ... determined by the column.

	$\mu_1,\mu_2 \notin \mathbb{Z}$	$\mu_1 \in \mathbb{Z}$, $\mu_2 \notin \mathbb{Z}$	$\mu_1,\mu_2 \in \mathbb{Z}$	
			$\mu_1 \cdot \mu_2 > 0$	$\mu_1 \cdot \mu_2 < 0$
$\mu_1-\mu_2 \notin \mathbb{K}(t)$,	I(0), Eq. (3.2)	I(1), Eq. (3.2)	I$(2,--)$ and I$(2,++)$, Eq. (3.3)	I$(2,-+)$, Eq. (3.3)
$\mu_1-\mu_2 \in \mathbb{K}(t)$, $t \notin \mathbb{Q}$	II(0), Eq. (3.4)	II(1), Eq. (3.4)	—	—
$\mu_1-\mu_2 \in \mathbb{K}(t)$, $t \in \mathbb{Q}$, $\mu_1-\mu_2 \notin \mathbb{Z}$, $(\mu_1-\mu_2)/t \notin \mathbb{Z}$	$\text{III}_\pm(0)$, Eq. (3.7)	$\text{III}_\pm(1)$, Eq. (3.9) and (3.10)	—	—
$\mu_1-\mu_2 \in \mathbb{K}(t)$, $t \in \mathbb{Q}$, $\mu_1-\mu_2 \in \mathbb{Z}$, $(\mu_1-\mu_2)/t \notin \mathbb{Z}$	$\text{III}^0_\pm(0)$, Eq. (3.11)	—	$\text{III}^0_\pm(2,--)$, Eq. (3.13), and $\text{III}^0_\pm(2,++)$	$\text{III}^0_\pm(2,-+)$, Eq. (3.32)
$\mu_1-\mu_2 \in \mathbb{K}(t)$, $t \in \mathbb{Q}$, $\mu_1-\mu_2 \notin \mathbb{Z}$, $(\mu_1-\mu_2)/t \in \mathbb{Z}$		$\text{III}^0_\pm(1)$, Eq. (3.12)	—	—
$\mu_1-\mu_2 \in \mathbb{K}(t)$, $t \in \mathbb{Q}$, $\mu_1-\mu_2 \in \mathbb{Z}$, $(\mu_1-\mu_2)/t \in \mathbb{Z}$	$\text{III}^{00}_\pm(0)$	—	$\text{III}^{00}_\pm(2,--)$, Eq. (3.35), and $\text{III}^{00}_\pm(2,++)$	$\text{III}^{00}_\pm(2,-+)$, Eq. (3.40)

Example: case $\mathbf{III}^0_+(2,-+)$



Introduction to localization on the affine flag variety

Let $\mathring{\mathfrak{g}}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$. The celebrated theorem of Beilinson-Bernstein '81 states that the functor of global sections is an exact equivalence of categories between the $\mathcal{D}$-modules on the flag variety twisted by the line bundle associated to a regular dominant weight and the $\mathring{\mathfrak{g}}$-modules of the corresponding central character.

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In particular, we need to associate a “flag variety” to $\mathfrak{g}$.

A first possibility, in case $\mathfrak{g}$ is untwisted, is to consider the thin flag variety defined as a quotient of the loop group by the Iwahori group scheme $X^{\text{thin}} = \mathrm{L}\,\mathring{G} / \mathrm{L}^+ I$ (Beilinson-Drinfeld, Pappas-Rapoport '08 and others). Here $\mathring{G}$ is a semisimple algebraic group.

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Theorem (Beilinson-Drinfeld, P. Shan '11)

Let $\lambda + \rho$ be regular antidominant. The functor $\Gamma(X^{\text{thin}}, \cdot)$ between the λ-twisted right $\mathcal{D}$-modules on X^{thin} and $\mathfrak{g}\,\mathrm{mod}$ is exact and faithful.

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The basic open question is to describe the essential image of this functor in $\mathfrak{g}\,\mathrm{mod}$ (conjectural description by Beilinson '02, I. Shapiro '09).

A second possibility is to consider the Kashiwara flag scheme X. As we will detail later, it is a scheme, not locally of finite type, but having an open cover by affine spaces of countable dimension. The finite dimensional Schubert cells X_w can be defined as subschemes of X and one again has a notion of twisted $\mathcal{D}$-modules on X and a functor of global sections.

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Theorem (Kashiwara-Tanisaki '95)

The global sections of the $*$- and $!$-direct image from X_w identify with $\mathrm{M}(w \cdot \lambda)^\vee$ and $\mathrm{M}(w \cdot \lambda)$ respectively when $\lambda + \rho$ is regular antidominant.

The last two theorems can be combined into

Theorem (Frenkel-Gaitsgory '09)

Let $\lambda + \rho$ be regular antidominant. $\Gamma(X^{\mathrm{thin}}, \cdot)$ defines an exact equivalence between the category of λ-twisted right $\mathcal{D}$-modules on X^{thin} that are equivariant for the pro-unipotent radical of $\mathsf{L}^+ I$ and the block of category $\mathcal{O}$ of $\mathfrak{g}$ defined by λ.

Setup

Affine Kac-Moody algebras

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- $\Phi = \Phi^{>0} \sqcup \Phi^{<0}$ positive and negative roots of $\mathfrak{g}$

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Kashiwara flag scheme

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The Kashiwara flag scheme is defined as $X = G/B$ (quotient by a locally free action). The so-called Tits extension $\widetilde{W}$ of W acts on G and X. On T-invariant subsets of X, the action of $\widetilde{W}$ factors through W.

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- $X = \bigcup_{w \in W} X^{\leq w}$, where $X^{\leq w} = \bigcup_{v \leq w} N(X_v)$ is B^{-}-invariant and quasi-compact and $\leq$ is the Bruhat partial order
- There is a line bundle $\mathcal{O}_X(\lambda)$ on X associated to $\lambda \in P$.

For fixed w and all $l \in \mathbb{Z}_{>0}$ large enough U_l^- acts locally freely on $X^{\leq w}$. The quotient $X_l^{\leq w} = U_l^- \backslash X^{\leq w}$ is a smooth quasi-projective variety (Shan-Varagnolo-Vasserot '14).

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We have for $l_1 \geq l_2$ large enough a commutative diagram

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The fibers of the projection $p_{l_2}^{l_1}$ are affine spaces and $\hookrightarrow$ are closed embeddings. Subsequently we will always assume that l is large enough.

Twisted $\mathcal{D}$-modules on X

$\mathrm{Hol}(\mathcal{D}_{X_I^{\leq w}}(\lambda), \overline{X_w})$ Category of holonomic right $\mathcal{D}$-modules on $X_I^{\leq w}$ twisted by $\mathcal{O}_{X_I^{\leq w}}(\lambda)$, $\lambda \in P$, with support in $\overline{X_w}$

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- It has a contravariant exact auto-equivalence $\mathbb{D}$, the holonomic duality.
- $p_{I_2 *}^{I_1} : \mathrm{Hol}(\mathcal{D}_{X_{I_1}^{\leq w}}(\lambda), \overline{X_w}) \to \mathrm{Hol}(\mathcal{D}_{X_{I_2}^{\leq w}}(\lambda), \overline{X_w})$ exact equivalence

$$\mathrm{Hol}(\lambda, \overline{X_w}, X^{\leq w}, l_0) \ni \mathcal{M} = \Big((\mathcal{M}_l)_{l \geq l_0}, (\gamma^{l_1}_{l_2})_{l_1 \geq l_2 \geq l_0} \Big)$$

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Taking limits we get rid of the auxiliary choices $\overline{X_w}$, $X^{\leq w}$ and l_0 and define the category $\mathrm{Hol}(\lambda)$ of λ-twisted holonomic right $\mathcal{D}$-modules on X.

Cohomology groups

For $\mathcal{M} \in \mathsf{Hol}(\lambda)$ and $j \in \mathbb{Z}_{\geq 0}$ define

$$\mathrm{H}^j(X, \mathcal{M}) = \varprojlim_I \mathrm{H}^j(X_I^{\leq w}, \mathcal{M}_I) ,$$

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Define

$$\overline{\mathrm{H}^j}(X, \mathcal{M}) = \bigoplus_{\mu \in \mathfrak{h}^*} \mathrm{H}^j(X, \mathcal{M})_\mu ,$$

where $(\cdot)_\mu$ denotes the generalized weight space associated to μ. This is a $\mathfrak{g}$-submodule of $\mathrm{H}^j(X, \mathcal{M})$.

Coordinates on $N(X_w)$

For $w \in W$ abbreviate

$$U_w^- = U^-(\Phi^{<0} \cap w\Phi^{<0}) \subseteq U^-$$
$$U_w = U(\Phi^{>0} \cap w\Phi^{<0}) \subseteq U \,.$$

The map $(u_1, u_2) \mapsto u_1 u_2 w 1B$ defines a T-equivariant isomorphism of schemes

$$U_w^- \times U_w \xrightarrow{\cong} N(X_w) \,.$$

The image of $1 \times U_w$ is the (finite dimensional) Schubert cell X_w in X.

Let w be such that $s_i w < w$.

Lemma

We have $X_w \cap s_i X_w = s_i X_w \setminus X_{s_i w} = X_w \setminus s_i X_{s_i w}$. In the above coordinates on $N(X_w)$ and $s_i N(X_w) = N(X_{s_i w})$ the identity map $s_i X_w \setminus X_{s_i w} \to X_w \setminus s_i X_{s_i w}$ is the isomorphism

$$(U^-(-\alpha_i) \setminus 1) \times U_{s_i w} \to (U(\alpha_i) \setminus 1) \times^{s_i} U_{s_i w}$$
$$(e^{zf_i}, h_i(z)^{-1} u h_i(z)) \mapsto (e^{z^{-1} e_i}, \dot{s}_i^{-1} \widetilde{u}(z) \dot{s}_i) \, .$$

Here $z \in \mathbb{G}_\mathrm{m}$ and $\dot{s}_i = e^{e_i} e^{-f_i} e^{e_i}$. h_i is considered as a group homomorphism $\mathbb{G}_\mathrm{m} \to T$. Given u and z, $\widetilde{u}(z) \in U_{s_i w}$ is uniquely determined by the condition $e^{ze_i} u \in \widetilde{u}(z) U(\Phi^{>0} \cap s_i w \Phi^{>0})$.

For $s_i w < w$ consider the locally closed affine embedding

$$i_{w,I} : X_w \cap s_i X_w \hookrightarrow X_I^{\leq w} \ .$$

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Definition

Define the right $\mathcal{D}_{X_I^{\leq w}}(\lambda)$-module for $\lambda \in P$ and $\alpha \in \mathbb{C}$

$$\mathcal{R}_{?w}(\lambda, \alpha)_I = i_{w,I?}\left(\left(\Omega_{\mathbb{C}^\times}^{(\alpha)} \boxtimes \Omega_{s_i U_{s_i w}}\right) \otimes i_{w,I}^* \mathcal{O}_{X_I^{\leq w}}(\lambda)\right) \quad ? \in \{*, !\}.$$

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Here we introduced the right $\mathcal{D}_{\mathbb{C}^\times}$-module $\Omega_{\mathbb{C}^\times}^{(\alpha)} = \mathcal{D}_{\mathbb{C}^\times}/(x\partial_x - \alpha)\mathcal{D}_{\mathbb{C}^\times}$. The coordinate x on $\mathbb{C}^\times$ is the one of $U(\alpha_i)$. Thus $x = \infty$ corresponds to $X_{s_i w}$.

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Then $\mathcal{R}_{?w}(\lambda, \alpha) = (\mathcal{R}_{?w}(\lambda, \alpha)_l)_{l \geq l_0} \in \mathrm{Hol}(\lambda)$, where the $\gamma^{l_1}_{l_2}$ are induced.

Before describing the cohomology of these $\mathcal{D}$-modules, let us pause briefly and explain that $X_w \cap s_i X_w$ can be understood as orbits for the subgroup $\mathrm{L}^+ I \cap^{s_i} \mathrm{L}^+ I$ of the Iwahori group $\mathrm{L}^+ I$ acting on X^{thin}.

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Indeed, for $s_i w > w$ the $\mathrm{L}^+ I$-orbit X_w is also a $\mathrm{L}^+ I \cap^{s_i} \mathrm{L}^+ I$-orbit, as is $s_i X_w$. For $s_i w < w$ the $\mathrm{L}^+ I$-orbit X_w splits into two $\mathrm{L}^+ I \cap^{s_i} \mathrm{L}^+ I$-orbits

$$X_w = (X_w \cap s_i X_w) \sqcup s_i X_{s_i w} \ .$$

The case of $\widehat{\mathfrak{sl}_2}$

$$\begin{tikzcd}
X_1 & & X_{s_0} \arrow[ll] & & X_{s_0 s_1} \arrow[ll] \arrow[dlll] & & X_{s_0 s_1 s_0} \arrow[ll] \arrow[dlll] \\
& X_{s_1} \cap s_1 X_{s_1} \arrow[ul] \arrow[dl] & & X_{s_1 s_0} \cap s_1 X_{s_1 s_0} \arrow[ll] \arrow[ul] \arrow[dl] & & X_{s_1 s_0 s_1} \cap s_1 X_{s_1 s_0 s_1} \arrow[ll] \arrow[ul] \arrow[dl] & & \cdots \\
s_1 X_1 & & s_1 X_{s_0} \arrow[ll] & & s_1 X_{s_0 s_1} \arrow[ll] \arrow[ulll] & & s_1 X_{s_0 s_1 s_0} \arrow[ll] \arrow[ulll]
\end{tikzcd}$$

The arrows indicate the closure relations.

Overview of results

Plan

We will identify the **global sections of** $\mathcal{R}_{?w}(\lambda, \alpha)$ as a $\mathfrak{g}$-module for some choices of the parameters $?, w, \lambda, \alpha$ following the methods of Kashiwara-Tanisaki '95. We thereby extend several of their results to a class of non-highest weight representations which we call relaxed highest weight because they generalize the $\widehat{\mathfrak{sl}_2}$-representations that we have seen earlier.

Plan

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We identify these candidate $\mathfrak{g}$-modules with the (dual of the) $\mathfrak{g}$-module of global sections.

$\mathfrak{h}$-module structure of global sections

Theorem

We have isomorphisms of $\mathfrak{h}$-modules

1. $\mathrm{H}^0(X_I^{\leq w}, \mathcal{R}_{*w}(\lambda, \alpha)_I) \cong \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathrm{S}(\mathfrak{n}_i^- / \mathfrak{n}_I^-) \otimes_{\mathbb{C}} \mathbb{C}_{s_i w \cdot \lambda + \alpha\alpha_i}$
2. $\overline{\mathrm{H}^0}(X, \mathcal{R}_{*w}(\lambda, \alpha)) \cong \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathrm{S}\, \mathfrak{n}_i^- \otimes_{\mathbb{C}} \mathbb{C}_{s_i w \cdot \lambda + \alpha\alpha_i}$

Here z has weight α_i.

Sketch of proof

Consider the following factorization of $i_{w,I} : X_w \cap s_i X_w \hookrightarrow X_I^{\leq w}$

$$X_w \cap s_i X_w \hookrightarrow X_w \hookrightarrow N(X_w)_I \cong (U_I^- \backslash U_w^-) \times U_w \hookrightarrow X_I^{\leq w} .$$

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By definition $\mathrm{H}^0(X_I^{\leq w}, \mathcal{R}_{*w}(\lambda, \alpha)_I) \xrightarrow{\cong} \mathrm{H}^0(N(X_w)_I, \mathcal{R}_{*w}(\lambda, \alpha)_I)$ and there is the explicit description of the $*$-direct image w.r.t. the second embedding $\kappa_{w,I} : X_w \hookrightarrow N(X_w)_I$

$$\kappa_{w,I*}\mathcal{M} = \kappa_{w,I\cdot}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_1, \ldots, \partial_r] \ ,$$

where $\mathcal{M}$ is any right $\mathcal{D}$-module on X_w and $r = \dim U_I^- \backslash U_w^-$.

Cohomology vanishing

Lemma

$\underline{\mathrm{H}}^j(X_I^{\leq w}, \mathcal{R}_{*w}(\lambda, \alpha)_I) = 0$ *and consequently* $\overline{\mathrm{H}}^j(X, \mathcal{R}_{*w}(\lambda, \alpha)) = 0$ *for* $j > 0$.

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This is again proven using the fact that $\mathrm{H}^j(X_I^{\leq w}, \mathcal{R}_{*w}(\lambda,\alpha)_I) \xrightarrow{\cong} \mathrm{H}^j(N(X_w)_I, \mathcal{R}_{*w}(\lambda,\alpha)_I)$ and the above explicit description of the $*$-direct image.

Definition

Define the $\mathfrak{sl}_2$-module for $\Lambda, \alpha \in \mathbb{C}$

$$\mathsf{R}^{\mathfrak{sl}_2}(\Lambda, \alpha) = \mathcal{U}\,\mathfrak{sl}_2 / (h + 2\alpha + \Lambda, ef + (\alpha + \Lambda)(\alpha + 1)) \ .$$

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When $\Lambda \in \mathbb{Z}_{\geq 2}$, $\alpha \in \mathbb{Z}$ and $1 - \Lambda \leq \alpha \leq -1$ this is a single isomorphism class denoted by $\mathrm{R}^{\mathfrak{sl}_2}(\Lambda)$ (case $(2, -+)$). Its weight diagram is

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The generalization of $R_{\mu_1,\mu_2,t}$ for arbitrary $\mathfrak{g}$ is

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Here $\mathfrak{p}_i$ acts on $\mathbb{C}_\lambda \otimes_{\mathbb{C}} \mathsf{R}^{\mathfrak{sl}_2}(\lambda(h_i), \alpha)$ via the projection

$$\mathfrak{p}_i \twoheadrightarrow \mathfrak{p}_i/\mathfrak{n}_i = \mathfrak{g}_i = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0\} \oplus \mathfrak{g}_i' .$$

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Put $\mathrm{R}^{\mathfrak{sl}_2}(\lambda(h_i))$ to get the isomorphism class $\mathrm{R}(\lambda)$.

The first observation one can make about this definition is that the underlying $\mathfrak{h}$-module of $\mathsf{R}(w \cdot \lambda, \alpha)$ coincides with the $\mathfrak{h}$-module $\overline{\mathsf{H}^0}(X, \mathcal{R}_{*w}(\lambda, \alpha))$ described earlier.

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In the rest of the presentation we will explain the cases in which we can prove that this induced module is indeed the (dual of the) $\mathfrak{g}$-module of global sections.

The case $w = s_i$

Let $\mathcal{M}$ be a holonomic right $\mathcal{D}$-module on $\overline{X_{s_i}} \cong \mathbb{P}^1$ twisted by $\mathcal{O}_{\mathbb{P}^1}(-\lambda(h_i))$. Let $i'_{s_i,l} : \overline{X_{s_i}} \hookrightarrow X_l^{\leq s_i}$ be the closed embedding. Then $i'_{s_i *}\mathcal{M} = (i'_{s_i,l*}\mathcal{M})_{l \geq l_0} \in \mathrm{Hol}(\lambda)$.

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$\overline{\mathrm{H}^j}(X, i'_{s_i *}\mathcal{M}) \cong \mathcal{U}\,\mathfrak{g} \otimes_{\mathcal{U}\,\mathfrak{p}_i} \mathrm{H}^j(\mathbb{P}^1, \mathcal{M})$ *as* $\mathfrak{g}$*-module*

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Together with the description of the cohomology of twisted $\mathcal{D}$-modules obtained as direct images from $X_{s_i} \cap s_i X_{s_i} \cong \mathbb{C}^\times \hookrightarrow \mathbb{P}^1$ (arXiv:1509.05299 [math.RT])

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Exact auto-equivalence $\widetilde{s}_{i*}$ of $\mathrm{Hol}(\lambda)$

The automorphism $s := \widetilde{s}_i = e^{e_i} e^{-f_i} e^{e_i}$ of X descends to affine morphisms $s_l^{l+\Delta} : X_{l+\Delta}^{\leq w} \to X_l^{\leq w}$ for w such that $s_i w < w$ and $\Delta \geq 4$.

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Theorem

Let $\mathcal{M} \in \mathrm{Hol}(\lambda)$. Then $\mathrm{H}^j(X, \widetilde{s}_{i*}\mathcal{M}) \cong \mathrm{H}^j(X, \mathcal{M})^{\widetilde{s}_i}$, where $(\cdot)^{\widetilde{s}_i}$ is the twist of the $\mathfrak{g}$-module by the automorphism $\widetilde{s}_i = e^{e_i} e^{-f_i} e^{e_i}$ of $\mathfrak{g}$.

The case of $\mathcal{R}_{*w}(\lambda)$

Let us abbreviate the isomorphism class $\mathcal{R}_{*w}(\lambda) = \mathcal{R}_{*w}(\lambda, \alpha)$ when $\alpha \in \mathbb{Z}$ (trivial monodromy).

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Theorem

Let $\lambda + \rho$ be regular antidominant. Then $\overline{\mathrm{H}^0}(X, \mathcal{R}_{*w}(\lambda)) \cong \mathrm{R}(w \cdot \lambda)^\vee$ *as $\mathfrak{g}$-module.*

Sketch of proof

Lemma

We have an isomorphism of $\mathfrak{g}_i$-modules

$$\bigoplus_{\mu \in \mathbb{Z}\alpha_i + w\cdot\lambda} \overline{\mathrm{H}^0}(X, \mathcal{R}_{*w}(\lambda))_\mu^\vee \cong \mathbb{C}_{w\cdot\lambda} \otimes \mathrm{R}^{\mathfrak{sl}_2}((w\cdot\lambda)(h_i))$$

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Thus we have an induced morphism $\phi : \mathsf{R}(w\cdot\lambda) \to \overline{\mathsf{H}^0}(X, \mathcal{R}_{*w}(\lambda))^\vee$ of $\mathfrak{g}$-modules. Source and target coincide as $\mathfrak{h}$-modules. In order to prove that ϕ is an isomorphism it suffices to prove that it injects.

We have a short exact sequence

$$0 \to \widetilde{s}_{i*}\mathcal{B}_w(\lambda) \to \mathcal{R}_{*w}(\lambda) \to \mathcal{B}_{s_i w}(\lambda) \to 0$$

in $\mathrm{Hol}(\lambda)$. Here $\mathcal{B}_w(\lambda)$ is the $*$-direct image from the Schubert cell X_w.

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We have $\overline{\mathsf{H}^1}(X, \widetilde{s}_{i*}\mathcal{B}_w(\lambda)) \cong \overline{\mathsf{H}^1}(X, \mathcal{B}_w(\lambda))^{\widetilde{s}_i} = 0$. We get a surjection $\overline{\mathsf{H}^0}(X, \mathcal{R}_{*w}(\lambda)) \twoheadrightarrow \overline{\mathsf{H}^0}(X, \mathcal{B}_{s_i w}(\lambda))$ and hence an injection

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Similarly, we get an injection $\psi^{\widetilde{s}_i} : \mathrm{M}(s_i w \cdot \lambda)^{\widetilde{s}_i} \hookrightarrow \overline{\mathrm{H}^0}(X, \mathcal{R}_{*w}(\lambda))^\vee$.

Lemma

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This lemma implies

Proposition

Any nonzero submodule of $\mathrm{R}(w \cdot \lambda)$ *intersects the submodule* $\mathrm{M}(s_i w \cdot \lambda) \oplus \mathrm{M}(s_i w \cdot \lambda)^{\widetilde{s_i}}$ *nontrivially.*

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Apply the proposition to $\ker \phi$. Note that $\phi| \mathrm{M}(s_i w \cdot \lambda)$ is a nonzero multiple of ψ and similarly for $\psi^{\widetilde{s_i}}$ to conclude $\ker \phi = 0$.